HYBRID BEAMFORMING WITH TWO BIT RF PHASE SHIFTERS IN SINGLE GROUP MULTICASTING

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ABSTRACT

In this paper, an efficient hybrid beamforming architecture combining analog and digital beamforming is proposed to reduce the number of radio frequency (RF) chains. It provides a good compromise between the higher degree of freedom of digital beamforming and hardware cost and complexity. In this hybrid system, two bit RF phase shifters are used as analog beamformers due to the fact that this enables the conversion of the combinatorial optimization problem to a continuous programming formulation. The overall optimization problem for the joint design of digital and analog beamforming weights is formulated as a quadratic-cost problem which can be solved iteratively by exact penalty and semidefinite programming. Simulation results show that the proposed method designs hybrid beamformer effectively and it performs better than antenna selection for the given multicasting problem.

***Index Terms*—** Transmit beamforming, multicast beamforming, hybrid beamforming, semidefinite programming

1. INTRODUCTION

Multicast beamforming is a well known space diversity technique for transmitting common information to different groups of subscribers. It is a part of the Evolved Multimedia Broadcast Multicast Service (eMBMS) in the Long-Term Evolution (LTE) standard for efficient delivery of multicast services such as audio and video streaming. It utilizes channel state information (CSI) together with an antenna array at the transmitter side to steer power in the desired directions effectively [1].

In this paper, a special case of multicast beamforming, namely single group multicast or broadcast beamforming is considered. In single group multicasting, there is only one group with each user subscribing for the same data stream. In the design of beamformer coefficients, the main goal is to minimize the transmitted power while ensuring that the received signal-to-noise ratio (SNR) of each user is above a certain threshold. This problem was initially studied in [2]. A simple and effective solution for digital multicast beamforming is proposed recently in [3]. Although the full capacity is achieved with digital beamforming, it requires a separate RF chain for each antenna and hence its cost and complexity are high in several applications. Different methods are proposed in the literature to overcome this limitation. As the antenna technology and fabrication techniques develop, antennas become cheaper and antenna selection strategy is shown to be a good low-cost alternative to increase spatial diversity [1], [4]. Another efficient method is the use of analog beamforming structure where only one RF chain is needed with RF phase shifters and amplifiers [5]. In practice, the elements of the beamformer weight vector can take values from finite discrete sets in this scenario. This problem is solved in an optimum manner in [5] by converting the original nonlinear integer programming problem to mixed integer linear form.

Recently, hybrid beamformer structures with analog and digital parts are proposed as a good alternative to decrease hardware cost while maintaining comparable performance with respect to the completely digital beamformer [6], [7]. In this paper, we propose a special hybrid beamforming structure as shown in Fig. 1 for single group multicast beamforming. To the best of our knowledge, this is the first work which considers hybrid beamforming in the context of single group multicasting. In this hybrid structure, the number of RF chains is less than the antennas and each RF chain is followed by several RF phase shifters. The signal is first processed by the digital beamformer which is then transferred to the analog beamformer composed of RF phase shifters. Most practical RF phase shifters can supply only discrete phase changes [7]. In the proposed system, two bit RF phase shifters are used due to simplicity, low cost and effective problem formulation for the joint beamformer design. The use of two bits allows us to express the complex combinatorial problem as a continuous optimization form leading to effective solution with semidefinite programming. The resulting problem has a semidefinite programming structure except the nonconvex rank one constraint. The common technique for the solution of such a problem is semidefinite relaxation (SDR) [2], [8]. Although SDR is an efficient method for most problems in communication and signal processing, its performance is shown to degrade significantly as the number of variables increases in multicasting [9]. In our case, the non-differentiable rank constraint is converted to an equivalent quadratic constraint which is more manageable. This new constraint is moved to the objective function by exact penalty method and then the problem is solved iteratively. The solution at each iteration is optimum and the convergence is guaranteed. Several simulations are done to show the performance of the proposed method.

2. SYSTEM MODEL

Hybrid beamforming structure as shown in Fig. 1 is considered in a multicasting scenario. The transmitter broadcasts a common information to each user with a single antenna. Hybrid beamformer consists of two stages, namely digital and analog beamformers which should be jointly designed for an effective power utilization. This structure presents a trade-off between the performance and the number of RF chains. When the number of RF chains is the same as the number of antennas (completely digital beamformer), the best performance is achieved. If the number of RF chains is less than the number of antennas (i.e., hybrid beamformer), the system cost is decreased while there is a certain performance loss. In Fig. 1, there are L RF chains. Each RF chain is followed by M RF phase shifters where each one is connected to a separate antenna. The total number of antennas is LM.

In this paper, digital and analog beamformer coefficients are

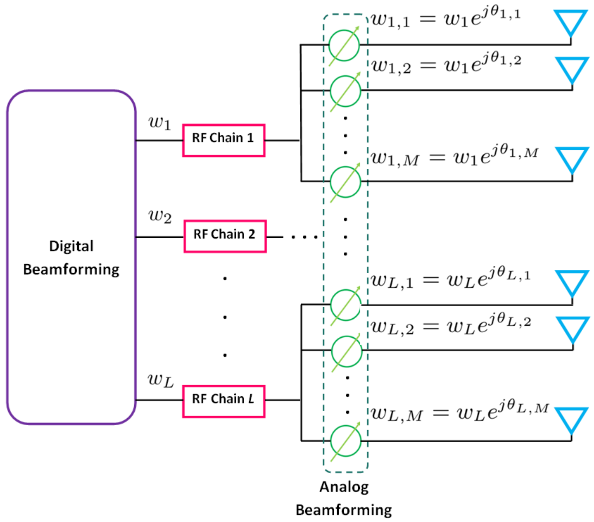

Fig. 1. Hybrid Beamforming System

jointly designed such that the quality of service (QoS) constraints are satisfied with minimum transmit power. The transmitted signal is $\mathbf{x}(t) = s(t)\mathbf{w}$ where $s(t)$ is the information signal and $\mathbf{w} = [\, w_{1,1} \, ... \, w_{1,M} \; w_{2,1} \, ... \, w_{2,M} \, ... \, w_{L,1} \, ... \, w_{L,M} \,]^T$ is the $LM \times 1$ complex beamformer weight vector where $w_{l,m} = w_l e^{j\theta_{l,m}}$. w_l is the digital beamformer coefficient corresponding to the l^{th} RF chain as in Fig. 1. $\theta_{l,m}$ corresponds to the phase of the m^{th} RF phase shifter following the l^{th} RF chain. The received signal at the k^{th} user is given as,

$$y_k(t) = \mathbf{h}_k^H \mathbf{x}(t) + n_k(t) \quad k = 1, ..., N \tag{1}$$

where $\mathbf{h}_k^H$ is the $1 \times LM$ complex channel vector for the k^{th} user and n_k is the additive noise uncorrelated with the information signal. The noise variance is σ_k^2. SNR for the k^{th} user can be written as,

$$SNR_k = \frac{\sigma_s^2 |\mathbf{w}^H \mathbf{h}_k|^2}{\sigma_k^2} \tag{2}$$

where σ_s^2 is the signal variance. $\sigma_s^2 = 1$ is selected without loss of generality throughout the paper.

In the QoS based multicast beamforming, the beamforming weight vector, $\mathbf{w}$, is chosen to minimize the transmitted power such that the SNR need of each user is satisfied. Furthermore, the elements of the beamforming weight vector, $\mathbf{w}$, should be the phase shifted versions of the digital weights $w_1, w_2, ..., w_L$. Note that the amplitude of the complex weights inside each phase shifter group should be the same, i.e., $|w_l| = |w_{l,m}|$ for $m = 1, ..., M$ and $l = 1, ..., L$. Note also that $\theta_{l,1} = 0$ (i.e., $w_{l,1} = w_l$) for $l = 1, ..., L$ can be selected without loss of generality since this phase can be added to the digital beamformer coefficient w_l. RF phase shifters usually provide discrete set of phase angles. Small number of bits in phase shifters is advantageous in terms of hardware complexity and stability. Furthermore, the system cost and the complexity are decreased for large scale antenna systems generating a convenient structure applicable to different areas such as massive MIMO and mmWave systems [7]. It turns out that the joint beamformer design has an important simplification when two bit RF phase shifters are used. This is due to the fact that for the two bit case, the discrete constraints can be written in terms of linear equality and inequalities. This is unique to the two bit structure and cannot be extended to higher bits easily.

In a two bit phase system, there are four possible discrete values for the phase shifters, i.e., $w_{l,m}/w_{l,1} \in \{1, j, -1, -j\}$, $m = 2, ..., M$, $l = 1, ..., L$.

Assuming that the CSI is available at the base station, the QoS based optimization problem can be expressed as,

$$\min_{\mathbf{w} \in \mathbb{C}^{LM}} \quad \mathbf{w}^H \mathbf{w} \tag{3.a}$$

$$s.t. \quad \mathbf{w}^H \mathbf{R}_k \mathbf{w} \geq \gamma_k \sigma_k^2, \quad k = 1, ..., N \tag{3.b}$$

$$\frac{w_{l,m}}{w_{l,1}} \in \{1, j, -1, -j\}, \quad m = 2, ..., M, \; l = 1, ..., L \tag{3.c}$$

where γ_k is the minimum SNR need for the k^{th} user and $\mathbf{R}_k = \mathbf{h}_k \mathbf{h}_k^H$. The above problem is not convex and has a combinatorial nature. The common technique to solve such problems is the matrix lifting by introducing $\mathbf{W} = \mathbf{w}\mathbf{w}^H$. In this case, the above problem can be written as,

$$\min_{\mathbf{W} \in \mathbb{C}^{LM \times LM}} \quad Tr\{\mathbf{W}\} \tag{4.a}$$

$$s.t. \quad Tr\{\mathbf{R}_k \mathbf{W}\} \geq \gamma_k \sigma_k^2, \quad k = 1, ..., N \tag{4.b}$$

$$\frac{W_{l,l}(m,1)}{W_{l,l}(1,1)} \in \{1, j, -1, -j\}, \tag{4.c}$$

$$m = 2, ..., M, \; l = 1, ..., L$$

$$\mathbf{W} \succeq 0 \tag{4.d}$$

$$rank(\mathbf{W}) = 1 \tag{4.e}$$

where $W_{l_1,l_2}(m_1, m_2)$ denotes the (m_1, m_2)-th entry of the (l_1, l_2)-th $M \times M$ submatrix of $\mathbf{W}$. The optimization problem in (4) is still nonconvex due to (4.c) and (4.e).

Lemma 1: The constraints in (4.c) can be expressed as linear equality and inequalities as follows,

$$\frac{-W_{l,l}(1,1)}{\sqrt{2}} \leq \mathrm{Re}(W_{l,l}(m,1)e^{j\pi/4}) \leq \frac{W_{l,l}(1,1)}{\sqrt{2}} \tag{5.a}$$

$$\frac{-W_{l,l}(1,1)}{\sqrt{2}} \leq \mathrm{Im}(W_{l,l}(m,1)e^{j\pi/4}) \leq \frac{W_{l,l}(1,1)}{\sqrt{2}} \tag{5.b}$$

$$W_{l,l}(m,m) = W_{l,l}(1,1), \quad m = 2, ..., M, \; l = 1, ..., L \tag{5.c}$$

Proof: $|W_{l,l}(m,1)| = W_{l,l}(m,m) = W_{l,l}(1,1)$ from (4.e) and (5.c). Hence, $(\mathrm{Re}(W_{l,l}(m,1)e^{j\pi/4}))^2 + (\mathrm{Im}(W_{l,l}(m,1)e^{j\pi/4}))^2 = W_{l,l}(1,1)^2$. In addition, $(\mathrm{Re}(W_{l,l}(m,1)e^{j\pi/4}))^2 \leq W_{l,l}(1,1)^2/2$ and $(\mathrm{Im}(W_{l,l}(m,1)e^{j\pi/4}))^2 \leq W_{l,l}(1,1)^2/2$ by (5.a-b). It turns out that (5.a-c) imply that $\mathrm{Re}(W_{l,l}(m,1)e^{j\pi/4}) = \pm W_{l,l}(1,1)/\sqrt{2}$ and $\mathrm{Im}(W_{l,l}(m,1)e^{j\pi/4}) = \pm W_{l,l}(1,1)/\sqrt{2}$. As a result, $W_{l,l}(m,1)/W_{l,l}(1,1) \in \{1, j, -1, -j\}$ which is the condition in (4.c). ■

When (4.c) is replaced by (5), the optimization problem in (4) can be solved using semidefinite relaxation by dropping the rank condition [10]. Let us denote the solution found by semidefinite relaxation as $\hat{\mathbf{W}}$ and the principal eigenvector of $\hat{\mathbf{W}}$ as $\hat{\mathbf{w}} = \mathcal{P}(\hat{\mathbf{W}})$. If the solution matrix $\hat{\mathbf{W}}$ has rank one then $\hat{\mathbf{w}}\hat{\mathbf{w}}^H = \hat{\mathbf{W}}$ and $\hat{\mathbf{w}}$ is the optimum beamforming weight vector. In SDR, rank one solution is not guaranteed, and it may return unacceptable solutions in certain problems including (4). In [9], an effective approach is presented for the semidefinite programming problems with rank one constraint. In this paper, the joint optimization problem is converted to a suitable form in order to employ the technique in [9]. This process does not change the number of variables.

3. EQUIVALENT PROBLEM

In this paper, the original problem in (4) with rank condition is converted into an equivalent form which admits more flexible and manageable solutions. In order to obtain this equivalent form, the approach presented in [9] is used. Here, only the results are given and the details of the method can be found in [9].

Lemma 2: For a Hermitian symmetric, positive semidefinite matrix $\mathbf{W}$, the condition in (6) necessitates $\mathbf{W}$ being a rank one matrix.

$$(Tr\{\mathbf{W}\})^2 - Tr\{\mathbf{W}^2\} \leq 0 \tag{6}$$

Using Lemma 2, the rank constraint in (4.e) can be replaced by (6). The only nonconvex constraint (6) can be moved into the objective function using exact penalty approach [11], [12], [13]. The following lemma establishes the equivalency of the new form and (4).

Lemma 3: ([12], page 487): The problem in (4) is equivalent to the problem in (7) for $\mu > \mu_0$ with μ_0 being a finite positive value in the sense that any local minimum of the problem in (4), which satisfies the second order sufficiency conditions, is also a local minimum of the problem in (7).

$$\min_{\mathbf{W}\in\mathbb{C}^{LM\times LM}} Tr\{\mathbf{W}\} + \mu \max(0, (Tr\{\mathbf{W}\})^2 - Tr\{\mathbf{W}^2\}) \tag{7}$$
$$s.t. \text{ (4.b), (4.d), (5.a), (5.b), (5.c)}$$

$\max(0, (Tr\{\mathbf{W}\})^2 - Tr\{\mathbf{W}^2\})$ corresponds to an exact penalty function [12], [14]. Note that $\max(0, (Tr\{\mathbf{W}\})^2 - Tr\{\mathbf{W}^2\}) = (Tr\{\mathbf{W}\})^2 - Tr\{\mathbf{W}^2\}$, and (7) can be expressed as,

$$\min_{\mathbf{W}\in\mathbb{C}^{LM\times LM}} Tr\{\mathbf{W}\} + \mu((Tr\{\mathbf{W}\})^2 - Tr\{\mathbf{W}^2\}) \tag{8}$$
$$s.t. \text{ (4.b), (4.d), (5.a), (5.b), (5.c)}$$

The problem in (8) is still nonconvex. However, the objective and constraint functions are twice differentiable and hence more manageable than the initial problem in (4). Alternating minimization can be used to solve the problem in (8) using convex optimization [15], [16]. At the iteration k, with the fixed $\mathbf{W^{k-1}}$, we can obtain $\mathbf{W^k}$ by solving the following semidefinite programming problem,

$$\begin{aligned}\min_{\mathbf{W}\in\mathbb{C}^{LM\times LM}} & Tr\{\mathbf{W}\} + \mu(Tr\{\mathbf{W^{k-1}}\}Tr\{\mathbf{W}\} \\ & -Tr\{\mathbf{W^{k-1}W}\}) \\ s.t.\ & \text{(4.b), (4.d), (5.a), (5.b), (5.c)}\end{aligned} \tag{9}$$

Then we alternate the fixed variable and find $\mathbf{W^{k+1}}$ while fixing $\mathbf{W^k}$. This alternating optimization is continued until convergence.

4. ALTERNATING MINIMIZATION ALGORITHM

In the previous parts, the problems in (4) and (8) are shown to be equivalent in the sense that they have the same local minima under certain constraint qualifications [14]. Furthermore, it is shown that (8) can be solved with alternating minimization. The convergence of this approach is guaranteed [9]. The steps for the alternating minimization algorithm for the hybrid beamforming can be presented as follows,

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Hybrid Beamforming Algorithm (HBA)

Let $\lambda_{max}(\mathbf{W})$ be the maximum eigenvalue of the matrix $\mathbf{W}$.

Initialization: $k = 0$,

Solve (9) for $\mathbf{W^0}$ while fixing $\mathbf{W^{-1}}$ as zero matrix. Set a proper μ.

Iterations: $k = k + 1$

1) Solve (9) for $\mathbf{W^k}$ while fixing $\mathbf{W^{k-1}}$. If $rank(\mathbf{W^k}) = 1$ go to step 4.

2) If $\frac{\lambda_{max}(\mathbf{W^k})}{Tr\{\mathbf{W^k}\}} \geq \beta \frac{\lambda_{max}(\mathbf{W^{k-1}})}{Tr\{\mathbf{W^{k-1}}\}}$ (improved solution), where $\beta > 1$ is a proper positive threshold value (Ex: 1.5), keep the value of μ same. Otherwise, increase μ (Ex: $\mu \to 2\mu$)

3) Terminate if the maximum iteration number, $k = k_{max}$, is reached.

End:

4) If $rank(\mathbf{W^k}) = 1$, take the beamformer weight vector as the principal eigenvector of the matrix $\mathbf{W^k}$. Otherwise, select the elements of the beamformer weight vector as,

$$w_{l,1} = \sqrt{W^k_{l,l}(1,1)}e^{\angle(W^k_{l,1}(1,1)/W^k_{1,1}(1,1))} \tag{10.a}$$

$$w_{l,m} = w_{l,1}e^{\hat{\theta}(\angle(W^k_{l,l}(m,1)/W^k_{l,l}(1,1)))} \tag{10.b}$$
$$m = 2, ..., M, \quad l = 1, ..., L$$

where $\hat{\theta}(\angle(W^k_{l,l}(m,1)/W^k_{l,l}(1,1)))$ is the quantized angle such that $\hat{\theta}(\angle(W^k_{l,l}(m,1)/W^k_{l,l}(1,1))) \in \{0, \pi/2, \pi, 3\pi/2\}$.

5) If necessary, scale $\mathbf{w}$ properly such that all SNR constraints are satisfied.

HBA searches the rank one solution in the neighborhood of the solution found by the semidefinite relaxation at the initial step of the algorithm. This process does not guarantee to find the global optimum solution.

It can be easily shown that the worst case complexity of HBA at each iteration using interior point methods is $O(\sqrt{LM}log(1/\epsilon))$ iterations where ϵ is the accuracy of the solution at termination. Each iteration requires at most $O((LM)^6 + (5L(M-1)+N)(LM)^2)$ arithmetic operations [17].

5. SIMULATION RESULTS

The proposed method, HBA, is the first approach in the literature for the solution of multicast beamforming problem in the context of a hybrid structure. HBA is an effective algorithm and can be implemented easily using standard convex programming solvers such as CVX [18]. Several simulations are done in order to show the performance of the proposed approach.

The minimum SNR threshold and the noise variance for each user are selected as $\gamma_k = 10$ and $\sigma_k^2 = 1$, respectively. The initial value of the penalty parameter is $\mu = 1$ and the threshold parameter is $\beta = 1.5$ in HBA. HBA terminates after the maximum number of iterations $k_{max} = 25$ is exceeded. The channel coefficients are chosen from the circularly symmetric complex Gaussian distribution with zero mean and unit variance. The average of 100 random channel realizations is presented for each experiment.

Fig. 2 shows the transmitted power for different number of users, N, and RF chains, L. There are $LM = 32$ antennas and the number of antennas is kept constant for all experiments. $L = 32$, $M = 1$ case is for the full digital beamformer whereas $L = 1$, $M = 32$ is for a complete analog beamformer. The remaining lines in this figure correspond to different hybrid beamformers. The performance of the proposed algorithm is compared with SDR lower bound for full digital beamforming, i.e., $L = 32$, $M = 1$. SDR lower bound presents the optimum objective value of (4) without rank condition. While this bound may not be achievable, and the solution can have rank greater than one, it allows us to evaluate the performance of the proposed approach effectively. As it is seen from Fig. 2, full digital beamformer performance is very close to the SDR lower bound. Furthermore, the results for different L and M are uniformly aligned indicating the effectiveness of the proposed approach. $LM = 32$ is kept constant. Hence, as the number of RF chains, L, increases, the transmitted power decreases as expected. However, the decrease in L together with an increase in M compensates the required power to a certain extent allowing effective hybrid structures with small number of RF chains. Another important observation is that the gap between full analog and digital beamforming increases with the number of users, reaching approximately 6 dB for $N = 20$ users.

In Fig. 3, a similar setup with Fig. 2 is used while the number of antennas is changed, i.e., $LM = 64$. A similar characteristics for the performance difference between the SDR lower bound and full digital beamforming is observed. When $LM = 64$, transmitted power decreases almost 3 dB compared to $LM = 32$ case. The performance gap between the full analog and digital beamforming is less than that of $LM = 32$ antenna case. In fact, there is an approximately 5 dB difference for $N = 20$ users.

In Fig. 4, the number of users is kept constant at $N = 12$ and the effect of the increase in the number of phase shifters per RF chain, M, is presented. As M increases, transmitted power decreases for different number of RF chains, L. As it is seen from this figure, hybrid beamformer is an effective structure to decrease the number of RF chains. For example, $L = 1$ RF chain with $M = 16$ phase shifters per RF chain, has a close performance to $L = 4$ RF chains with $M = 1$ phase shifter per RF chain.

In Fig. 5, the proposed hybrid beamforming approach is compared with one of the best performing antenna selection techniques in the literature [4]. While this comparison is not completely fair due to the differences in hardware, it gives a good idea about the performance of the proposed method. There are $LM = 32$ antennas and L RF chains are used. While antenna selection uses only L antennas, hybrid beamforming employs all antennas with the help of phase shifters following each RF chain. As it is seen from this figure, hybrid beamforming results significant power saving with the use of cost efficient and simple two bit phase shifters.

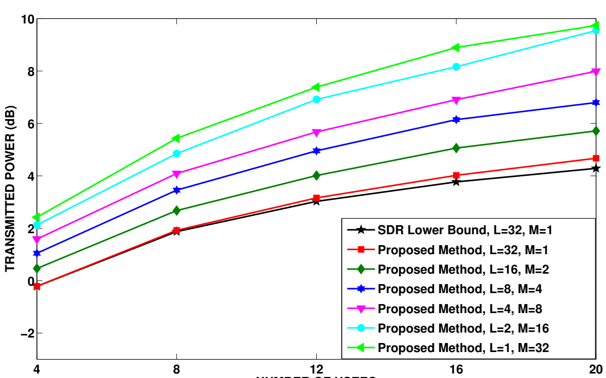

Fig. 2. Transmitted power for different number of RF chains and users for an array of $LM = 32$ antennas.

6. CONCLUSION

In this paper, a hybrid beamforming structure is proposed for single group multicasting and the joint design of analog and digital beamformers is considered. The optimization problem is converted to a quadratic-cost problem with linear constraints over a semidefinite matrix. An alternating minimization algorithm is presented for the solution of the resulting problem where semidefinite programming is used at each iteration. The proposed method is efficient in terms of both hardware complexity and the performance. Simulation results show that it is a good low-cost alternative to full digital beamforming.

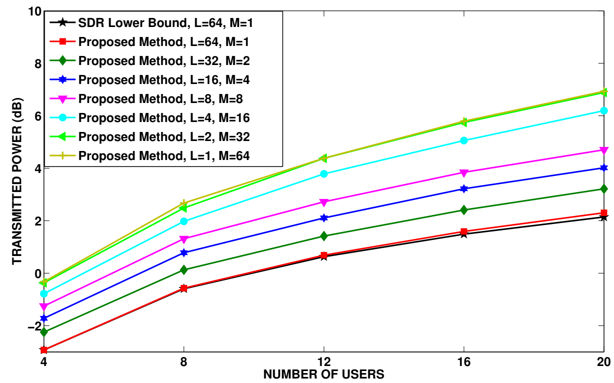

Fig. 3. Transmitted power for different number of RF chains and users for an array of $LM = 64$ antennas.

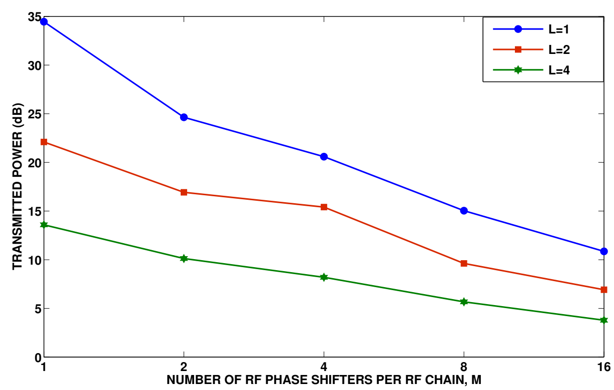

Fig. 4. Transmitted power for different number of RF chains and phase shifters for $N = 12$ users.

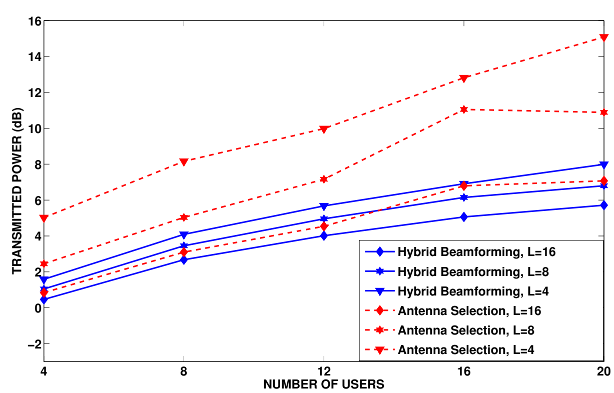

Fig. 5. Comparison of hybrid beamforming and antenna selection in terms of transmitted power.

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