ADAPTIVE BOOLEAN COMPRESSIVE SENSING BY USING MULTI-ARMED BANDIT

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ABSTRACT

A new method for solving adaptive Boolean compressive sensing is proposed. By greedy maximization of an expected information gain, a conventional method controls the pool-size for adaptive Boolean compressive sensing. However, the conventional greedy method has the drawback that it has no guarantee of convergence to the optimal strategy. To solve the problem, based on the multi-armed bandit, the proposed method controls the pool-size adaptively. The information gain of the conventional greedy method is rewritten as the reward of the multi-armed bandit, and the multi-armed bandit is introduced into adaptive Boolean compressive sensing. Experimental results indicate that the correct rate of exact recovery of the proposed method converges to 1 fast without prior knowledge about the number of defective items and that the proposed method outperforms the conventional greedy method in the case that the number of defective items is large.

***Index Terms*—** adaptive group-testing, Boolean compressive sensing, information gain, multi-armed bandit

1. INTRODUCTION

Group-testing is a well-known approach for discovering a sparse subset of defective items in a large set of items by using a small number of tests. In group-testing, each test consists of three processing steps: (1) selecting items for a pool on the basis of a certain method, (2) mixing the selected items into the pool, and (3) observing a single Boolean result by testing the pool. When the proportion of defective items is small, a small number of the tests on the mixed pool are sufficient to detect the defective items; that is, all the items need not be tested directly. Group-testing dates back to the work of Dorfman [1] in 1943, during the Second World War. Dorfman developed this approach in order to test blood for detecting sick soldiers. Nowadays, it is commonly known that group-testing has a lot of applications such as blood screening, deoxyribonucleic acid screening, and anomaly detection in computer networks [2].

Traditionally, group-testing has been regarded as a combinatorial problem. As for this problem, many researches about the upper and lower bounds on the number of tests required to find all the defective items have been done. A set of information-theoretic bounds for group-testing with random mixing was established by Malyutov [3, 4], Atia and Saligrama [5], Sejdinovic and Johnson [6], and Aldridge *et al.* [7]. In addition, several tractable approximation algorithms, such as one based on belief propagation [6] and one based on matching pursuit [8], have been proposed.

In recent years, group-testing has drawn interest from the active research area of compressive sensing. Compressive sensing solves a kind of underdetermined linear equation, namely, $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{x}$ is an unknown high-dimensional vector to be estimated, $\boldsymbol{A}$ is a given mixing matrix, and $\boldsymbol{y}$ is a given low-dimensional observed vector. The problem with compressive sensing is similar to that with group-testing from the viewpoint that both of them are underdetermined problems such that an unknown high-dimensional vector is decoded from an observed low-dimensional vector. However, while compressive sensing is defined in a real vector space, group-testing is defined in a Boolean vector space. To improve the performance of group-testing by using compressive sensing, Malioutov and Malyutov [9] proposed a method for converting group-testing into compressive sensing through linear-programming relaxation. As for this conversion method, ℓ_1 minimization imposes the sparsity constraint to the solution and solves the uncertainty of the underdetermined problem. It thus outperforms other methods (i.e., the method based on belief propagation [6], the method based on matching pursuit [8], etc.). However, the method based on linear-programming relaxation is defined in non-adaptive group-testing, which cannot choose the pool adaptively based on observation data. In particular, the optimal size of the pool depends on the number of defective items, and the number of defective items is unknown; therefore, in the case that Malioutov's method is applied, a larger number of tests are required when the pool-size is not optimal. To solve the problem that the number of positive elements is unknown in advance, approaches changing the mixing matrix adaptively, called adaptive Boolean compressive sensing, have been proposed by Kawaguchi *et al.* [10] and Kawasaki *et al.* [11]. Kawaguchi's conventional method [10] is based on greedy maximization of an expected information gain. However, the conventional greedy method has the drawback that it has no guarantee of convergence to the optimal strategy. Kawasaki's method improves the reconstruction performance by solution space reduction [11]. However, Kawasaki's method assumes that test results are not deteriorated by noise [11], and Kawasaki's method can not be applied for noisy cases.

To solve the problem that Kawaguchi's conventional greedy method [10] has no guarantee of convergence to the optimal strategy, a method for adaptive Boolean compressive sensing is proposed here. Based on the multi-armed bandit, the proposed method controls the pool-size adaptively. The information gain of the conventional greedy method [10] is rewritten as the reward of the multi-armed bandit, and the multi-armed bandit is introduced into adaptive Boolean compressive sensing. Experimental results indicate that the correct rate of exact recovery of the proposed method converges to 1 fast without prior knowledge about the number of defective items and that the proposed method outperforms the non-adaptive method [9] and the conventional greedy method [10] in the case that the number of defective items is large.

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2. PROBLEM STATEMENT

To state the problem, first, the following notation is fixed. N is the number of items, of which a subset of size K is defective. Defective items are called "positive", and non-defective items are called "negative". $x_n = 1$ indicates that the n-th item is positive, and $x_n = 0$ indicates that the n-th item is negative. For convenience, $\boldsymbol{x} = [x_1, x_2, \cdots, x_N]^T$ is written. T tests, where $T < N$, are then performed. As explained above, in each test, some items are selected from all the items, and they are mixed into the same pool. This selection is defined by a mixing matrix, $\boldsymbol{A}$, which is a $T \times N$ binary matrix. The element of the t-th row and the n-th column of $\boldsymbol{A}$ is given as a_{tn}, where $a_{t,n} = 1$ indicates that the n-th item is mixed into the pool of the t-th test, and $a_{t,n} = 0$ indicates that the n-th item is not mixed into the pool of the t-th test. The observed signal of each test, t, is a single Boolean value, $y_t \in \{0, 1\}$. y_t is obtained by taking the Boolean sum of $\{x_n | a_{tn} = 1\}$. For convenience, $\boldsymbol{y} = [y_1, y_2, \cdots, y_T]^T$ is written. The vector notation

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \tag{1}$$

is used in the following. The problem of group-testing is to estimate unknown vector $\boldsymbol{x}$ from given $\boldsymbol{A}$ and $\boldsymbol{y}$. In addition, the noise of the observation is considered. The noise includes both the false positive and the false negative. The former represents the case that $y_t = 1$ even when the Boolean sum of $\{x_n | a_{tn} = 1\}$ is 0. The latter represents the case that $y_t = 0$ even when the Boolean sum of $\{x_n | a_{tn} = 1\}$ is 1. This observation with noise is represented by

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \otimes \boldsymbol{v}, \tag{2}$$

where $\boldsymbol{v}$ is the Boolean vector of errors, and $\otimes$ means the XOR operation.

A number of works have studied the design of $\boldsymbol{A}$ [2]. For example, K-separating and K-disjunct are well-known properties of $\boldsymbol{A}$. When these properties hold, $\boldsymbol{x}$ can be recovered exactly. However, such design is often unsuitable for practical situations because it assumes that the exact number of the positive items (K) is known before group-testing. Moreover, if all T tests cannot be carried out, the performance of the method will not be guaranteed [7]. Therefore, in many works, $\boldsymbol{A}$ is simply designed by the Bernoulli random design, where each element of $\boldsymbol{A}$ is generated independently at random with a probability p corresponding with the size of the pool g. That is, a_{tn} is 1 with probability p, and a_{tn} is 0 with probability $1 - p$. Bernoulli random design is also used in this study.

One of the problems of non-adaptive group-testing is that, although the number of positive items K is unknown, the optimal pool-size g largely depends on K. To solve this problem, the present study thus focuses on adaptive group-testing. In adaptive group-testing, the mixing vector of the next test after each observation is determined according to $\boldsymbol{A}$ and $\boldsymbol{y}$. In the present study, because Bernoulli random design is used, g is controlled according to $\boldsymbol{A}$ and $\boldsymbol{y}$, and $a_{T+1,n}$ is determined randomly as follows:

$$a_{T+1,n} \overset{\text{i.i.d}}{\sim} \text{Bernoulli}(p_{T+1}) \; s.t. \; g = F_T(\boldsymbol{A}, \boldsymbol{y}), \tag{3}$$

where F_T is a function to determine the next Bernoulli probability, p_{T+1}, after the T-th test. In Section 4, a new F_T is proposed.

3. BOOLEAN COMPRESSIVE SENSING FOR GROUP-TESTING

Malioutov and Malyutov [9] proposed a conversion of group-testing into compressive sensing through a linear-programming relaxation. This method is the basis of our method, which is explained in this section.

Equation (1) is not a linear equation in a real vector space but a Boolean equation. However, it is shown in [9] that (1) can be replaced with a closely related linear formulation: $\mathbf{1} \leq \boldsymbol{A}_{\mathcal{I}}\boldsymbol{x}$, and $\mathbf{0} = \boldsymbol{A}_{\mathcal{I}}\boldsymbol{x}$, where $\mathcal{I} = \{t | y_t = 1\}$ is the set of positive test results, and $\mathcal{J} = \{t | y_t = 0\}$ is the set of negative test results. A linear-programming formulation is therefore given as

$$\begin{aligned} &\min_{\boldsymbol{x}, \boldsymbol{\xi}} \left\{ \sum_n x_n + \alpha \sum_t \xi_t \right\} \\ \text{subject to} \quad & \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}, \quad \mathbf{0} \leq \boldsymbol{\xi}_{\mathcal{I}} \leq \mathbf{1}, \quad \mathbf{0} \leq \boldsymbol{\xi}_{\mathcal{J}}, \\ & \boldsymbol{A}_{\mathcal{I}}\boldsymbol{x} + \boldsymbol{\xi}_{\mathcal{I}} \geq \mathbf{1}, \quad \boldsymbol{A}_{\mathcal{J}}\boldsymbol{x} = \boldsymbol{\xi}_{\mathcal{J}}, \end{aligned} \tag{4}$$

where $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_T]$ is the vector composed of the slack variables for preventing degradation in the case that the test-results $\boldsymbol{y}$ include noise, and α is the regularization parameter that balances the noise tolerance and the sparsity of the solution.

4. PROPOSED METHOD

The proposed method for controlling the pool-size g in adaptive Boolean compressive sensing is described as follows.

Here, similarly to the conventional control [10], expected information gain of the next $(T+1)$-th test is introduced as

$$I_{T+1}(g) = \gamma I_{\text{NEG}} + (1 - \gamma) I_{\text{POS}}, \tag{5}$$

where $I_{T+1}(g)$ is the expected information gain corresponding to the pool-size g, namely, the number of the non-zero elements of $\boldsymbol{a}_{T+1}$, γ is the probability that the result of the $(T+1)$-th test is negative, I_{NEG} is the information gain of the negative test, and I_{POS} is the information gain of the positive test. The negative test means that all the items of the pool are negative, so γ is given by

$$\gamma = \frac{\begin{pmatrix} N - |\boldsymbol{x}|_0 \\ g \end{pmatrix}}{\begin{pmatrix} N \\ g \end{pmatrix}}. \tag{6}$$

The negative test gives the information that all the items of the pool are negative, so I_{NEG} is the sum of the current entropy of the g items of the pool; therefore, I_{NEG} is given by

$$I_{\text{NEG}} = g \left\{ -r \log r - (1 - r) \log(1 - r) \right\}, \tag{7}$$

where $r = |\boldsymbol{x}|_0 / N$ is the probability that each item is positive. The positive test gives the information that there is at least one positive item in the pool; therefore, I_{POS} is given as

$$I_{\text{POS}} = \left\{ -r^g \log r^g - (1 - r^g) \log(1 - r^g) \right\}. \tag{8}$$

The temporary estimate of $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$, is obtained by using T tests, and the conventional control [10] optimizes g by maximizing $I_{T+1}(g)$ of (5) based on $|\hat{\boldsymbol{x}}|_0$. However, $\hat{\boldsymbol{x}}$ may include an estimation error because $\hat{\boldsymbol{x}}$ is only a temporary result based on a small number of tests. The greedy method for controlling g is degraded by the estimation error, so the greedy method has no guarantee of convergence to the optimal strategy. To solve the problem, the multi-armed bandit approach is introduced into the greedy method here. The multi-armed bandit was introduced by Robbins [12]. The multi-armed bandit is a method for solving the trade-off between to gain new knowledge by exploring an environment and to exploit a current reliable knowledge

[13]. There are several approaches for the multi-armed bandit. One of the approaches that have a guarantee of convergence to the optimal strategy is Upper-Confidence-Bounds (UCB) algorithm [14]. Here, the multi-armed bandit problem is defined by a random variable $R(t) \in [0,1]$ for $t \geq 1$, where $R(t)$ is called a "reward" and is yielded from the i-th machine selected at each t-th play. $R(t)$ at each play is independent and identically distributed following an unknown expected value μ_i. UCB algorithm selects the next machine to play based on the sequence of past plays and obtained rewards. At $(T+1)$-th play, UCB algorithm selects the i-th machine such that the following function $f_i(T+1)$ is maximized:

$$f_i(T+1) = \bar{\mu}_i(T+1) + c\sqrt{\frac{2\log T}{T_i}}, \tag{9}$$

where $\bar{\mu}_i(T+1)$ is the average of $R(t)$ obtained from the i-th machine, and T_i is the number of times that the i-th machine has been selected, and c is a constant positive value. Then the regret after T-th play is defined by

$$\mathbb{E}\left[\mu^* T - \sum_{t}^{T} R(t)\right], \tag{10}$$

where $\mu^* = \max_i \mu_i$. Auer *et al.* [14] show that the regret at the T-th play is bounded by:

$$8\left[\sum_{i:\mu_i<\mu^*} \frac{\log T}{\Delta_i}\right] + \left(1+\frac{\pi^2}{3}\right)\sum_i \Delta_i, \tag{11}$$

where $\Delta_i = \mu^* - \mu_i$. Here, to introduce UCB algorithm into the conventional pool-size control method, the reward $R(t)$ is replaced by the information gain, and the index of machine i is interpreted as the pool-size g. In addition, to accelerate the convergence, $\bar{\mu}_i(T+1)$ is replaced by the predicted information gain $\hat{\mu}_g(T+1)$ that can be calculated by

$$\hat{\mu}_g(T+1) = \sum_K p(|\boldsymbol{x}|_0 = K \mid |\hat{\boldsymbol{x}}|_0) \frac{I_{T+1}(g)}{\max_g I_{T+1}(g)}, \tag{12}$$

where $p(|\boldsymbol{x}|_0 = K \mid |\hat{\boldsymbol{x}}|_0)$ is the following probability distribution function of the binomial distribution:

$$\begin{aligned} & p\left(|\boldsymbol{x}|_0 = K \mid |\hat{\boldsymbol{x}}|_0\right) \\ = & \left(\begin{array}{c} N \\ |\hat{\boldsymbol{x}}|_0 \end{array}\right)\left[\frac{K}{N}\right]^{|\hat{\boldsymbol{x}}|_0}\left[1-\frac{K}{N}\right]^{N-|\hat{\boldsymbol{x}}|_0}. \end{aligned} \tag{13}$$

Therefore, the proposed method selects the pool-size g such that the following function $f_g(T+1)$ is maximized:

$$f_g(T+1) = \hat{\mu}_g(T+1) + c\sqrt{\frac{2\log T}{T_i}}. \tag{14}$$

The UCB-based proposed method also has the guarantee of convergence to the optimal strategy.

5. EXPERIMENTAL RESULTS

We evaluated the performance of the proposed method by simulation. We computed the averaged probability of the correct estimation over 100 trials as a function of T, for $N = 150$. The N items were generated independently for each trial. In this experiment, we considered the case of $\hat{\boldsymbol{x}} = \boldsymbol{x}$ as correct. We compared the non-adaptive method [9], the conventional greedy maximization of (5) [10], and the proposed method. In order to evaluate the robustness against the difference of the number of the positive items K, we conducted the experiment in the case of $K = 2$ and that of $K = 6$. Also, we calculated the optimal p for $K = 2$, i.e. $p = 0.31$, that for $K = 4$, i.e. $p = 0.2$, and that for $K = 6$, i.e. $p = 0.14$ by simulation. In the non-adaptive method, these fixed optimal p was used. In the conventional greedy method and the proposed method, the adaptively-determined p_T was used.

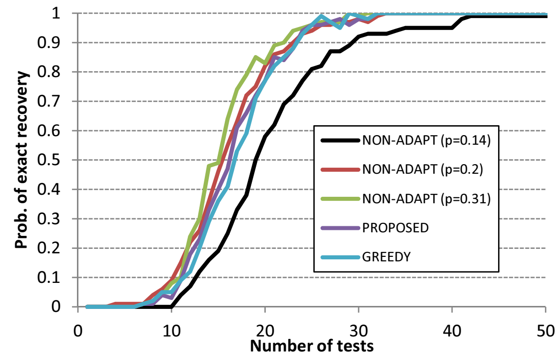

Fig. 1. Probability of exact recovery in the noiseless case as a function of the number of tests T. NON-ADAPT means the non-adaptive method [9], PROPOSED means the proposed method, and GREEDY means the conventional greedy maximization of (5) [10]. $N = 150$, $K = 2$.

First, we computed the performance in the case of no noise. Figure 1 shows the case of $K = 2$, and Fig. 2 shows the case of $K = 6$. In both cases, the convergence of the proposed method was faster than that of the worst cases of the non-adaptive method, and the correct rate after convergence was 1. The convergence speed of the proposed method was on the same level with the optimal pool-size. Also, as Fig. 2 shows, in the case that K was 6, the convergence of the proposed method was faster than that of the conventional greedy method. The results in Fig. 2 indicate that the exploration of the multi-armed bandit works well in the case that K is large. These results indicate that the control of the pool-size of the proposed method is effective.

Second, we computed the performance in the noisy case. We added noise with i.i.d 5% probability of flipping each bit of $\boldsymbol{y}$. Figure 3 shows the case of $K = 2$, and Fig. 4 shows the case of $K = 6$. Also, in these results, the convergence of the proposed method was faster than that of the worst cases of the non-adaptive method, and the correct rate after convergence was 1. The convergence speed of the proposed method was on the same level with the optimal pool-size. Also, as Fig. 4 shows, in the case that K was 6, the convergence of the proposed method was faster than that of the conventional greedy method. The results in Fig. 4 indicate that the exploration of the multi-armed bandit works well in the case that K is large under noisy conditions. These results indicate that the control of the pool-size of the proposed method is effective even under noisy conditions.

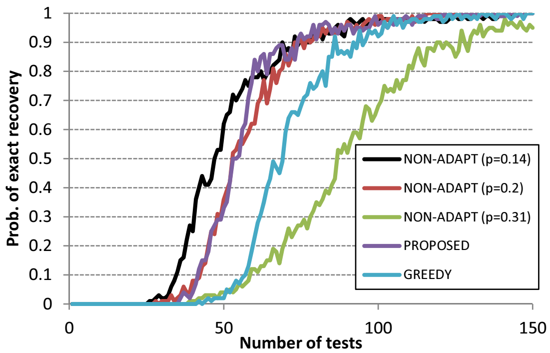

Fig. 2. Probability of exact recovery in the noiseless case as a function of the number of tests T. NON-ADAPT means the non-adaptive method [9], PROPOSED means the proposed method, and GREEDY means the conventional greedy maximization of (5) [10]. $N = 150$, $K = 6$.

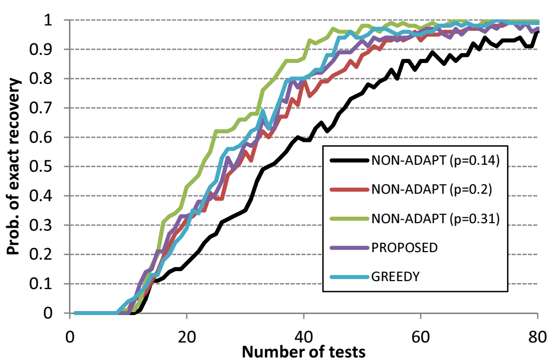

Fig. 3. Probability of exact recovery in the noisy case as a function of the number of tests T. NON-ADAPT means the non-adaptive method [9], PROPOSED means the proposed method, and GREEDY means the conventional greedy maximization of (5) [10]. $N = 150$, $K = 2$.

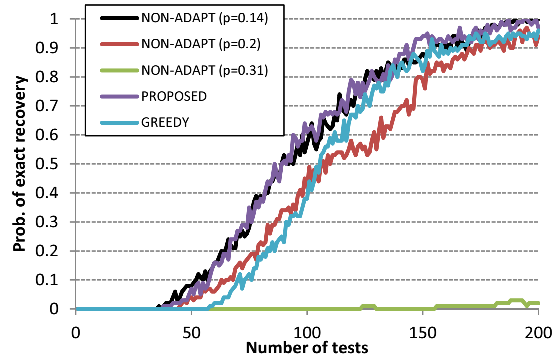

Fig. 4. Probability of exact recovery in the noisy case as a function of the number of tests T. NON-ADAPT means the non-adaptive method [9], PROPOSED means the proposed method, and GREEDY means the conventional greedy maximization of (5) [10]. $N = 150$, $K = 6$.

6. CONCLUSION

A new method for solving adaptive Boolean compressive sensing was proposed. To achieve the guarantee of convergence to the optimal strategy, based on the multi-armed bandit, the proposed method controls the pool-size adaptively. The information gain of the conventional greedy method was rewritten as the reward of the multi-armed bandit, and the multi-armed bandit was introduced into adaptive Boolean compressive sensing. In an experimental evaluation of the method, it was shown that the correct rate of exact recovery of the proposed method converges to 1 fast without prior knowledge about the number of defective items and that the proposed method outperforms the non-adaptive method and the conventional greedy method in the case that the number of defective items is large.

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