SPARSITY-BASED DIRECTION-OF-ARRIVAL ESTIMATION FOR STRICTLY NON-CIRCULAR SOURCES

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ABSTRACT

Direction of arrival (DOA) estimation via sparse signal recovery (SSR) has recently attracted a considerable research interest due to its various advantages over the conventional DOA estimation methods. Yet, the performance of the SSR-based algorithms can be further enhanced by exploiting the structure of strictly non-circular (NC) signals. In this paper, we present a novel strategy to take the NC signal structure into account for the SSR, which results in a two-dimensional SSR problem. Thereby, the known benefits associated with NC sources can be achieved. Moreover, we address the 2-D off-grid problem by proposing a low-complexity procedure that estimates the sources' grid offset from the closest neighboring grid points. For a single off-grid source, we show analytically that the 2-D offset estimation problem is separable, allowing to perform the offset estimation in both dimensions independently. We also propose a numerical procedure for the joint estimation of the grid offsets of two closely-spaced sources. The effectiveness of the proposed methods is demonstrated via simulations.

Index Terms— Compressed sensing, sparse signal recovery, non-circular sources, off-grid model, DOA estimation.

1. INTRODUCTION

Estimating the directions of arrival (DOAs) of signals captured by an antenna array has long been of great research interest [1] due to its wide applications in radar, sonar, channel sounding, and wireless communications. Previous work has shown that if the signals exhibit a strictly second-order (SO) non-circular (NC) structure [2], exploiting this property can improve the performance of the conventional parameter estimation algorithms [1]. Consequently, a great number of DOA estimation algorithms have been proposed [3]-[7] that take the signals' non-circularity into account to improve the estimation accuracy and double the number of identifiable sources. Such strictly non-circular signals are used in various digital modulation schemes such as BPSK, ASK, Offset-QPSK, PAM, etc. The more general case of coexisting circular and strictly non-circular sources has been addressed in [8]-[10].

In recent years, the new concept of addressing the conventional DOA estimation problem via sparse signal recovery (SSR) or compressed sensing (CS) [11] has attracted significant research attention. According to this representation, the array response is modeled as the superposition of few wavefronts in an overcomplete basis, i.e., the received signal power is sparse in the angular domain. Various sparsity-based DOA estimation algorithms have been proposed of

late [12]-[15]. It has been observed that these sparsity-based algorithms, when compared to the conventional methods, can provide benefits in challenging scenarios such as a high source correlation, a low sample size, unknown model order, etc. However, one common problem they face is the required sampling of the continuous angular domain with a predefined grid in order to construct a finite dictionary. As a consequence, the true DOAs mostly lie off the discretized grid, which results in a performance degradation due to the model mismatch. Various solutions to the off-grid problem include an adaptive refinement of the grid [12], statistical modeling and fitting of the mismatch error [16], and a low-complexity analytical solution by explicitly estimating the grid offset [17].

The concept of exploiting the non-circularity property has recently been introduced for sparsity-based DOA estimation [18], [19]. While [18] proposes a sparse covariance matrix representation of the SO statistics of the non-circular data, in [19], the authors adopt a strategy, which relies on a sparsity-based fitting of the NC subspaces. However, both algorithms require a rather complex setting of the sparsity-inducing parameters depending on the scenario, are limited to the case of uncorrelated sources, and do not deal with the critical off-grid problem.

In this paper, we present a new approach to exploiting the signals' strict non-circularity via sparse recovery. As NC signals usually have an unknown rotation phase, we need to introduce another finite dictionary for the rotation phase domain along with that for the spatial domain. Therefore, the recovery problem results in a twodimensional (2-D) sparse power spectrum estimation. By adding this additional dimension to the original SSR problem, the known benefits associated with NC sources [3]-[7] can be achieved analogously via sparse signal representation. As a result, even closely-spaced NC sources can be distinguished by their phase discrimination. In order to handle the resulting 2-D off-grid problem, we first show analytically by considering a single NC off-grid source that the 2-D grid offset estimation is separable, which enables the application of the low-complexity grid offset estimation [17] in both dimensions independently. Additionally, we propose a numerical joint offset estimation procedure for two closely-spaced NC sources. Simulation results demonstrate the effectiveness of the proposed methods.

2. SYSTEM MODEL AND NC PREPROCESSING

Suppose that d narrow-band planar wavefronts from stationary sources in the far field are captured by an M-element sensor array. The noise-corrupted array output at T subsequent snapshots can be collected in the measurement matrix

$$\boldsymbol{X} = \boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{S} + \boldsymbol{N} \in \mathbb{C}^{M \times T}, \tag{1}$$

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where $A(\mu) = [a(\mu_1), \ldots, a(\mu_d)] \in \mathbb{C}^{M \times d}$ contains the array steering vectors for the spatial frequencies $\mu = [\mu_1, \ldots, \mu_d]^T$, $S \in \mathbb{C}^{d \times T}$ represents the symbols, and $N \in \mathbb{C}^{M \times T}$ consists of the additive white Gaussian sensor noise samples. In the case of strictly non-circular sources, S can be written as $S = \Psi S_0$, where $\Psi =$ diag $\{e^{j\varphi_i}\}_{i=1}^d$ contains the rotation phase shifts $\varphi = [\varphi_1, \ldots, \varphi_d]^T$ on its diagonal and $S_0 \in \mathbb{R}^{d \times T}$ is a real-valued symbol matrix.

In order to exploit the structure of the strictly non-circular sources, the following preprocessing scheme is applied to (1) to form the augmented measurement matrix $X^{(nc)} \in \mathbb{C}^{2M \times T}$ [6], [7]

$$\boldsymbol{X}^{(\mathrm{nc})} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{\Pi}_{M} \boldsymbol{X}^{*} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{\mu}) \boldsymbol{\Psi} \\ \boldsymbol{\Pi}_{M} \boldsymbol{A}^{*}(\boldsymbol{\mu}) \boldsymbol{\Psi}^{*} \end{bmatrix} \boldsymbol{S}_{0} + \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{\Pi}_{M} \boldsymbol{N}^{*} \end{bmatrix}$$
(2)

$$= \boldsymbol{A}^{(\mathrm{nc})}(\boldsymbol{\mu}, \boldsymbol{\varphi})\boldsymbol{S}_0 + \boldsymbol{N}^{(\mathrm{nc})},$$
(3)

where Π_M is the $M \times M$ matrix with ones on its anti-diagonal. It has been shown that processing $\mathbf{X}^{(nc)}$ instead of \mathbf{X} reduces the estimation error and doubles the number of resolvable sources [7].

3. SPARSE NC SIGNAL RECOVERY

In this section, we apply the preprocessing scheme for NC sources from the previous section to sparsity-based DOA estimation.

We start with the common sparse signal representation of (1), which interprets X as a set of multiple measurement vectors (MMV) that is *d*-sparse in an overcomplete basis obtained by discretizing the array manifold. Thus, an equivalent representation of (1) is given by

$$\boldsymbol{X} = \tilde{\boldsymbol{A}}(\tilde{\boldsymbol{\mu}})\tilde{\boldsymbol{S}} + \boldsymbol{N} \in \mathbb{C}^{M \times T}, \tag{4}$$

where $\tilde{A}(\tilde{\mu}) \in \mathbb{C}^{M \times N_{\mu}}$ contains the sampling of the spatial frequency range $[0, 2\pi]$ at the N_{μ} grid points $\tilde{\mu} = [\tilde{\mu}_1, \dots, \tilde{\mu}_{N_{\mu}}]^{\mathrm{T}}$. Typically, $N_{\mu} = MP_{\mu}$, where $P_{\mu} > 1$ is the oversampling factor such that $N_{\mu} > M > d$. For simplicity, we consider uniform sampling with $\tilde{\mu}_{n_{\mu}} = (n_{\mu} - 1)\Delta_{\mu}, n_{\mu} = 1, \dots, N_{\mu}$, where $\Delta_{\mu} = 2\pi/N_{\mu}$ is the grid spacing. Moreover, $\tilde{S} \in \mathbb{C}^{N_{\mu} \times T}$ is the row-sparse matrix of interest, i.e., its columns share the same support. The support of the non-zero rows of \tilde{S} then corresponds to the locations of the DOAs on the spatial grid.

3.1. Preprocessing for Non-Circular Sources

In order to benefit from the above-mentioned advantages of processing NC sources in SSR based on (4), we exploit the NC structure of \tilde{S} by applying a preprocessing step similar to that in (3). However, since the array steering vectors in $A^{(nc)}(\mu, \varphi)$ in (3) depend on the unknown matrix Ψ , we need to introduce a second grid by discretizing the rotation phase domain, resulting in a 2-D grid. Then, a sparse representation of $X^{(nc)}$ accounting for the NC structure is given by

$$\boldsymbol{X}^{(\mathrm{nc})} = \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\varphi}})\tilde{\boldsymbol{S}}_0 + \boldsymbol{N}^{(\mathrm{nc})} = \boldsymbol{X}_0^{(\mathrm{nc})} + \boldsymbol{N}^{(\mathrm{nc})}, \quad (5)$$

where $\tilde{A}^{(nc)}(\tilde{\mu}, \tilde{\varphi}) \in \mathbb{C}^{2M \times N_{\mu}N_{\varphi}}$ is the new dictionary and $\tilde{S}_0 \in \mathbb{R}^{N_{\mu}N_{\varphi} \times T}$ is the corresponding real-valued row-sparse matrix. The 2-D grid embedded in $\tilde{A}^{(nc)}(\tilde{\mu}, \tilde{\varphi})$ is defined by the $N_{\mu}N_{\varphi}$ tuples $(\tilde{\mu}_{n_{\mu}}, \tilde{\varphi}_{n_{\varphi}}), n_{\mu} = 1, \ldots, N_{\mu}; n_{\varphi} = 1, \ldots, N_{\varphi}$. Thus, the rotation phase range $[0, \pi]$ is sampled at $\tilde{\varphi} = [\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{N_{\varphi}}]^T$ with the uniform grid $\tilde{\varphi}_{n_{\varphi}} = (n_{\varphi} - 1)\Delta_{\varphi}, n_{\varphi} = 1, \ldots, N_{\varphi}$, where $\Delta_{\varphi} = \pi/N_{\varphi}$ and $N_{\varphi} = MP_{\varphi}$ with $P_{\varphi} > 1$.

Then, the overcomplete dictionary $ilde{A}^{(\mathrm{nc})}(ilde{\mu}, ilde{arphi})$ is defined as

$$\tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\varphi}}) = \begin{bmatrix} \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\boldsymbol{\mu}}, \tilde{\varphi}_1) & \cdots & \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\boldsymbol{\mu}}, \tilde{\varphi}_{N_{\varphi}}) \end{bmatrix}, \quad (6)$$

where $\tilde{A}^{(nc)}(\tilde{\mu}, \tilde{\varphi}_{n_{\varphi}}) = [A^{T}(\tilde{\mu})e^{j\tilde{\varphi}_{n_{\varphi}}}, (\Pi_{M}A^{*}(\tilde{\mu}))^{T}e^{-j\tilde{\varphi}_{n_{\varphi}}}]^{T} \in \mathbb{C}^{2M \times N_{\mu}}$. Therefore, the effective $N_{\mu}N_{\varphi}$ -point sampling grid is

given by the points $k = (n_{\varphi} - 1)N_{\mu} + n_{\mu}$. Note that the extended row dimensions of the dictionary $\tilde{A}^{(nc)}(\tilde{\mu}, \tilde{\varphi})$ can be interpreted as a virtual doubling of the number of sensor elements.

If the array is centro-symmetric, i.e., it is symmetric with respect to its centroid, and its phase reference is at the center, the property $\Pi_M \tilde{A}^*(\tilde{\mu}) = \tilde{A}(\tilde{\mu})$ holds. Then, the 2-D dictionary $\tilde{A}^{(\rm nc)}(\tilde{\mu}, \tilde{\varphi})$ in (6) can be compactly expressed as

$$\tilde{A}^{(\mathrm{nc})}(\tilde{\mu}, \tilde{\varphi}) = B(\tilde{\varphi}) \otimes \tilde{A}(\tilde{\mu}),$$
(7)

where $B(\tilde{\varphi}) = [b(\tilde{\varphi}_1), \dots, b(\tilde{\varphi}_{N_{\varphi}})] \in \mathbb{C}^{2 \times N_{\varphi}}$ with $b(\tilde{\varphi}_{n_{\varphi}}) = [e^{j\tilde{\varphi}_{n_{\varphi}}}, e^{-j\tilde{\varphi}_{n_{\varphi}}}]^{\mathrm{T}}$ and \otimes denotes the Kronecker product.

3.2. NC Signal Recovery Problem

In SSR, solving the ℓ_0 -problem to estimate the signal support, i.e., min $\|\tilde{S}_0\|_{2,0}$, where $\|X\|_{p,q} = [\sum_{j=1}^N (\sum_{i=1}^n |X_{j,i}|^p)^{q/p}]^{1/q}$, is NP-hard. Therefore, the original ℓ_0 -problem is usually approximated by a convex ℓ_1 -problem. The augmented MMV problem based on (5) can be formulated as

$$\min_{\tilde{\boldsymbol{S}}_0 \in \mathbb{R}^{N_\mu N_\varphi \times T}} \|\tilde{\boldsymbol{S}}_0\|_{2,1}$$
(8)

s. t.
$$\|\boldsymbol{X}^{(\mathrm{nc})} - \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\varphi}})\tilde{\boldsymbol{S}}_0\|_F \leq \sqrt{\beta},$$
 (9)

where β is chosen as $\beta = \text{Tr}\{\mathbb{E}\{N^{(\text{nc})}N^{(\text{nc})^{H}}\}\} = 2MN\sigma_{n}^{2}$. Problem (9) can be solved by any sparse recovery algorithm, e.g., ℓ_1 -type algorithms such as the basis pursuit denoising (BPDN) algorithm [20] or greedy algorithms such as the orthogonal matching pursuit (OMP) [21]. Due to the effective sampling grid, closelyspaced NC sources (even if they are on the same grid point) are well-separated as long as they have a rotation phase discrimination. Therefore, not only the support estimation is improved, but also the estimated amplitudes of the sparse components in the spectrum. The support in both dimensions can be found by matching the effective grid into the 2-D grid.

3.3. Number of Resolvable Sources

It was shown in [12] through simulations and later proven in [13] that at most M - 1 sources can be resolved via the conventional MMV model in (4). Supported by strong numerical evidence, this suggests that due to the doubling of the dimensions in (5) at most 2(M - 1)NC sources can be uniquely resolved.

4. 2-D OFFGRID ESTIMATION

If all the NC sources lie on a grid point, the model in (5) holds exactly. However, this assumption is highly unrealistic in practice, leading to a 2-D off-grid problem. In order to handle this model mismatch, we extend the previous work on the 1-D off-grid problem [17] to the NC case as well as the MMV case. As a result, we propose two efficient solutions for a single NC off-grid source and two closelyspaced NC off-grid sources that are used as a post-processing step after the support estimation via SSR. We show analytically for a single off-grid source that the atoms in the 2-D grid are separable, i.e., the off-grid estimation can be performed in both dimensions separately. Based on these findings, two simple low-complexity estimators are presented that require a considerably lower computational complexity compared to the SSR.

Let us first introduce the off-grid model for the *i*-th source located at the pair (μ_i, φ_i) , $i = 1, \ldots, d$, in both dimensions as

$$\mu_{i} = \tilde{\mu}_{L_{\mu_{i}}} + \epsilon_{i} (\tilde{\mu}_{L_{\mu_{i}}+1} - \tilde{\mu}_{L_{\mu_{i}}}) = (L_{\mu_{i}} - 1 + \epsilon_{i})\Delta_{\mu}, \quad (10)$$

$$\varphi_i = \tilde{\varphi}_{L_{\varphi_i}} + \delta_i (\tilde{\varphi}_{L_{\varphi_i}+1} - \tilde{\varphi}_{L_{\varphi_i}}) = (L_{\varphi_i} - 1 + \delta_i) \Delta_{\varphi}, \quad (11)$$

where we have used the defined uniform sampling grid on the right hand side. Moreover, L_{μ_i} and L_{φ_i} are the respective nearest left grid points obtained from the support estimation, and ϵ_i , $\delta_i \in [0, 1]$ model the grid offset. It should be noted that for $\epsilon_i = \delta_i = 0$, (10) and (11) reduce to the on-grid model in (5).

For simplicity, we assume a uniform linear array (ULA) of isotropic sensors as an example for centro-symmetric arrays. The steering vectors are given by $\boldsymbol{a}(\mu_i) = \left[e^{-j\frac{M-1}{2}\mu_i}, \ldots, e^{j\frac{M-1}{2}\mu_i}\right]^T$, i.e., the phase reference coincides with the array centroid. Under this assumption, the augmented steering vector $\boldsymbol{a}^{(nc)}(\mu_i, \varphi_i)$ can be expressed according to (7) as $\boldsymbol{a}^{(nc)}(\mu_i, \varphi_i) = [e^{j\varphi_i}, e^{-j\varphi_i}]^T \otimes \boldsymbol{a}(\mu_i)$.

4.1. Single Off-Grid Source

To gain more insights into the 2-D off-grid problem, we first consider a single NC off-grid source. In [17], it has been shown that each 1-D off-grid source can be well approximated by the atoms corresponding to the two closest grid points. As a consequence, sparse recovery algorithms concentrate the signal power at exactly those grid points. This suggests that a similar approach can be used in the 2-D case for an NC off-grid source. Thus, based on the relative height of the peaks at the two neighboring atoms in both grid dimensions, we can estimate the corresponding two offsets to approximate the NC offgrid source.

In the noiseless case, the model (3) for a single NC source simplifies to $\mathbf{X}_{0}^{(\mathrm{nc})} = \mathbf{a}^{(\mathrm{nc})}(\epsilon, \delta) \mathbf{s}_{0}^{\mathrm{T}}$, where $\mathbf{a}^{(\mathrm{nc})}(\epsilon, \delta) = \mathbf{a}^{(\mathrm{nc})}(\tilde{\mu}_{L} + \epsilon \Delta_{\mu}, \tilde{\varphi}_{L} + \delta \Delta_{\varphi}) \in \mathbb{C}^{2M \times 1}$ is the true steering vector and $\mathbf{s}_{0} \in \mathbb{R}^{T \times 1}$. Thus, we represent $\mathbf{a}^{(\mathrm{nc})}(\epsilon, \delta)$ by the linear model

$$\boldsymbol{a}^{(\mathrm{nc})}(\epsilon,\delta) \approx \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\mu}_L,\tilde{\mu}_{L+1},\tilde{\varphi}_L,\tilde{\varphi}_{L+1})\boldsymbol{\alpha}, \qquad (12)$$

where the matrix $\tilde{A}^{(nc)}(\tilde{\mu}_L, \tilde{\mu}_{L+1}, \tilde{\varphi}_L, \tilde{\varphi}_{L+1}) = [\tilde{a}^{(nc)}(\tilde{\mu}_L, \tilde{\varphi}_L), \tilde{a}^{(nc)}(\tilde{\mu}_{L+1}, \tilde{\varphi}_L), \tilde{a}^{(nc)}(\tilde{\mu}_{L+1}, \tilde{\varphi}_{L+1})] \in \mathbb{C}^{2M \times 4}$ contains the neighboring grid points and $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_4]^{\mathrm{T}}$ contains the coefficients. The coefficients $\boldsymbol{\alpha}$ can be found by solving the least squares problem

$$\min_{\boldsymbol{\alpha}} \quad \left\| \boldsymbol{X}_{0}^{(\mathrm{nc})} - \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\mu}_{L}, \tilde{\mu}_{L+1}, \tilde{\varphi}_{L}, \tilde{\varphi}_{L+1}) \boldsymbol{\alpha} \boldsymbol{s}_{0}^{\mathrm{T}} \right\|_{F}^{2}, \quad (13)$$

where $s_0^{\rm T}$ on both sides can be neglected. The solution is given by

$$\hat{\boldsymbol{\alpha}}(\epsilon,\delta) = \tilde{\boldsymbol{A}}^{(\mathrm{nc})}(\tilde{\mu}_L, \tilde{\mu}_{L+1}, \tilde{\varphi}_L, \tilde{\varphi}_{L+1})^+ \boldsymbol{a}^{(\mathrm{nc})}(\epsilon, \delta), \qquad (14)$$

where $(\cdot)^+$ denotes the Moore-Penrose pseudo inverse. After expanding the pseudo inverse in (14), we note that for instance $a_{\tilde{\mu}L}^H a_{\tilde{\mu}L+1} = \sum_{m=-(M-1)/2}^{(M-1)/2} e^{jm\Delta_{\mu}} = D(\Delta_{\mu})$, which is similar for the combinations of $\tilde{\mu}_L, \tilde{\mu}_{L+1}, \tilde{\mu}_L + \epsilon \Delta_{\mu}$, and where we have defined

$$D(y) = \begin{cases} M & \text{if } y = 0\\ \sin(yM/2)/\sin(y/2) & \text{otherwise.} \end{cases}$$
(15)

Then, denoting $\boldsymbol{d}(x) = [D(x\Delta_{\mu}), D((x-1)\Delta_{\mu})]^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$ and $\boldsymbol{c}(x) = [\cos(x\Delta_{\varphi}), \cos((x-1)\Delta_{\varphi})]^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$ as well as $\boldsymbol{D}(x) = [\boldsymbol{d}(x), \boldsymbol{d}(x+1)] \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{C}(x) = [\boldsymbol{c}(x), \boldsymbol{c}(x+1)] \in \mathbb{R}^{2\times 2}$, we obtain

$$\hat{\boldsymbol{\alpha}}(\epsilon,\delta) = \left(\boldsymbol{C}(0) \otimes \boldsymbol{D}(0)\right)^{-1} \left(\boldsymbol{c}(\delta) \otimes \boldsymbol{d}(\epsilon)\right)$$
(16)

$$= \boldsymbol{\alpha}(\delta) \otimes \boldsymbol{\alpha}(\epsilon), \tag{17}$$

where $\boldsymbol{\alpha}(\delta) = [\alpha_1(\delta), \alpha_2(\delta)]^T$ and $\boldsymbol{\alpha}(\epsilon) = [\alpha_1(\epsilon), \alpha_2(\epsilon)]^T$ with

$$\alpha_1(\delta) = \frac{\cos(\delta\Delta_{\varphi}) - \cos(\Delta_{\varphi})\cos((\delta - 1)\Delta_{\varphi})}{1 - \cos^2(\Delta_{\varphi})} \qquad (18)$$

$$\alpha_2(\delta) = \frac{\cos((\delta - 1)\Delta_{\varphi}) - \cos(\Delta_{\varphi})\cos(\delta\Delta_{\varphi})}{1 - \cos^2(\Delta_{\varphi})}$$
(19)

$$\alpha_1(\epsilon) = \frac{MD(\epsilon\Delta_{\mu}) - D(\Delta_{\mu})D((\epsilon - 1)\Delta_{\mu})}{M^2 - D^2(\Delta_{\mu})}$$
(20)

$$\alpha_1(\epsilon) = \frac{MD((\epsilon - 1)\Delta_{\mu}) - D(\Delta_{\mu})D(\epsilon\Delta_{\mu})}{M^2 - D^2(\Delta_{\mu})}.$$
 (21)

Thus, the 2-D offset estimation is separable in both grid dimensions. As it can be shown that $\alpha_n(\epsilon)$ and $\alpha_n(\delta)$, n = 1, 2, become linear in ϵ and δ with increasing P_{μ} and P_{φ} , the simple 1-D estimator from [17]

$$\hat{\epsilon} = \frac{\alpha_2(\epsilon)}{\alpha_1(\epsilon) + \alpha_2(\epsilon)}, \quad \hat{\delta} = \frac{\alpha_2(\delta)}{\alpha_1(\delta) + \alpha_2(\delta)}$$
(22)

can be applied in both dimensions independently. Arranging the elements of $\hat{\alpha}$ in a matrix \boldsymbol{B} , we obtain $\boldsymbol{B} = \boldsymbol{\alpha}(\delta)\boldsymbol{\alpha}(\epsilon)^{\mathrm{T}}$, which is of rank 1.

In the noisy case, the matrix $X_0^{(nc)}$ in (13) needs to be replaced by $X^{(nc)}$. As a consequence, B becomes an estimate that is not rank one anymore. However, a good rank-one approximation can be obtained from the SVD of \hat{B} , i.e., $\hat{B} = uv^{\rm H} = \hat{\alpha}(\delta)\hat{\alpha}^{\rm T}(\epsilon)$. Based on $\hat{\alpha}(\delta)$ and $\hat{\alpha}(\epsilon)$, the two estimators in (22) are used to determine the grid offsets.

4.2. Two Off-Grid Sources

In the case of two sources, their mutual influence depends on the correlation between the array steering vectors. If the source separation is much larger than P_{μ} grid points, which corresponds to the Rayleigh resolution limit $2\pi/M$, they are separable. Due to the effective grid, this is the case for two closely-spaced NC sources if their phases discriminate. Therefore, the two sources can be treated independently and the estimators (22) can be applied for each source separately.

However, this approach fails for closely-spaced sources with the same rotation phase. Thus, inspired by [17], we propose a numerical joint off-grid estimation procedure for two NC sources with the same rotation phase. Assuming that the correct support has been estimated via SSR, we approximate each of the two sources located at (μ_i, φ_i) , i = 1, 2, by the four respective neighboring grid points $L_{\mu_i}, L_{\mu_i} + 1, L_{\varphi_i}$, and $L_{\varphi_i} + 1$ as an extension of (12). Denote $\tilde{A}_i^{(nc)} \in \mathbb{C}^{2M \times 4}$ as the matrix containing the neighboring grid points of the *i*-th source as in (12). Then, in analogy to the noiseless case in (14), we have $G(\epsilon, \delta) = \tilde{A}_{1,2}^{(nc)+} X_0^{(nc)} = \tilde{A}_{1,2}^{(nc)+} A^{(nc)}(\epsilon, \delta) S_0$, where $\tilde{A}_{1,2}^{(nc)} = [\tilde{A}_1^{(nc)}, \tilde{A}_2^{(nc)}] \in \mathbb{C}^{2M \times 8}$ and $G(\epsilon, \delta) \in \mathbb{R}^{8 \times T}$ is the matrix of coefficients that depends on $\epsilon = [\epsilon_1, \epsilon_2]^T$ and $\delta = [\delta_1, \delta_2]^T$. It can be shown that $G(\epsilon, \delta)$ can be expressed as $G(\epsilon, \delta) = D_0^{-1} D(\epsilon, \delta) S_0$, where

$$\boldsymbol{D}_{0} = \begin{bmatrix} \boldsymbol{C}(0) \otimes \boldsymbol{D}(0) & \boldsymbol{C}(d_{\varphi}) \otimes \boldsymbol{D}(d_{\mu}) \\ \boldsymbol{C}(d_{\varphi})^{\mathrm{T}} \otimes \boldsymbol{D}(d_{\mu})^{\mathrm{T}} & \boldsymbol{C}(0) \otimes \boldsymbol{D}(0) \end{bmatrix} \in \mathbb{R}^{8 \times 8},$$
$$\boldsymbol{D}(\boldsymbol{\epsilon}, \boldsymbol{\delta}) = \begin{bmatrix} \boldsymbol{c}(\delta_{1}) \otimes \boldsymbol{d}(\epsilon_{1}) & \boldsymbol{c}(\delta_{2} + d_{\varphi}) \otimes \boldsymbol{d}(\epsilon_{2} + d_{\mu}) \\ \boldsymbol{c}(\delta_{1} - d_{\varphi}) \otimes \boldsymbol{d}(\epsilon_{1} - 1) & \boldsymbol{c}(\delta_{2}) \otimes \boldsymbol{d}(\epsilon_{2}) \end{bmatrix} \in \mathbb{R}^{8 \times 2},$$

where $d_{\mu} = L_{\mu_2} - L_{\mu_1}$ and $d_{\varphi} = L_{\varphi_2} - L_{\varphi_1}$. Next, we consider the noisy case, which yields $\hat{G} = \tilde{A}_{1,2}^{(nc)+} X^{(nc)}$, and define $\bar{G} = D_0 \hat{G}$. For the comparison of the coefficients \bar{G} obtained from the measurements with the analytical approximation $G(\epsilon, \delta)$, we note that the columns of $G(\epsilon, \delta)$ are a linear combination of the columns of $D(\epsilon, \delta)$. Thus, we aim at maximizing the overlap between \overline{G} and the column space of $D(\epsilon, \delta)$. To this end, we propose to estimate the off-grid parameters by minimizing the cost function

$$J(\boldsymbol{\epsilon}, \boldsymbol{\delta}) = \left\| \bar{\boldsymbol{G}} - \boldsymbol{D}(\boldsymbol{\epsilon}, \boldsymbol{\delta}) \boldsymbol{D}^{+}(\boldsymbol{\epsilon}, \boldsymbol{\delta}) \bar{\boldsymbol{G}} \right\|_{F}^{2}.$$
 (23)

We have observed that the function $J(\epsilon, \delta)$ is smooth and convex in the parameter range $\epsilon_i, \delta_i \in [0, 1]$ with a unique minimum. Therefore, (23) can be minimized by any local optimization method, e.g., the gradient descent algorithm.

It is worth mentioning that the presented joint estimation procedure can be extended straightforwardly if a group of more than two closely-spaced sources with the same rotation phase is present.

5. SIMULATION RESULTS

In this section, we present simulations to assess the performance of the proposed SSR algorithm for strictly non-circular sources. To this end, we use the OMP algorithm for the SSR step and compare the proposed offset estimation scheme "NC OMP Joint" according to (23) to its non-NC counterpart from [17]. For the stopping criteria of OMP, the sparsity level d is assumed known. Note that the OMP method can be replaced by any other SSR algorithm. We also consider the deterministic CRB "Det CRB" as well as the deterministic CRB for NC sources "Det NC CRB" [22]. For the computation of the mean square error (MSE), we only take the estimation error in the spatial frequency domain μ into account as the estimation of the rotation phases is not of primary interest, but can be added straightforwardly. For the numerical results, we adopt a ULA of M = 8isotropic sensors with half-wavelength spacing. The phase reference is at the centroid. The symbols are drawn from a real-valued Gaussian distribution and the noise is circularly symmetric white complex Gaussian with $\sigma_n^2 = 1/SNR$. We have used 300 Monte Carlo trials.

In Fig. 1, we display the MSE versus the SNR for a scenario, where $P_{\mu} = 8$, $P_{\varphi} = 6$, and d = 2 uncorrelated sources are located at $(15.1\Delta_{\mu}, 10.2\Delta_{\varphi})$ and $(17.5\Delta_{\mu}, 34.2\Delta_{\varphi})$, respectively, i.e., we have $\Delta \mu = \mu_2 - \mu_1 = 0.3P_{\mu}\Delta_{\mu}$ and $\Delta \varphi = \varphi_2 - \varphi_1 = 24\Delta_{\varphi} = \pi/2$. The number of snapshots is T = 20. Note that in such a setting of two closely-spaced uncorrelated NC sources with a phase discrimination of $\pi/2$, the maximum NC gain can be achieved [7]. It can be seen from Fig. 1 that the proposed algorithm for NC sources provides a significantly lower estimation error compared to its non-NC counterpart. The "NC OMP Joint" algorithm successfully estimates the grid offset and achieves the deterministic NC CRB.

Fig. 2 illustrates the MSE versus the spatial separation $\Delta \mu$, where we have d = 2 uncorrelated sources at $\mu_1 = 20.2\Delta_{\mu}$ and $\mu_2 = \mu_1 + \Delta \mu$. The SNR is fixed to 40 dB and the remaining parameters are kept the same. We observe again that the NC scheme outperforms its counterpart as it is constant for all the distances.

In Fig. 3, we show the MSE versus the rotation phase separation $\Delta \varphi$ for d = 2 sources at $\mu = [2.1, 4.5] \Delta_{\mu}$ with $\varphi_1 = 5.1 \Delta_{\varphi}$ and $\varphi_2 = \varphi_1 + \Delta \varphi$. All the parameters are kept the same as before. It can be seen that the NC method provides the best performance for a phase discrimination of $\Delta \mu = \pi/2$ while its non-NC counterpart remains constant.

6. CONCLUSION

In this paper, we have presented a novel strategy to take the NC signal structure into account for the SSR-based DOA estimation, which results in a two-dimensional SSR problem. Thereby, the benefits associated with NC sources can be achieved. Moreover, we have addressed the 2-D off-grid problem by proposing a low-complexity



Fig. 1. MSE versus SNR for M = 8, $P_{\mu} = 8$, $P_{\varphi} = 6$, T = 20, d = 2 sources at $\mu = [15.1, 17.5]\Delta_{\mu}$ with $\varphi = [10.2, 34.2]\Delta_{\varphi}$.



Fig. 2. MSE versus $\Delta \mu$ for M = 8, $P_{\mu} = 8$, $P_{\varphi} = 6$, T = 20, d = 2 sources at $\mu_1 = 20.2\Delta_{\mu}$ and $\mu_2 = \mu_1 + \Delta \mu$.



Fig. 3. MSE versus $\Delta \varphi$ for M = 8, $P_{\mu} = 8$, $P_{\varphi} = 6$, T = 20, d = 2 sources at $\mu = [2.1, 4.5]\Delta_{\mu}$ with $\varphi_1 = 5.1\Delta_{\varphi}$ and $\varphi_2 = \varphi_1 + \Delta \varphi$.

procedure that estimates the sources' grid offset from the closest neighboring grid points. For a single off-grid source, we have shown analytically that the 2-D offset estimation problem is separable, which allows to perform this step in both dimensions independently. We have also proposed a numerical scheme for the joint estimation of the grid offsets of two closely-spaced sources. The effectiveness of the proposed methods has been demonstrated via simulations.

7. REFERENCES

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