DIRECTION-OF-ARRIVAL ESTIMATION BASED ON TOEPLITZ COVARIANCE MATRIX RECONSTRUCTION

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ABSTRACT

This paper addresses the issue of direction-of-arrival (DOA) estimation with an objective to eliminate the off-grid effect of the sparsity-based methods and enlarge the maximum number of distinguishable signals in the subspace-based methods. We first reconstruct the covariance matrix of the array output in the Toeplitz structure and then employ the reconstructed covariance matrix together with root-MUSIC to estimate the DOAs. The proposed covariance matrix reconstruction approach (CMRA) can be used for uniform and sparse linear arrays. It can also estimate the DOAs of multiple signals that are larger than the number of sensors by taking advantage of the array geometry. In contrast to the sparsity-based methods, CMRA is formulated in the continuous angle space rather than the discretized one, and hence it is immune to the off-grid effect. Simulations are carried out to verify the effectiveness of our method.

***Index Terms*—** DOA estimation, sparse signal representation (SSR), Toeplitz structure, atomic norm

1. INTRODUCTION

The sparsity-based methods for DOA estimation [1–3] have attracted much interest in the past decade. Compared with the subspace-based methods and maximum likelihood (ML) [4], the sparsity-based methods exhibit several advantages: robustness to noise, no requirement of source number, and improved resolution. However, the disadvantages cannot be ignored. These methods formulate signal representation on the predefined discrete dictionary, i.e., the continuous angle space is reduced to a set of discrete grids under the assumption that the true DOAs of the sources lie exactly on the predefined finite discrete grids. This discretization strategy may degrade the performance of sparsity-based methods since there is often an unavoidable basis mismatch between the true DOA and the assumed grid. To alleviate the effect of basis mismatch, an approach is proposed in [5] based on structured matrix completion [6], where the problem is formulated into a structured Toeplitz matrix completion. However, it is not easy to choose a satisfying regularization parameter in this method.

In this paper, a discretization-free method for DOA estimation named as covariance matrix reconstruction approach (CMRA) is proposed for both uniform linear array (ULA) and sparse linear array (SLA). Unlike the approach in [5], the regularization parameter is set automatically in the proposed CMRA. We first formulate a trace minimization problem to estimate the covariance matrix. Then the DOAs can be estimated from the reconstructed covariance matrix using root-MUSIC method [7]. As a byproduct, the number of signals can be easily obtained as the rank of the estimated covariance matrix. We then show that the CMRA can be connected to the *atomic norm* [8], and our method is superior to the sparsity-based algorithms and can deal with more signals than the number of sensors in the SLA case.

2. CMRA WITH ULA

Suppose that K narrowband far-field signals impinge onto an array with N ($N > K$) equal-spaced omnidirectional sensors from directions of $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_K\}$ simultaneously. The array output at time t, which is corrupted by additive circular complex Gaussian white noise, can be expressed as,

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} \boldsymbol{a}(\theta_k) s_k(t) + \boldsymbol{v}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{v}(t), \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t), \cdots, x_N(t)]^T$ is the array output, $\boldsymbol{s}(t) = [s_1(t), \cdots, s_K(t)]^T$ is the vector of source signals, $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_K)]$ is the array manifold matrix,

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with $\boldsymbol{a}(\theta_k) = [e^{j2\pi f_0 \tau_{k,1}}, \cdots, e^{j2\pi f_0 \tau_{k,N}}]^T$ being a vector of the time-delayed versions of the kth signal received at each sensor relative to the reference sensor, $\boldsymbol{v}(t)$ is the complex independent white Gaussian noise with zero mean. We assume that both the source signals and the noises are uncorrelated spatially as well as temporarily, i.e.,

$$E\left[\boldsymbol{s}(t_1)\boldsymbol{s}^H(t_2)\right] = \mathrm{diag}(\boldsymbol{p})\delta_{t_1,t_2}, \tag{2}$$

$$E\left[\boldsymbol{v}(t_1)\boldsymbol{v}^H(t_2)\right] = \mathrm{diag}(\boldsymbol{\sigma})\delta_{t_1,t_2}, \tag{3}$$

where $\boldsymbol{p} = [p_1, \ldots, p_K]^T$ denotes the source power parameter, $\boldsymbol{\sigma} = [\sigma_1, \ldots, \sigma_N]^T$ denotes the noise variance parameter and δ_{t_1,t_2} equals 1 if $t_1 = t_2$ or 0 otherwise. Based on the signal and noise models, the covariance matrix of the array output can be obtained as

$$\boldsymbol{R} = T(\boldsymbol{u}) + \mathrm{diag}(\boldsymbol{\sigma}), \tag{4}$$

where $T(\boldsymbol{u}) = \boldsymbol{A}\mathrm{diag}(\boldsymbol{p})\boldsymbol{A}^H$ is a Hermitian Toeplitz matrix and $\boldsymbol{u} = [u_1, \cdots, u_N]^T$ is the first column of $T(\boldsymbol{u})$. Moreover, $T(\boldsymbol{u}) \geq \mathbf{0}$ and $\mathrm{rank}[T(\boldsymbol{u})] = K \leq N-1$ can also be concluded.[1]

In practical applications, the covariance matrix is estimated with L snapshots as follows,

$$\hat{\boldsymbol{R}} = \frac{1}{L}\sum_{l=1}^{L} \boldsymbol{x}(t_l)\boldsymbol{x}^H(t_l), \tag{5}$$

which is error-contaminated due to finite snapshots. We denote the estimation error matrix as

$$\boldsymbol{E} = \hat{\boldsymbol{R}} - \boldsymbol{R} = \hat{\boldsymbol{R}} - T(\boldsymbol{u}) - \mathrm{diag}(\boldsymbol{\sigma}), \tag{6}$$

where $\boldsymbol{E}$ consists of signal-signal, signal-noise cross correlation terms which are not 0 due to finite snapshot effect. For large L, $\boldsymbol{E}$ can be assumed to have a small Frobenius norm since the cross correlation terms become smaller as L increases [9]. When $T(\boldsymbol{u})$ is obtained, the unknown DOAs can be well estimated by using conventional methods such as MUSIC. Hence, we then propose a two-stage approach, consisting of a low-rank recovery stage to estimate $T(\boldsymbol{u})$, followed by the root-MUSIC method to estimate the DOAs from the recovered covariance matrix. In the first stage, based on the low-rank matrix recovery (LRMR) theory [5], the covariance matrix can be reconstructed by solving the following optimization problem

$$\min_{\boldsymbol{u},\boldsymbol{\sigma}\succeq\mathbf{0}} \mathrm{rank}\left[T(\boldsymbol{u})\right] \quad \text{s.t. } \|\boldsymbol{E}\|_2^2 \leq \beta,\ T(\boldsymbol{u}) \geq \mathbf{0}, \tag{7}$$

where $\boldsymbol{\sigma} \succeq \mathbf{0}$ means that every entry of $\boldsymbol{\sigma}$ is nonnegative, and β is a user-specific bound which is difficult to determine. In fact, directly calculating β as the expectation of $\|\boldsymbol{E}\|_2^2$ is not recommended since it is likely that a particular realization will have $\|\boldsymbol{E}\|_2^2 \geq \beta$ [10]. In what follows, we give an alternative constraint which enables the true solution to fall inside the feasible region with a high probability.

According to [11], the vectorization form of $\boldsymbol{E}$ satisfies the asymptotic normal distribution as follows (see [11] for more details),

$$\mathrm{vec}(\boldsymbol{E}) \sim \mathrm{AsN}(\mathbf{0}, \boldsymbol{W}) \tag{8}$$

where $\boldsymbol{W} = \frac{1}{L}\boldsymbol{R}^T \otimes \boldsymbol{R}$, with $\otimes$ being the Kronecker matrix product. In practice, $\boldsymbol{W}$ can be approximately estimated as $\hat{\boldsymbol{W}} = \frac{1}{L}\hat{\boldsymbol{R}}^T \otimes \hat{\boldsymbol{R}}$. From (8), it can be deduced that

$$\hat{\boldsymbol{W}}^{-\frac{1}{2}}\mathrm{vec}(\boldsymbol{E}) \sim \mathrm{AsN}(\mathbf{0}, \boldsymbol{I}_{N^2}), \tag{9}$$

which directly results in

$$\left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}\mathrm{vec}(\boldsymbol{E})\right\|_2^2 \sim \mathrm{As}\chi^2(N^2), \tag{10}$$

where $\mathrm{As}\chi^2(N^2)$ denotes the asymptotic chi-square distribution with N^2 degrees of freedom. We introduce a parameter η so that the confidence interval $[0, \eta]$ integrates to probability of $1-p$, i.e.,

$$\left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}\mathrm{vec}(\boldsymbol{E})\right\|_2^2 \leq \eta \tag{11}$$

with a high probability $1-p$ (p is very small). Replacing the first constraint of (7) by (11), we propose the following rank-minimization problem for DOA estimation,

$$\min_{\boldsymbol{u},\boldsymbol{\sigma}\succeq\mathbf{0}} \mathrm{rank}\left[T(\boldsymbol{u})\right] \quad \text{s.t. } \left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}\mathrm{vec}(\boldsymbol{E})\right\|_2^2 \leq \eta,\ T(\boldsymbol{u}) \geq \mathbf{0}. \tag{12}$$

Nonetheless, this problem is NP-hard and hence, of little practical value. To avoid the nonconvexity, we utilize convex relaxation to relax the pseudo rank norm to the nuclear norm or equivalently the trace norm for a positive semidefinite matrix, i.e., to replace $\mathrm{rank}\left[T(\boldsymbol{u})\right]$ by $\mathrm{tr}\left[T(\boldsymbol{u})\right]$, where $\mathrm{tr}\left[T(\boldsymbol{u})\right]$ denotes the trace of the matrix $T(\boldsymbol{u})$. Consequently, the convex relaxation form of problem (12) can be given as,

$$\min_{\boldsymbol{u},\boldsymbol{\sigma}\succeq\mathbf{0}} \mathrm{tr}\left[T(\boldsymbol{u})\right] \quad \text{s.t. } \left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}\mathrm{vec}(\boldsymbol{E})\right\|_2^2 \leq \eta,\ T(\boldsymbol{u}) \geq \mathbf{0}. \tag{13}$$

This constrained trace minimization problem can be solved by any optimization toolbox such as CVX or SeDuMi. After obtaining the optimal solution $\boldsymbol{u}^*$, the estimate of $T(\boldsymbol{u})$ can be given as $T(\boldsymbol{u}^*)$. As for the number of signals, recall that $\mathrm{rank}\left[T(\boldsymbol{u})\right] = K$, hence the estimate $\hat{K}$ can be easily formulated as $\hat{K} = \mathrm{rank}\left[T(\boldsymbol{u}^*)\right]$. In practical applications, based on the eigenvalue decomposition of $T(\boldsymbol{u}^*)$, $\hat{K}$ can be determined as the number of eigenvalues that are greater than a predefined threshold ϵ.

In the second stage, we adopt the root-MUSIC method for the ensuing DOA estimation after obtaining $T(\boldsymbol{u}^*)$ and $\hat{K}$. It should be noted that, since $T(\boldsymbol{u}^*)$ has a Toeplitz structure, DOA estimation can also be efficiently performed based on the classical Vandermonde decomposition lemma for positive semidefinite Toeplitz matrices [12]. Readers are referred to [13] for more information.

[1] Matrix $\boldsymbol{A} \geq \mathbf{0}$ indicates that $\boldsymbol{A}$ is positive semidefinite.

3. CMRA WITH SLA

In this section, we extend the CMRA method to the SLA case, which can be regarded as a subset of a ULA. We define the sensor index set of an SLA as $\mathbf{\Omega} \subset \{1, \cdots, N\}$ for better illustration, where $|\mathbf{\Omega}| = M$ denotes the size of the array. In this paper, we are mainly interested in the minimum redundancy arrays (MRA), i.e., the spacings between each two sensors form the set $\{d, 2d, \cdots, (N-1)d\}$ where d is the minimum interval. Similar to the ULA case, the corresponding steering vector for the kth signal is $\boldsymbol{a}_{\mathbf{\Omega}}(\theta_k) = [e^{j2\pi f_0 \tau_{k,\Omega_1}}, \cdots, e^{j2\pi f_0 \tau_{k,\Omega_M}}]^T$. We denote $\mathbf{\Gamma}_{\mathbf{\Omega}} \in \{0,1\}^{M \times N}$ as a selection matrix such that the jth row of $\mathbf{\Gamma}_{\mathbf{\Omega}}$ contains all 0s but a single 1 at the Ω_jth position. It is clear that the steering vector of the kth signal is $\boldsymbol{a}_{\mathbf{\Omega}}(\theta_k) = \mathbf{\Gamma}_{\mathbf{\Omega}} \boldsymbol{a}(\theta_k)$ and the output of the SLA at time t is

$$\begin{aligned} \boldsymbol{x}_{\mathbf{\Omega}}(t) &= \boldsymbol{A}_{\mathbf{\Omega}} \boldsymbol{s}(t) + \boldsymbol{v}_{\mathbf{\Omega}}(t) \\ &= \mathbf{\Gamma}_{\mathbf{\Omega}} \boldsymbol{A} \boldsymbol{s}(t) + \mathbf{\Gamma}_{\mathbf{\Omega}} \boldsymbol{v}(t), \end{aligned} \tag{14}$$

where $\boldsymbol{A}_{\mathbf{\Omega}} = [\boldsymbol{a}_{\mathbf{\Omega}}(\theta_1), \cdots, \boldsymbol{a}_{\mathbf{\Omega}}(\theta_K)]$. The covariance matrix is thus formulated as

$$\begin{aligned} \boldsymbol{R}_{\mathbf{\Omega}} &= \lim_{L \to +\infty} \frac{1}{L} \boldsymbol{x}_{\mathbf{\Omega}}(t_l) \boldsymbol{x}_{\mathbf{\Omega}}^H(t_l) \\ &= \mathbf{\Gamma}_{\mathbf{\Omega}} T(\boldsymbol{u}) \mathbf{\Gamma}_{\mathbf{\Omega}}^T + \mathbf{\Gamma}_{\mathbf{\Omega}} \text{diag}(\boldsymbol{\sigma}) \mathbf{\Gamma}_{\mathbf{\Omega}}^T \\ &\triangleq T_{\mathbf{\Omega}}(\boldsymbol{u}) + \text{diag}(\boldsymbol{\sigma}_{\mathbf{\Omega}}), \end{aligned} \tag{15}$$

where $T_{\mathbf{\Omega}}(\boldsymbol{u}) \triangleq \mathbf{\Gamma}_{\mathbf{\Omega}} T(\boldsymbol{u}) \mathbf{\Gamma}_{\mathbf{\Omega}}^T$ and $\text{diag}(\boldsymbol{\sigma}_{\mathbf{\Omega}}) \triangleq \mathbf{\Gamma}_{\mathbf{\Omega}} \text{diag}(\boldsymbol{\sigma}) \mathbf{\Gamma}_{\mathbf{\Omega}}^T$. We then denote the sample covariance matrix as $\hat{\boldsymbol{R}}_{\mathbf{\Omega}} = \frac{1}{L}\sum_{l=1}^{L} \boldsymbol{x}_{\mathbf{\Omega}}(t_l) \boldsymbol{x}_{\mathbf{\Omega}}^H(t_l)$, the error matrix as $\boldsymbol{E}_{\mathbf{\Omega}} = \hat{\boldsymbol{R}}_{\mathbf{\Omega}} - \boldsymbol{R}_{\mathbf{\Omega}}$ and the covariance matrix of $\boldsymbol{E}_{\mathbf{\Omega}}$ as $\hat{\boldsymbol{W}}_{\mathbf{\Omega}} = \frac{1}{L} \hat{\boldsymbol{R}}_{\mathbf{\Omega}}^T \otimes \hat{\boldsymbol{R}}_{\mathbf{\Omega}}$. It can be concluded in the same manner as the ULA case that

$$\left\| \hat{\boldsymbol{W}}_{\mathbf{\Omega}}^{-\frac{1}{2}} \text{vec}(\boldsymbol{E}_{\mathbf{\Omega}}) \right\|_2^2 \sim \text{As}\chi^2(M^2). \tag{16}$$

Similarly, we propose the following trace minimization problem for DOA estimation in the SLA case,

$$\min_{\boldsymbol{u}, \boldsymbol{\sigma}_{\mathbf{\Omega}} \succeq \mathbf{0}} \text{tr}\left[T(\boldsymbol{u})\right] \quad \text{s.t.} \ \left\| \hat{\boldsymbol{W}}_{\mathbf{\Omega}}^{-\frac{1}{2}} \text{vec}(\boldsymbol{E}_{\mathbf{\Omega}}) \right\|_2^2 \leq \eta', \ T(\boldsymbol{u}) \geq \mathbf{0}, \tag{17}$$

where η' is defined in the same manner as η except that the asymptotic chi-square distribution is of M^2 degrees of freedom. After obtaining the solution $T(\boldsymbol{u}^*)$, the number of sources $\hat{K}$ can be given as the rank of $T(\boldsymbol{u}^*)$ and the DOA estimates can be also determined from $T(\boldsymbol{u}^*)$ and $\hat{K}$ by utilizing the root-MUSIC method. Note that, when $K \geq M$, the partial covariance matrix $T_{\mathbf{\Omega}}(\boldsymbol{u})$ is no longer low rank, hence directly applying root-MUSIC to $T_{\mathbf{\Omega}}(\boldsymbol{u})$ for DOA estimation is impossible. Different from root-MUSIC, our method recovers the complete $T(\boldsymbol{u})$ in advance and so has a potential to recover more signals than sensors by taking advantage of the array geometry, e.g., the co-prime array [14]. Moreover, our method is able to well estimate $\hat{K}$ as long as $K < N$ while any classical model-order selection strategy like AIC and MDL [15] can work only when $K < M$.

4. RELATION TO SPARSITY-BASED METHODS

In this section, we show that the proposed CMRA method is closely related to sparsity-based methods. In particular, CMRA can be considered as a gridless version of the covariance matrix sparse representation (CMSR) method in [3]. To see this, suppose that the whole angle space $[-\pi, \pi)$ is divided into a uniform finite grid set of directions, i.e., $\boldsymbol{\vartheta} = \{\vartheta_1, \cdots, \vartheta_{\bar{N}}\}$, where $\bar{N}$ denotes the size of the grid set. Without loss of generality, we consider the SLA case as an example. The ULA case can be regarded as a special case when $\mathbf{\Omega} = \{1, \cdots, N\}$. Denote the corresponding manifold matrix by $\bar{\boldsymbol{A}}_{\mathbf{\Omega}}$ and the corresponding power vector by $\bar{\boldsymbol{p}}$, then we have $\bar{\boldsymbol{r}}_{\mathbf{\Omega}} = \text{vec}(\hat{\boldsymbol{R}}_{\mathbf{\Omega}} - \sigma \boldsymbol{I}_{\mathbf{\Omega}})$,[2] $\tilde{\boldsymbol{A}}_{\mathbf{\Omega}} = \bar{\boldsymbol{A}}_{\mathbf{\Omega}}^* \odot \bar{\boldsymbol{A}}_{\mathbf{\Omega}}$. The constraint in CMSR may make the true solution fall outside the feasible region, possibly deteriorating the estimation performance [10]. We then formulate the following sparsity-based model for DOA estimation and name it as the modified CMSR (mCMSR),

$$\min_{\bar{\boldsymbol{p}} \succeq \mathbf{0}} \|\bar{\boldsymbol{p}}\|_1 \quad \text{s.t.} \ \left\| \hat{\boldsymbol{W}}_{\mathbf{\Omega}}^{-\frac{1}{2}} \left(\bar{\boldsymbol{r}}_{\mathbf{\Omega}} - \tilde{\boldsymbol{A}}_{\mathbf{\Omega}} \bar{\boldsymbol{p}} \right) \right\|_2^2 \leq \eta'. \tag{18}$$

When the sampling grids are fine enough such that the true DOAs exactly lie on the grids, it can be easily concluded that, $\|\bar{\boldsymbol{p}}\|_1 = \sum_{k=1}^{K} p_k = \frac{1}{N} \text{tr}[T(\boldsymbol{u})]$ and

$$\begin{aligned} &\left\| \hat{\boldsymbol{W}}_{\mathbf{\Omega}}^{-\frac{1}{2}} \left(\bar{\boldsymbol{r}}_{\mathbf{\Omega}} - \tilde{\boldsymbol{A}}_{\mathbf{\Omega}} \bar{\boldsymbol{p}} \right) \right\|_2^2 \\ =& \left\| \hat{\boldsymbol{W}}_{\mathbf{\Omega}}^{-\frac{1}{2}} \text{vec}\left(\hat{\boldsymbol{R}}_{\mathbf{\Omega}} - \sigma \boldsymbol{I}_{\mathbf{\Omega}} - \bar{\boldsymbol{A}}_{\mathbf{\Omega}} \text{diag}(\bar{\boldsymbol{p}}) \bar{\boldsymbol{A}}_{\mathbf{\Omega}}^H \right) \right\|_2^2 \\ =& \left\| \hat{\boldsymbol{W}}_{\mathbf{\Omega}}^{-\frac{1}{2}} \text{vec}(\boldsymbol{E}_{\mathbf{\Omega}}) \right\|_2^2, \end{aligned} \tag{19}$$

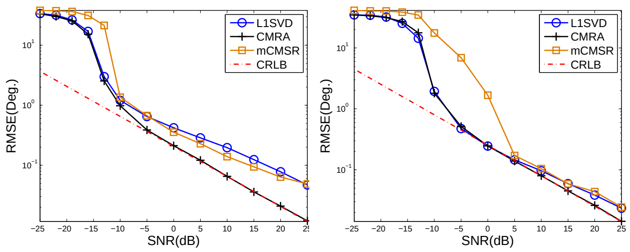
which indicates that, when $\bar{N} \to +\infty$, mCMSR is equivalent to the proposed CMRA, or equivalently, mCMSR can be considered as a discretized version of CMRA.

Without loss of generality, the noiseless covariance matrix $\boldsymbol{R} = \sum_{k=1}^{K} p_k \boldsymbol{a}(\theta_k) \boldsymbol{a}^H(\theta_k)$ can be regarded as a positive weighted combination of K unit-norm rank-one matrices $\left\{ \boldsymbol{B}(\theta_k) = \frac{1}{N} \boldsymbol{a}(\theta_k) \boldsymbol{a}^H(\theta_k) : k = 1, \cdots, K \right\}$. Motivated by this finding, we define a set of atoms as $\mathcal{A} = \left\{ \boldsymbol{B}(\theta) = \frac{1}{N} \boldsymbol{a}(\theta) \boldsymbol{a}^H(\theta) : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}) \right\}$ to represent the noiseless covariance matrix $\boldsymbol{R}$ as follows,

$$\boldsymbol{R} = N \sum_{k=1}^{K} p_k \boldsymbol{B}(\theta_k) \qquad \boldsymbol{B}(\theta_k) \in \mathcal{A}. \tag{20}$$

By contrast, the sparsity-based methods divide the angle space and result in the discretized set of atoms $\mathcal{A}_{\bar{N}} = \left\{ \bar{\boldsymbol{B}}(\vartheta) = \frac{1}{N} \boldsymbol{a}(\vartheta) \boldsymbol{a}^H(\vartheta) : \vartheta \in \boldsymbol{\vartheta} \right\}$. Obviously, $\boldsymbol{R}$ cannot be represented by the combination of $\bar{\boldsymbol{B}}(\vartheta)$ unless the true DOAs

[2] In the sparsity-based methods, the noise powers $\sigma_m (m = 1, \cdots, N)$ are assumed to be equal for simplicity which is true in most scenarios and can be approximately given by the minimum eigenvalue of $\hat{\boldsymbol{R}}_{\mathbf{\Omega}}$.



(a) 7-element ULA (b) 4-element SLA

Fig. 1. RMSE comparison of L1-SVD, mCMSR and CMRA with $N = 7$, $L = 400$.

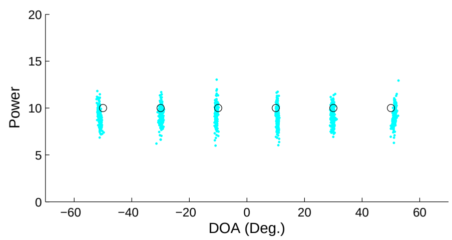
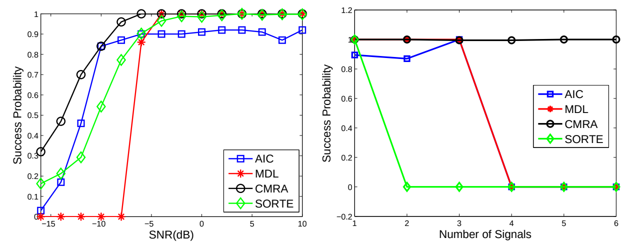


Fig. 2. DOA and power estimates of CMRA using MRA with $N = 7$, $M = 4$, $L = 400$, SNR$= 10$dB.



(a) 7-element ULA (b) 4-element SLA

Fig. 3. Detection probabilities of CMRA, AIC, MDL and SORTE.

are assumed to lie on the grid. Our method completely avoids this potential conflict by working directly on the continuous parameter space for estimating the continuous DOAs. The resulting dictionary is an infinite dictionary with continuously many atoms and arbitrarily high correlation between candidate atoms. Interestingly, the continuous dictionary $\mathcal{A}$ is closely related to the *atomic norm*, which generalizes the nuclear norm (or the trace norm) for low-rank matrix completion [8]. Further study will be carried out in the future to exploit the relationship between CMRA and the atomic norm.

5. SIMULATION RESULTS

In this section, we first evaluate the estimation performance of CMRA with comparison to L1-SVD [1] and mCMSR by simulations. The sparsity-based algorithms L1-SVD and mCMSR divide the $[-90°, 90°]$ space with interval $\Delta\theta = 2°$ and employ the iterative grid refinement (IGR) procedure for accuracy improvement [1]. In our simulation, parameter η (or η') in CMRA can be calculated using MATLAB routine $\texttt{chi2inv}(1 - p, N^2$ (or M^2)), where p is set to 0.001 in general.

First, suppose two equal-power narrowband signals impinge onto a 7-element ULA or a 4-element SLA with $\mathbf{\Omega} = \{1, 2, 5, 7\}$ from $[-6°+v, 6°+v]$ with v chosen randomly and uniformly within $[-1°, 1°]$. We compare the estimation performance with $L = 400$ and different SNRs. The statistical results are derived from 200 independent trials and shown in Fig.1. It can be seen that CMRA coincides with the CRLB [4] when SNR > -5dB and enjoys the best performance of the three both in the ULA and SLA cases, while L1-SVD and mCMSR lose their super-resolution ability when SNR becomes large.

Next, we attempt to estimate the DOAs and powers of 6 signals impinged onto the aforementioned SLA from different directions, where the powers can be estimated by solving a least squares problem provided that the corresponding DOA estimates are first obtained. We carry out 500 trials and show the results in Fig. 2. Black circles denote the positions of the true powers and corresponding DOAs. Blue dots denote the estimated ones.

We also demonstrate the signal detection performance of CMRA with comparison to AIC, MDL and SORTE [16] in Fig.3. The left figure is based on the 7-element ULA while the right one is based on the 4-element SLA. From Fig.3(a) we can see that the detection performance of CMRA is superior to other compared methods. AIC cannot achieve 100% detection probability when SNR is high. Fig.3(b) indicates that AIC and MDL can only detect at most 3 signals while CMRA can detect up to 6 signals, which is the maximum number of signals the SLA can estimate. SORTE can estimate only one signal since the maximum number of its distinguishable signals is limited by $M - 3$.

6. CONCLUSIONS

The DOA estimation problem without employing discretization has been studied in this paper. The CMRA method has been proposed by working directly on the continuous angle space for ULA/SLA. It has been shown that the CMRA is closely related to sparsity-based methods as well as the atomic norm. In particular, CMRA can be regarded as the gridless version of the sparsity-based method mCMSR. Computer simulation has shown that CMRA is able to detect more signals than sensors in the SLA case.

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