DETERMINISTIC MAXIMUM LIKELIHOOD METHOD FOR DIRECTION-OF-ARRIVAL ESTIMATION OF STRICTLY NONCIRCULAR SIGNALS

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ABSTRACT

In this paper, a noncircular deterministic maximum likelihood (NC-DML) estimator for direction-of-arrival estimation of strictly NC signals is devised. Unlike the conventional DML solution for arbitrary signals, the NC-DML exploits the NC properties of the sources by reconstructing the parameter set, significantly decreasing the number of parameters to be considered. For computing the NC-DML, we present a novel NC alternating projection (NC-AP) approach. The NC-AP solution is carried out based on an augmented virtual array structure. Moreover, it also takes the impact of the initial phase shift of the NC signals into account. Simulation results are included to illustrate the superiority of the proposed method.

Index Terms— noncircular, deterministic maximum likelihood, direction-of-arrival, alternating projection.

1. INTRODUCTION

The problem of estimating the direction-of-arrival (DOA) of narrow-band signals has been an extensive research topic in a variety of areas such as radar, sonar and wireless communications. The deterministic maximum likelihood (DML) method was one of the first to be investigated [1], nevertheless, due to the high computational load involved, it did not become popular. Instead, suboptimal techniques with reduced computational load have dominated the field, such as the extrema-searching [2], polynomial-rooting [3] and matrix-shifting [4] techniques. The relationship between these techniques and DML estimator was investigated in [5], and it is shown that the performance of these techniques is inferior to that of the DML.

As mentioned above, the DML has not attracted as much attention as other suboptimal DOA estimation methods because its computational burden grows dramatically with increasing number of impinging signals. Fortunately, several efficient DML solutions have been proposed to reduce the computational burden while maintaining a relatively high level of estimation performance, such as alternating projection (AP) [6], expectation maximization [7], method of direction estimation [1] and spatial aliasing [8] methods.

Apart from the high computational load, another limitation of the DML is the asymptotic efficiency problem. It is shown in [1] that the DML estimator is not statistically efficient if the number of snapshots is small. Furthermore, even if the number of snapshots is large, the DML can achieve the Cramer-Rao bound only if the number of sensors is increased, namely, the array aperture should be large enough.

To further enhance the performance of the DOA estimators, the temporal properties of the signals, such as the noncircular (NC) property, can also be employed. The NC signals such as BPSK, offset-QPSK, PAM and ASK-modulated signals, have been widely used in many modern communication systems. By taking advantage of the NC properties of the received signals, a number of improved subspace-based DOA estimators have been proposed, such as the NC-MUSIC [9], NC-root-MUSIC [10], NC-ESPRIT [11], NC unitary ESPRIT [12, 13] and SLS-NC-ESPRIT [14].

In this paper, by exploiting the NC properties of the arriving signals, we devise a NC DML (NC-DML) estimator for strictly NC sources. The derivation is based on the fact that the real and imaginary parts of the strictly NC sources are linearly dependent. As a result, unlike the conventional DML scheme, the real and imaginary parts of the strictly NC signals cannot be treated as independent random variables. Moreover, in analogy to the AP method [6], we present a novel and computationally attractive NC AP (NC-AP) approach for computing the NC-DML estimator.

The paper is organized as follows. In Section II, we formulate the problem and review the DML estimator. In Section III, we derive the NC-DML and present the NC-AP algorithm. Simulation results are provided for performance comparison in Section IV. Finally, Section V draws the conclusion.

The following notations are used throughout the paper. Both Matrices and vectors are represented by bold-faced letter. Superscripts $(\cdot)^T$, $(\cdot)^H$ and $(\cdot)^*$ stand for transpose, conjugate transpose and conjugate, respectively. The $||\cdot||$ and $\mathrm{tr}\{\cdot\}$ denote Euclidean norm and trace, respectively. $\boldsymbol{I}$ represents the identity matrix.

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2. PROBLEM FORMULATION

2.1. Signal Model

Let us consider an array of M sensors receiving d far-field and narrow-band signals. The array measurements are modeled as

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{1}$$

where $\boldsymbol{s}(t) = [s_1(t), \cdots, s_d(t)]^T$ is the signal vector, $\boldsymbol{n}(t) = [n_1(t), \cdots, n_M(t)]^T$ contains the additive sensor noise and $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_d)]$ is the steering matrix, which consists of d array steering vectors $\boldsymbol{a}(\theta_i),\ i = 1, \cdots, d$, with θ_i representing the DOA of the i-th source. Here, we assume that $\boldsymbol{n}(t)$ is a circular zero mean Gaussian random process with covariance matrix $\sigma^2\boldsymbol{I}$. Moreover, the assumption that d is known is made to simplify the exposition. The case of unknown d was also discussed in the literature [15] by using the information theoretic criterion, such as the AIC [16], MDL [17] or their computationally efficient variants [18]-[19].

Suppose that the signals are strictly NC. In other words, the complex symbol amplitudes of signals lie on a line in the I/Q diagram, which may correspond to BPSK, offset QPSK and PAM [13]. Consequently, the source snapshot vector can be decomposed as [12]

$$\boldsymbol{s}(t) = \boldsymbol{\Psi}\boldsymbol{s}_o(t), \tag{2}$$

where the diagonal matrix $\boldsymbol{\Psi} = \mathrm{diag}\{e^{j\varphi_1}, \ldots, e^{j\varphi_d}\}$ contains the initial complex phase shift of each source, and $\boldsymbol{s}_o(t)$ is the real-valued signal vector. As a result, the array observations are rewritten as

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{\Psi}\boldsymbol{s}_o(t) + \boldsymbol{n}(t). \tag{3}$$

2.2. DML

In the DML estimator case, the sources are regarded as unknown deterministic sequences instead of sample functions of random processes. Let $t_1, \cdots, t_N$ denote the time instant at which the snapshots are taken. Then the array measurements with N samples can be expressed as

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{N} = \boldsymbol{A}\boldsymbol{\Psi}\boldsymbol{S}_o + \boldsymbol{N} \tag{4}$$

where $\boldsymbol{X}$ and $\boldsymbol{N}$ are the $M \times N$ matrices

$$\boldsymbol{X} = [\boldsymbol{x}(t_1), \cdots .\boldsymbol{x}(t_N)] \tag{5}$$
$$\boldsymbol{N} = [\boldsymbol{n}(t_1), \cdots .\boldsymbol{n}(t_N)], \tag{6}$$

both $\boldsymbol{S}$ and $\boldsymbol{S}_o$ are the $d \times N$ matrices

$$\boldsymbol{S} = [\boldsymbol{s}(t_1), \cdots .\boldsymbol{s}(t_N)] \tag{7}$$
$$\boldsymbol{S}_o = [\boldsymbol{s}_o(t_1), \cdots .\boldsymbol{s}_o(t_N)]. \tag{8}$$

It follows from (1) that the joint density function of the sampled observations is therefore given by

$$f(\boldsymbol{X}) = \prod_{i=1}^{N} \frac{1}{(\pi)^M(\sigma_n^2)^M} \exp(-\frac{1}{\sigma_n^2}\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{s}(t_i)\|^2). \tag{9}$$

Thus, the negative log-likelihood, ignoring constant terms, can be expressed as

$$L = MN\log\sigma_n^2 + \frac{1}{\sigma_n^2}\sum_{i=1}^{N}\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{s}(t_i)\|^2. \tag{10}$$

According to [6], by minimizing the negative log-likelihood function, we obtain the well-known DML estimator

$$\hat{\boldsymbol{\Theta}}_{\text{DML}} = \arg\min_{\boldsymbol{\Theta}} \mathrm{tr}[\mathbf{P}_A^{\perp}\hat{\boldsymbol{R}}_x] \tag{11}$$

where $\boldsymbol{\Theta} = [\theta_1, \cdots, \theta_d]$ denotes the DOA parameters to be estimated,

$$\mathbf{P}_A^{\perp} = \boldsymbol{I} - \boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{(-1)}\boldsymbol{A}^H \tag{12}$$

and

$$\hat{\boldsymbol{R}}_x = \frac{1}{N}\boldsymbol{X}\boldsymbol{X}^H \tag{13}$$

corresponds to the sample covariance matrix.

3. PROPOSED ALGORITHM

In this section, a NC-DML estimator for the special case of strictly NC sources is proposed. Subsequently, we present an efficient NC-AP algorithm for computing this estimator.

3.1. NC-DML

In case of arbitrary signals, the number of parameters that needs to be considered for the DML is equal to $2Nd + d + 1$ [20][21]. However, for the strictly NC signals case in (2), the parameter set is different. By substituting (2) into (10), we have

$$L = MN\log\sigma_n^2 + \frac{1}{\sigma_n^2}\sum_{i=1}^{N}\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{\Psi}\boldsymbol{s}_o(t_i)\|^2. \tag{14}$$

It is observed from (14) that the set of parameters is given by the DOA parameters $\boldsymbol{\Theta} \in \mathbb{R}^{d\times 1}$, the real-valued signals $\mathrm{vec}\{\boldsymbol{S}_o\} \in \mathbb{R}^{Nd\times 1}$, the initial phase angles $\boldsymbol{\varphi} = [\varphi_1 \cdots, \varphi_d]^T \in \mathbb{R}^{d\times 1}$, and the noise power σ_n^2. Thus, the number of parameters is now equal to $Nd + 2d + 1$. As a result, it is required to derive a corresponding DML estimator for the strictly NC signals. In this context, in order to simplify the resulting model, we assume that the initial phase angles $\boldsymbol{\varphi}$, namely, $\boldsymbol{\Psi}$, are known. Thus, the number of parameters needed to be estimated reduces to $Nd + d + 1$. Note that several efficient solutions [22][23] for the estimation of $\boldsymbol{\Psi}$ have been provided in the literature.

To compute the NC-DML estimator, we have to minimize the cost function in (14) with respect to the unknown parameters. Fixing $\boldsymbol{\Theta}$ and $\boldsymbol{S}_o$, and then minimizing with respect to σ_n^2, we get

$$\hat{\sigma}_n^2 = \frac{1}{MN}\sum_{i=1}^{N}\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{\Psi}\boldsymbol{s}_o(t_i)\|^2. \tag{15}$$

Substituting (15) into (14), we fix $\boldsymbol{\Theta}$ and minimize with respect to $\boldsymbol{s}_o(t_i), i = 1, \cdots, N$. This yields the solution

$$\begin{aligned}\hat{\boldsymbol{s}}_o(t_i) =& (\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{\Psi} + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{A}^*\boldsymbol{\Psi}^*)^{-1} \\ &\times [\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{x}(t_i) + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{x}^*(t_i)]. \end{aligned} \tag{16}$$

In contrast to the case of arbitrary signals where $\hat{\boldsymbol{s}}_o(t_i)$ is only related to the array observations $\boldsymbol{x}(t_i)$, it is shown in (16) that $\hat{\boldsymbol{s}}_o(t_i)$ has some connection with both $\boldsymbol{x}(t_i)$ and $\boldsymbol{x}^*(t_i)$. Then substituting $\hat{\boldsymbol{s}}_o(t_i)$ back into the cost function we obtain the following minimization problem

$$\begin{aligned}\min_{\boldsymbol{\Theta}}\sum_{i=1}^{N}&\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{\Psi}(\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{\Psi} + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{A}^*\boldsymbol{\Psi}^*)^{-1} \\ &\times [\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{x}(t_i) + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{x}^*(t_i)]\|^2. \end{aligned} \tag{17}$$

Obviously, the estimator in (17) does not have the appealing geometric interpretation as the general DML scheme. However, it is observed that the minimizing problem in (17) is equivalent to minimizing the following cost function

$$\begin{aligned}J =& 2\sum_{i=1}^{N}\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{\Psi}(\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{\Psi} + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{A}^*\boldsymbol{\Psi}^*)^{-1} \\ &\times [\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{x}(t_i) + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{x}^*(t_i)]\|^2 \\ =& \sum_{i=1}^{N}\big\{\|\boldsymbol{x}(t_i) - \boldsymbol{A}\boldsymbol{\Psi}(\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{\Psi} + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{A}^*\boldsymbol{\Psi}^*)^{-1} \\ &\times [\boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{x}(t_i) + \boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{x}^*(t_i)]\|^2 \\ &+ \|\boldsymbol{x}^*(t_i) - \boldsymbol{A}\boldsymbol{\Psi}^*(\boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{A}^*\boldsymbol{\Psi}^* + \boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{\Psi})^{-1} \\ &\times [\boldsymbol{\Psi}^T\boldsymbol{A}^T\boldsymbol{x}^*(t_i) + \boldsymbol{\Psi}^H\boldsymbol{A}^H\boldsymbol{x}(t_i)]\|^2\big\} \\ =& \sum_{i=1}^{N}\left\|\tilde{\boldsymbol{x}}(t_i) - \tilde{\boldsymbol{A}}(\tilde{\boldsymbol{A}}^H\tilde{\boldsymbol{A}})^{-1}\tilde{\boldsymbol{A}}^H\tilde{\boldsymbol{x}}(t_i)\right\|^2 \end{aligned} \tag{18}$$

where $\tilde{\boldsymbol{x}}(t_i) = [\boldsymbol{x}^T(t_i), \boldsymbol{x}^H(t_i)]^T$, $\tilde{\boldsymbol{A}} = [\boldsymbol{\Psi}^T\boldsymbol{A}^T, \boldsymbol{\Psi}^H\boldsymbol{A}^H]^T$. In analogy to the DML estimator, according to (18), a similar form of the NC-DML is given by

$$\hat{\boldsymbol{\Theta}}_{\text{NC-DML}} = \arg\min_{\boldsymbol{\Theta}} \operatorname{tr}[\mathbf{P}_{\tilde{A}}^{\perp}\hat{\tilde{\boldsymbol{R}}}_x]. \tag{19}$$

Here,

$$\mathbf{P}_{\tilde{\boldsymbol{A}}}^{\perp} = \boldsymbol{I} - \tilde{\boldsymbol{A}}(\tilde{\boldsymbol{A}}^H\tilde{\boldsymbol{A}})^{(-1)}\tilde{\boldsymbol{A}}^H \tag{20}$$

and

$$\hat{\tilde{\boldsymbol{R}}}_x = \frac{1}{N}\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}^H \tag{21}$$

is the augmented sample covariance matrix with $\tilde{\boldsymbol{X}} = [\boldsymbol{X}^T, \boldsymbol{X}^H]^T$. Setting

$$\mathbf{P}_{\tilde{\boldsymbol{A}}} = \tilde{\boldsymbol{A}}(\tilde{\boldsymbol{A}}^H\tilde{\boldsymbol{A}})^{(-1)}\tilde{\boldsymbol{A}}^H, \tag{22}$$

the minimizing problem in (19) is equivalent to the following maximizing problem

$$\hat{\boldsymbol{\Theta}}_{\text{NC-DML}} = \arg\max_{\boldsymbol{\Theta}} \operatorname{tr}[\mathbf{P}_{\tilde{A}}\hat{\tilde{\boldsymbol{R}}}_x]. \tag{23}$$

3.2. NC-AP technique

We start by solving the problem for a single source. In this case, we yield the DOA estimate of the first source

$$\hat{\theta}_1^{(0)} = \arg\max_{\theta_1} \operatorname{tr}[\mathbf{P}_{\bar{\mathbf{a}}(\theta_1)}\hat{\tilde{\boldsymbol{R}}}_x] \tag{24}$$

where $\bar{\boldsymbol{a}}(\theta_1) = [\boldsymbol{a}^T(\theta_1), \boldsymbol{a}^H(\theta_1)]^T$. Next, we solve the second source, fixing the first source at $\hat{\theta}_1^{(0)}$,

$$\hat{\theta}_2^{(0)} = \arg\max_{\theta_2} \operatorname{tr}[\mathbf{P}_{[\bar{\mathbf{a}}(\hat{\theta}_1^{(0)}),\bar{\mathbf{a}}(\theta_2)]}\hat{\tilde{\boldsymbol{R}}}_x]. \tag{25}$$

The procedure is continued until all the initial values $\hat{\boldsymbol{\Theta}}^0 = [\hat{\theta}_1^{(0)}, \cdots, \hat{\theta}_d^{(0)}]$, are computed. As a result, the initial estimate of $\tilde{\boldsymbol{A}}^{(0)}$ is $[\boldsymbol{\Psi}^T\boldsymbol{A}^{(0)T}, \boldsymbol{\Psi}^H\boldsymbol{A}^{(0)H}]^T$.

Continuing in this fashion, the value of θ_i at the $(k+1)$-th iteration is obtained by solving the following problem

$$\hat{\theta}_i^{(k+1)} = \arg\max_{\theta_i^{(k)}} \operatorname{tr}[\mathbf{P}_{\tilde{\boldsymbol{A}}_i^k}\hat{\tilde{\boldsymbol{R}}}_x] \tag{26}$$

where

$$\tilde{\boldsymbol{A}}_i^{(k)} = \begin{bmatrix} e^{j\psi_1}\boldsymbol{a}(\hat{\theta}_1^{(k)}) & \cdots & e^{j\psi_d}\boldsymbol{a}(\hat{\theta}_d^{(k)}) \\ e^{-j\psi_1}\boldsymbol{a}^*(\hat{\theta}_1^{(k)}) & \cdots & e^{-j\psi_d}\boldsymbol{a}^*(\hat{\theta}_d^{(k)}) \end{bmatrix} \tag{27}$$

with $\theta_i^{(k)}$ replacing $\hat{\theta}_i^{(k)}$ in the i-th column.

4. SIMULATION RESULTS

In this section, we present a number of simulation examples that illustrate the superiority of the proposed NC-AP method. For comparison, the empirical results of the MUSIC [2], NC-MUSIC [9] and AP [6] methods are included. A uniform linear array consisting of $M = 8$ sensors with separation of half wavelength is considered in our simulations. In all examples, we assume that there are two equipowered BPSK signals, their initial phases are $0°$ and $10°$, and their DOAs are $5°$ and $10°$, respectively. Moreover, 100 independent Monte-Carlo trials are performed.

In the first example, we fix the number of snapshots at 60 and investigate the root mean square errors (RMSE) of the DOA estimates as a function of signal-to-noise ratio (SNR).

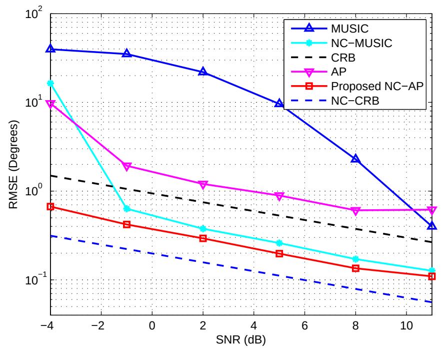

Fig. 1. RMSE of DOA estimates versus SNR for $\rho = 0$

Here, the source correlation coefficient is set as $\rho = 0$. Fig. 1 shows that the NC-AP performs much better than the other methods, whereas the MUSIC and NC-MUSIC fail to provide accurate DOA estimates especially at low SNR conditions.

Fig. 2 shows the RMSEs of the investigated methods versus SNR at $\rho = 0.5$. The number of snapshots is $N = 60$. It is observed that the value of ρ has little influence on the AP-based approaches. Moreover, we see that the output RMSEs of the MUSIC and AP algorithms are larger than the NC-based schemes because they do not exploit the NC properties of the signals.

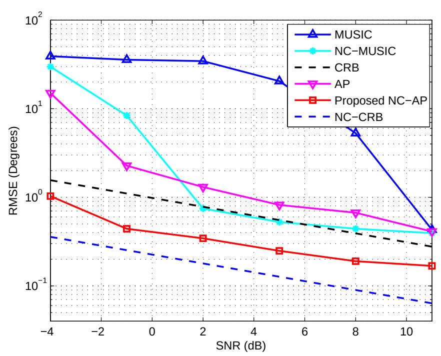

Fig. 2. RMSE of DOA estimates versus SNR for $\rho = 0.5$

The empirical results in Fig. 3 correspond to the scenario where SNR = 5dB and $\rho = 0$. It is seen that the proposed NC-AP technique still surpasses the other approaches even when the number of snapshots is small. Since the proposed approach is able to utilize the NC properties of the signals, it is superior to the AP in estimation accuracy.

In this example, we fix the value of ρ at 0.5 and investigate the RMSE of the DOAs as a function of the number of snapshots. it is observed from Fig. 4 that the NC-AP and NC-MUSIC, which exploit the NC properties of the signals, achieve a performance improvement compared with their

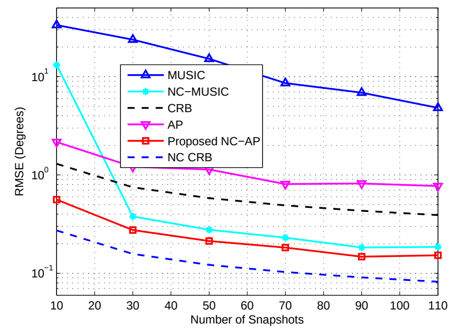

Fig. 3. RMSE of DOA estimates versus snapshot number for $\rho = 0$

counterparts, i.e., AP and MUSIC. Furthermore, we can see that the MUSIC-based algorithms with $\rho = 0.5$ have a distinctly performance degradation compared with the scenario of $\rho = 0$.

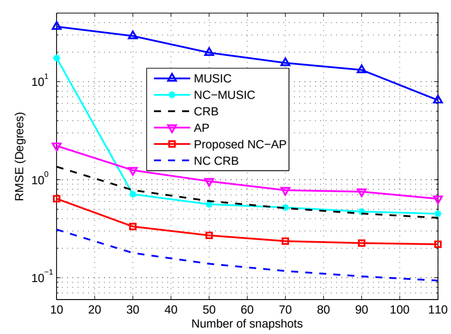

Fig. 4. RMSE of DOA estimates versus snapshot number for $\rho = 0.5$

5. CONCLUSION

A NC-DML estimator for DOA estimation of strictly NC signals has been proposed in this paper. The derivation is based on the fact that the number of parameters to be considered is reduced for the strictly NC signals. For computing the NC-DML, we present a novel NC-AP method. In analogy to the AP scheme, the NC-AP is implemented in an iterative fashion and converges to a local maximum eventually. However, unlike the AP solution, the NC-AP technique is performed based on the doubled virtual array structure. Moreover, it also has taken the impact of the initial phase shift of the NC signals into consideration. Simulation results show that the proposed NC-AP solution is superior to the state-of-the-art algorithms in terms of RMSE performance.

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