A NEW ARRAY GEOMETRY FOR DOA ESTIMATION WITH ENHANCED DEGREES OF FREEDOM

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ABSTRACT

This work presents a new array geometry, which is capable of providing $O(M^2N^2)$ degrees of freedom (DOF) using only MN physical sensors via utilizing the second-order statistics of the received data. This new array is composed of multiple, identical minimum redundancy subarrays, whose positions follow a minimum redundancy configuration. Thus the new array is a minimum redundancy array (MRA) of MRA subarrays, and is termed as *nested MRA*. The sensor positions, aperture length, and the number of DOF of the new array can be predicted if these parameters of MRA subarrays are given. Numerical simulations demonstrate the superiorities of the proposed array geometry in resolving more sources than sensors and DOA estimation.

Index Terms— Sensor arrays, minimum redundancy array, direction-of-arrival (DOA) estimation, co-array, nested array.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation is an important topic in various applications [1–3], such as radar and sonar. It is well known that the maximum number of sources that can be resolved by an N-element uniform linear array (ULA) using traditional DOA estimation methods, such as MUSIC [1] and ESPRIT [2], is $N-1$. The underdetermined DOA estimation problem, i.e. resolving more sources than the number of sensors, has received considerable interest in recent years [4–6]. An effective approach to solve this problem is to increase the number of degrees of freedom (DOF) under a *virtual array* equivalence, for example [5]. This virtual array is constructed by vectorizing the covariance matrix of the received data from a properly designed non-uniform linear array. The minimum redundancy array (MRA) [7] is such a linear array whose virtual array is a filled ULA with maximum possible aperture for a given number (N) of physical sensors. Unfortunately, there are no closed-form expressions for the sensor positions and achievable DOF as a function of N. Although MRAs for $N \leq 17$ sensors have been found by exhaustive search routines [8], there is no easy way to predict the design of larger MRAs.

In this paper, we propose a new array geometry dubbed *nested MRA (NMRA)* constituted of multiple, identical minimum redundancy subarrays. A key feature of the NMRA is that the positions of the subarrays follow a minimum redundancy configuration. Thus the NMRA is an MRA of MRA subarrays. The sensor positions and the number of DOF of an NMRA can be predicted when these parameters of MRA subarrays are known. It follows that given a known N-element MRA, it is possible to design a much larger N^2-element NMRA. By properly designing the spacing among the subarrays, its virtual array can also be a filled ULA.

2. SIGNAL MODEL

Consider a K-element linear antenna array with sensors located at $\mathbf{p} = [p_1, p_2, \cdots, p_K]d$, where $p_i\ (i = 1, 2, \cdots, K)$ are integers, and d is the unit inter-element spacing, usually equals to a half wavelength. Let Q uncorrelated narrowband sources impinge on the array from directions $\{\theta_q, q = 1, 2, \cdots, Q\}$. A steering vector is the array response to a unit strength source at angle θ,

$$\mathbf{a}(\theta) = \left[e^{j\kappa p_1 \sin\theta}, e^{j\kappa p_2 \sin\theta}, \cdots, e^{j\kappa p_K \sin\theta}\right]^T, \quad (1)$$

where $\kappa = 2\pi d/\lambda$, and λ is the signal wavelength.

The data received by the array can be expressed as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, 2, \cdots, T, \quad (2)$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_Q)]$ is the array manifold matrix and T is the number of snapshots. The source signals vector $\mathbf{s}(t) = [s_1(t), s_2(t), \cdots, s_Q(t)]^T$ is assumed unknown, but each element $s_q(t)$ follows the Gaussian distribution $\mathcal{CN}(0, \sigma_q^2)$. The sources are assumed to be temporally uncorrelated, so that the source autocorrelation matrix of $\mathbf{s}(t)$ is diagonal. The components of the noise vector $\mathbf{n}(t)$ are assumed to be independent and identically distributed (i.i.d.)

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additive white Gaussian noise with zero mean and variance σ_n^2, and are independent from the sources. Then the covariance matrix of the received data $\mathbf{x}(t)$ can be represented as

$$\begin{aligned}\mathbf{R_{xx}} &= E[\mathbf{x}\mathbf{x}^H] = \mathbf{A}\mathbf{R_{ss}}\mathbf{A}^H + \sigma_n^2\mathbf{I}_K \\ &= \sum_{q=1}^{Q}\sigma_q^2\mathbf{a}(\theta_q)\mathbf{a}^H(\theta_q) + \sigma_n^2\mathbf{I}_K,\end{aligned} \tag{3}$$

where $\mathbf{R_{ss}}$ is a $Q \times Q$ diagonal matrix with diagonal elements $\sigma_1^2, \sigma_2^2, \cdots, \sigma_Q^2$ and $\mathbf{I}_K$ is a $K \times K$ identity matrix. Vectorizing the matrix $\mathbf{R_{xx}}$, we obtain the $K^2 \times 1$ vector

$$\mathbf{z} = \mathrm{vec}(\mathbf{R_{xx}}) = \mathbf{B}\mathbf{c} + \sigma_n^2\mathbf{1}_n, \tag{4}$$

where $\mathbf{B} = [\mathbf{a}^*(\theta_1)\otimes\mathbf{a}(\theta_1), \cdots, \mathbf{a}^*(\theta_Q)\otimes\mathbf{a}(\theta_Q)]^T \in \mathbb{C}^{K^2\times Q}$, $\otimes$ stands for the Kronecker product and $\mathbf{1}_n = \left[\mathbf{e}_1^T, \mathbf{e}_2^T, \cdots, \mathbf{e}_K^T\right]^T$ with $\mathbf{e}_i^T$ denoting a vector of all zeros, except the i-th element which is equal to one. From (4) we notice that the vector $\mathbf{z}$ is equivalent to the received data from a *virtual array* with elements located at the position set

$$\mathbb{P} = \{(p_i - p_j)d, \quad i, j = 1, 2, \cdots, K\}. \tag{5}$$

The vector $\mathbf{c} = \left[\sigma_1^2, \sigma_2^2, \cdots, \sigma_Q^2\right]^T$ is interpreted as an equivalent source signal vector in the virtual array.

In the location set $\mathbb{P}$ of the virtual array, there are a total of K^2 elements, but some of them may be repeated (pairs of $p_i, p_j, i, j = 1, 2, \cdots, K$, for which $(p_i - p_j)$ are the same). In this case, multiple virtual elements are associated with the same virtual sensor location. We define the virtual array that has sensors located at distinct elements of $\mathbb{P}$, as a *difference co-array (DCA)* of an original array [5]. The DCA is symmetric due to the fact that for any $d_{ij} = (p_i - p_j)d$ in the position set $\mathbb{P}$ in (5), $-d_{ij} = (p_j - p_i)d$ is also in the set $\mathbb{P}$. The number of elements in the DCA is equal to the number of DOF [5, 9], and it can be larger than the number of physical sensors, if the original array is properly designed. Therefore, when performing DOA estimation, by using a part or the whole of the DCA instead of the original array, it is possible to resolve more sources than the number of physical sensors. The NMRA is such an array that designed to provide higher number of DOF than the number of physical sensors.

3. PROPOSED NESTED MINIMUM REDUNDANCY ARRAY

3.1. Array geometry

The proposed array is composed of N identical subarrays. Each subarray has M sensors with locations specified by the vector

$$\mathbf{u}_M = [m_1, m_2, \cdots, m_M]\, d, \tag{6}$$

where d is the minimum inter-element spacing, $m_1, m_2, \cdots, m_M$ are integers and $m_1 = 0$ without loss of generality. The subarray is an MRA and it is referred to as Subarray A. Assume

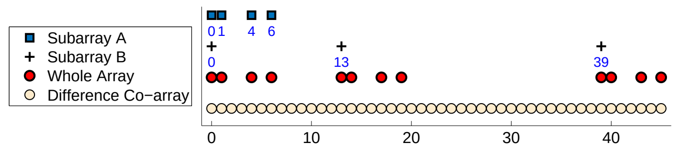

Fig. 1. An example geometry of NMRA consists of 3 identical 4-element minimum redundancy subarrays, in which $D = 13d$ and only the nonnegative part of the DCA is given since the symmetric property of the DCA.

now a second MRA of N elements, referred to as Subarray B, where N may or may not be equal to M. Let the element locations be

$$\mathbf{u}_N = [n_1, n_2, \cdots, n_N]\, D, \tag{7}$$

where $n_1, n_2, \cdots, n_N$ are integers, $n_1 = 0$, and $D > m_M \cdot d$. We now combine the two arrays by placing a Subarray A at each element location of Subarray B. Then the positions of all sensors form a cross summation set

$$\mathbb{S} = \{n_j \cdot D + m_i \cdot d, \quad 1 \le i \le M,\ 1 \le j \le N\}. \tag{8}$$

Let the mathematical sign $\oplus$ denote the cross summation of every element in $\mathbf{u}_N$ and every element in $\mathbf{u}_M$. In this way the sensor positions of the whole array may be expressed by the vector

$$\mathbf{v} = \mathbf{u}_N \oplus \mathbf{u}_M. \tag{9}$$

Note that the nesting method embodied in (9) between two MRA's is different from the nested array introduced in [5], which is a union of two uniform linear subarrays.

Fig. 1 depicts an example that illustrates the proposed array geometry. With $M = 4$ and $N = 3$, the NMRA has $MN = 12$ sensors. Subarray A, Subarray B, NMRA and its DCA are shown in Fig. 1, with $D = 13d$. We can see that the aperture length of the NMRA is $45d$, and the DCA is a filled ULA with $2 \times 45 + 1 = 91$ elements owing to the symmetric property of the DCA. Hence the number of DOF associated with the DCA of the NMRA is 91.

3.2. Properties of NMRA

Our design assumes the structures of Subarray A and Subarray B to be known. We denote the parameters of the two subarrays as follows. The aperture length of Subarray A with M sensors is $l_A \cdot d$, the location set of the DCA associated with Subarray A is $\mathbb{D}_A$ and the number of DOF obtained from its DCA is f_A. The corresponding parameters for Subarray B are $l_B \cdot D$, $\mathbb{D}_B$ and f_B, respectively. Next we derive the properties of an NMRA using the parameters of its components, Subarray A and Subarray B.

Proposition 1: The location set $\mathbb{D}_V$ of the DCA associated with an NMRA $\mathbf{v}$ can be obtained by the cross summation of location sets $\mathbb{D}_B$ and $\mathbb{D}_A$, that is

$$\mathbb{D}_V = \mathbb{D}_B \oplus \mathbb{D}_A. \tag{10}$$

Proof. Substituting (8) into (5), we obtain

$$\begin{aligned}\mathbb{D}_V &= \{(n_j \cdot D + m_i \cdot d) - (n_{j'} \cdot D + m_{i'} \cdot d)\,, \\ &\quad 1 \le i, i' \le M,\ 1 \le j, j' \le N\} \\ &= \{(n_j - n_{j'})D + (m_i - m_{i'})d\}.\end{aligned} \tag{11}$$

where $\{(n_j - n_{j'})D,\ 1 \le j, j' \le N\}$ and $\{(m_i - m_{i'})d,\ 1 \le i, i' \le M\}$ are the location set forming the DCA of Subarray B and Subarray A, respectively. The DCA of an MRA is a filled ULA with symmetric property, and both Subarray B and Subarray A are MRAs. Therefore $\mathbb{D}_B$ and $\mathbb{D}_A$ can be expressed respectively as

$$\begin{aligned}\mathbb{D}_B &= \{nD = nLd, \quad -l_B \le n \le l_B\}\,, \\ \mathbb{D}_A &= \{md, \quad -l_A \le m \le l_A\}\,.\end{aligned} \tag{12}$$

where $D = Ld$, L is an integer.

Combining (11) and (12), we can obtain

$$\begin{aligned}\mathbb{D}_V &= \{(nL + m)d, \quad -l_B \le n \le l_B, -l_A \le m \le l_A\} \\ &= \mathbb{D}_B \oplus \mathbb{D}_A.\end{aligned} \tag{13}$$

Therefore $\mathbb{D}_V$ can be calculated by the cross summation of $\mathbb{D}_B$ and $\mathbb{D}_A$. □

This proposition reveals the relationship between the DCA of an NMRA and DCAs of its components, Subarray B and Subarray A. According to the proposition, the DCA of a larger NMRA can be calculated using the DCAs of two smaller arrays, which simplifies its computation method. This proposition also inspires us to compute the aperture length and the number of DOF of the NMRA using parameters of Subarray A and Subarray B. The following proposition specifies the NMRA.

Proposition 2: If $D = f_A \cdot d$, then the following properties hold for the NMRA $\mathbf{v}$ constructed by (9).

(a) The DCA of the NMRA is a filled ULA.

(b) The aperture length of the NMRA is $l_V \cdot d = (l_B \cdot f_A + l_A)\, d$.

(c) The number of DOF obtained from the DCA of the NMRA is $f_V = f_A \cdot f_B$.

Proof. From (12) we can obtain the relationship between the number of DOF and the aperture length of Subarray B and Subarray A respectively, that is $f_A = 2l_A + 1$, $f_B = 2l_B + 1$. Next we prove Proposition 2 using the following three steps.

(a) There are $2l_A + 1\ (= f_A)$ possible values for m in (13). Thus the item $nL + m$ are consecutive integers when $L = f_A$, which leads to the fact that the DCA of the NMRA is a filled ULA if $D = f_A \cdot d$.

(b) From (8) we know that the aperture length of an NMRA is $l_V = l_B \cdot D + l_A \cdot d = (l_B \cdot f_A + l_A)\, d$.

(c) The DCA of an NMRA is a filled ULA within the range of $[-l_V, l_V]\, d$. Therefore the number of DOF is

$$\begin{aligned}f_V &= 2l_V + 1 = 2f_A \cdot l_B + 2l_A + 1 \\ &= 2f_A \cdot l_B + f_A = f_A \cdot (2l_B + 1) \\ &= f_A \cdot f_B.\end{aligned} \tag{14}$$

□

Subarray A and Subarray B are both MRAs and their numbers of DOF can be expressed respectively as [8]

$$\begin{aligned}f_A &= M^2 - M + 1 - M_R, \\ f_B &= N^2 - N + 1 - N_R,\end{aligned} \tag{15}$$

where $M_R,\ N_R$ are the number of redundancies for an M-sensor MRA and an N-sensor MRA, respectively. Substituting (15) into (14), we can obtain

$$\begin{aligned}f_V &= f_A \cdot f_B \\ &= \left(M^2 - M + 1 - M_R\right)\left(N^2 - N + 1 - N_R\right).\end{aligned} \tag{16}$$

Thereby it can be concluded that *the NMRA can provide $O(M^2N^2)$ DOF using only MN physical sensors.*

3.3. DOF Comparison of different array geometries

In this section we compare the number of DOF provided by our proposed NMRA with DOFs respectively provided by the MRA, the recently proposed nested array [5] and the coprime array with compressed inter-element spacing (CACIS) in [9]. These numbers of DOFs can be obtained using Proposition 2 and some relevant equations in [5, 9]. Let the total number of sensors K be some integers from 9 to 36. We get the comparison results in Table 1. We observe that our proposed NMRA has the highest DOF except the MRAs. However no such MRA with more than 17 sensors has been given by existing references [8], and hence their parameters are shown as 'NA' in the corresponding table.

Table 1. DOF comparison of different array geometries

K	9	12	16	18	20	24	27	30	32	36
MRA	59	101	181	NA	NA	NA	NA	NA	NA	NA
Nested Array	49	83	143	179	219	311	391	479	543	683
CACIS	29	65	111	145	189	277	341	437	495	631
NMRA	49	91	169	189	247	351	413	513	611	767

4. DOA ESTIMATION METHODS

In this section we introduce the DOA estimation methods applied to the DCA of an NMRA to resolve more sources than sensors. Recall the equivalent source signal vector $\mathbf{c} = [\sigma_1^2, \sigma_2^2, \cdots, \sigma_Q^2]^T$ in (4). It is composed of the powers σ_q^2 of the actual sources, and all elements in $\mathbf{c}$ are real values. Therefore, these equivalent sources behave like fully coherent sources in the DCA. Then the traditional subspace-based DOA estimation algorithms, such as MUSIC and ESPRIT, cannot be applied directly to the DCA to estimate DOAs. Various algorithms, such as spatial smoothing (SS) MUSIC [5, 9, 10], KR product based MUSIC [4], sparse signal reconstruction based methods [11, 12], have been proposed to implement underdetermined DOA estimation.

Because the DCA generated from an NMRA is a filled ULA, to which the SS-MUSIC algorithm can be directly applied to resolve more sources than sensors. The implementation method of SS-MUSIC algorithm can be found in [5, 9], which is used as the performance metric in this paper. It should be noted that the achievable DOF used for the SS-MUSIC algorithm is only $l_V + 1$ owing to the SS operation, which is roughly equal to half of the number of distinct elements in the DCA.

5. NUMERICAL EXAMPLES

In this section we conduct experiments to evaluate the DOA estimation performance using the SS-MUSIC algorithm described in Section 4. First we show the ability of the proposed array to resolve more sources than sensors. We use the NMRA with 12 physical sensors illustrated in Fig. 1 for example. The achievable DOF used for SS-MUSIC becomes 46 due to the fact that the SS operation will halve the number of DOF obtained from the DCA. We consider $Q = 37$ uncorrelated narrowband sources impinging on the array with equal power, whose spatial frequencies $\sin\theta$ are uniformly distributed between -0.95 and 0.95. The number of source Q is assumed to be known in our simulations. The MUSIC spectrum of the NMRA is shown in Fig. 2, where 1000 noise-free snapshots are used. It shows that the NMRA can resolve all the 37 sources correctly, which is much larger than the number of physical sensors (=12).

Next we use Monte Carlo simulations to analyze the average root-mean-square error (RMSE) of the estimated DOAs. We use three array geometries (the Nested Array, CACIS and NMRA) of 24 physical sensors and consider $Q = 16$ narrowband uncorrelated sources uniformly distributed between $-70°$ and $70°$. Fig. 3 plots the RMSEs of three array geometries as a function of SNR with 100 snapshots, which is obtained over 500 trials. It can be concluded that the DOA estimation performance improves with the increase of SNR. The CACIS has the highest RMSE because it has the shortest aperture, least number of DOF and its DCA is not a filled ULA. Our proposed NMRA achieves the best performances with the lowest RMSE, which shows the superiority in DOA estimation over other array geometries.

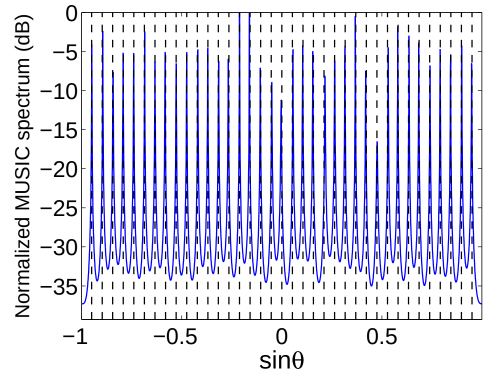

Fig. 2. MUSIC spectrum as a function of sine of the DOA (Sources number $Q = 37$), using the NMRA with 12 physical sensors. The vertical dash lines are the true positions of the sources.

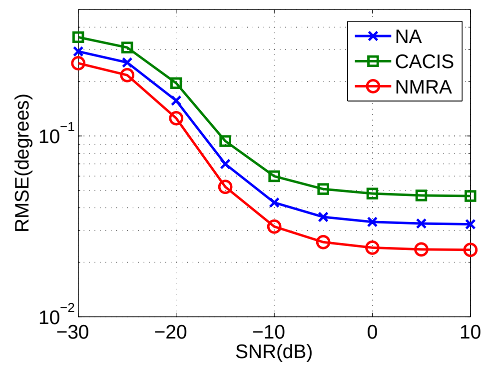

Fig. 3. RMSE versus SNR (100 snapshots) for different arrays with 24 physical sensors.

6. CONCLUSION

In this paper we proposed a new array geometry dubbed nested MRA (NMRA), which can be easily constructed by the cross summation of two MRA subarrays. It is possible to predict the sensor positions and the number of DOF when these parameters of the MRA subarrays are known. The NMRA has a larger aperture as well as a higher number of DOF than the nested array and the CACIS. We demonstrated the superiorities of the proposed array in resolving more sources than sensors and DOA estimation performance using spatial smoothing based MUSIC algorithm. The shortcoming of the proposed array geometry is that the new array relies on the structures of known MRAs, and therefore not all NMRA geometries with any number of sensors can be obtained. However the new array geometry provides a closed-form solution to generate a suboptimal co-array, and can easily obtain a larger array employing the known MRA.

7. REFERENCES

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *Antennas and Propagation, IEEE Transactions on*, vol. 34, no. 3, pp. 276–280, 1986.

[2] R. Roy and T. Kailath, "Esprit-estimation of signal parameters via rotational invariance techniques," *Acoustics, Speech and Signal Processing, IEEE Transactions on*, vol. 37, no. 7, pp. 984–995, 1989.

[3] Xin Yuan, "Direction-finding wideband linear fm sources with triangular arrays," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 48, no. 3, pp. 2416–2425, 2012.

[4] Wing-Kin Ma, Tsung-Han Hsieh, and Chong-Yung Chi, "Doa estimation of quasi-stationary signals with less sensors than sources and unknown spatial noise covariance: A khatri-rao subspace approach," *Signal Processing, IEEE Transactions on*, vol. 58, no. 4, pp. 2168–2180, 2010.

[5] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *Signal Processing, IEEE Transactions on*, vol. 58, no. 8, pp. 4167–4181, 2010.

[6] Zhao Tan, Y. C. Eldar, and A. Nehorai, "Direction of arrival estimation using co-prime arrays: A super resolution viewpoint," *Signal Processing, IEEE Transactions on*, vol. 62, no. 21, pp. 5565–5576, 2014.

[7] A. Moffet, "Minimum-redundancy linear arrays," *Antennas and Propagation, IEEE Transactions on*, vol. 16, no. 2, pp. 172–175, 1968.

[8] Harry L Van Trees, *Optimum array processing: part IV of detection, estimation, and modulation*, Wiley, New York, 2002.

[9] Si Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *Signal Processing, IEEE Transactions on*, vol. 63, no. 6, pp. 1377–1390, 2015.

[10] Chun-Lin Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray music," *Signal Processing Letters, IEEE*, vol. 22, no. 9, pp. 1438–1442, 2015.

[11] E. BouDaher, F. Ahmad, and M. G. Amin, "Sparsity-based direction finding of coherent and uncorrelated targets using active nonuniform arrays," *Signal Processing Letters, IEEE*, vol. 22, no. 10, pp. 1628–1632, 2015.

[12] Y. D. Zhang, M. G. Amin, and B. Himed, "Sparsity-based doa estimation using co-prime arrays," in *Acoustics, Speech and Signal Processing (ICASSP), 2013 IEEE International Conference on*, 2013, pp. 3967–3971.