# ROBUST ADAPTIVE BEAMFORMING BASED ON DOA SUPPORT USING DECOMPOSED COPRIME SUBARRAYS

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## ABSTRACT

In this paper, we propose a novel robust adaptive beamforming algorithm with direction-of-arrival (DOA) support for the coprime array. Specifically, by using the property of coprime number, we may estimate the DOAs of sources by matching two super-resolution spatial spectra of the pair of decomposed coprime subarrays. After that, the power of each source can be estimated via a covariance matrix joint estimation problem corresponding to the pair of decomposed coprime subarrays. Taking the estimated DOAs and their corresponding power as the support information, the interference-plus-noise covariance matrix for the coprime array can be reconstructed, from which the minimum variance distortionless response beamformer weight vector can be calculated. Simulation results show that the proposed adaptive beamforming algorithm is more robust to signal look direction mismatch than the existing algorithms.

***Index Terms—*** Coprime array, direction-of-arrival (DOA), joint estimation, robust adaptive beamforming.

## 1. INTRODUCTION

Compared with fixed beamforming and switch beamforming, adaptive beamforming can provide better resolution and much better interference suppression capability by calculating the beamforming weight from the array received signal. Hence, it has been widely applied in radar, sonar, acoustics, seismology, speech processing, wireless communications, and so on [1–7]. However, adaptive beamformers are also well known to be sensitive to model mismatch, especially when the training samples are contaminated by the desired signal. To reduce the sensitivity of adaptive beamformers, many robust algorithms were proposed in the past decades [8–12]. Among them, the recently proposed interference covariance matrix reconstruction-based adaptive beamforming algorithm [11, 12] performs much better than others. In spite of this, the existing adaptive beamfoming algorithms were mainly designed for the uniform linear array (ULA).

More recently, the emergence of coprime sampling [13], which utilizes a pair of coprime factors to undersample the signal, attracts the research interests of researchers. Among them, coprime array is a typical application of coprime sampling, where a large array aperture can be obtained by using far fewer antennas. However, existing researches on coprime array mainly focus on improving the degree-of-freedom (DOF) to estimate more sources than the actual physical antennas [14–18]. Up to now, there is few research on adaptive beamforming specially for the coprime array.

In this paper, we propose a novel adaptive beamforming algorithm based on DOA support for the coprime array, which is especially robust against signal look direction mismatch. Instead of generating a virtual ULA, we use the pair of decomposed coprime subarrays separately to reduce the computational complexity as we did in [19]. By exploiting the property of coprime number, the DOAs of sources can be estimated by matching two super-resolution spatial spectra of the pair of decomposed coprime subarrays. With the estimated DOAs, the source power can be subsequently estimated by solving a covariance matrix joint estimation problem. With the estimated DOA of the desired signal and the reconstructed interference-plus-noise covariance matrix, the adaptive beamforming weight vector can be readily calculated following minimum variance distortionless response (MVDR) principle. Numerical examples show the effectiveness and robustness of the proposed adaptive beamforming algorithm for the coprime array.

## 2. COPRIME ARRAY SIGNAL MODEL

The coprime array consists of two sparsely-spaced uniform linear subarrays with $M$ and $N$ physical sensors, respectively, where $M$ and $N$ are coprime numbers. In detail, the sensors of the first subarray locate at $\{0, Nd, 2Nd, \cdots, (M-1)Nd\}$ with the inter-element space $Nd$, while the sensors of the second subarray locate at $\{0, Md, 2Md, \cdots, (N-1)Md\}$ with the inter-element space $Md$, where $d$ is generally chosen to

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be a half-wavelength. According to the property of coprime number, the pair of sparse uniform linear subarrays does not overlap other than the first sensor. Hence, the coprime array has the aperture of $max\big((M-1)Nd, (N-1)Md\big)$ with only $M+N-1$ physical sensors.

The coprime array received signal at time $k$ can be modeled as

$$\mathbf{x}(k) = \mathbf{a}(\theta_s)s(k) + \mathbf{i}(k) + \mathbf{n}(k), \tag{1}$$

where $s(k)$ is the desired signal waveform and $\mathbf{a}(\theta_s) \in \mathbb{C}^{M+N-1}$ is the corresponding steering vector with DOA $\theta_s$, $\mathbf{i}(k)$ and $\mathbf{n}(k)$ are the statistically independent interference component and noise component, respectively.

The coprime array output is given by

$$y(k) = \mathbf{w}^H\mathbf{x}(k) \tag{2}$$

with a designed beamforming weight vector $\mathbf{w} \in \mathbb{C}^{M+N-1}$, where $(\cdot)^H$ denotes the Hermitian transpose. Among various beamforming criteria, the output signal-to-interference-plus-noise ratio (SINR) maximization

$$\max_{\mathbf{w}} \frac{\sigma_s^2\big|\mathbf{w}^H\mathbf{a}(\theta_s)\big|^2}{\mathbf{w}^H\mathbf{R}_{i+n}\mathbf{w}} \tag{3}$$

is the most popular one, where $\sigma_s^2 = E\big\{|s(k)|^2\big\}$ denotes the signal power, $\mathbf{R}_{i+n} = E\big\{\big(\mathbf{i}(k)+\mathbf{n}(k)\big)\big(\mathbf{i}(k)+\mathbf{n}(k)\big)^H\big\} \in \mathbb{C}^{(M+N-1)\times(M+N-1)}$ denotes the interference-plus-noise covariance matrix. Here, $E\{\cdot\}$ denotes the expectation operator. The output SINR maximization problem defined in (3) is equivalent to the MVDR problem [20] as

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{R}_{i+n}\mathbf{w} \quad \text{subject to} \quad \mathbf{w}^H\mathbf{a}(\theta_s) = 1, \tag{4}$$

which solution

$$\mathbf{w} = \frac{\mathbf{R}_{i+n}^{-1}\mathbf{a}(\theta_s)}{\mathbf{a}^H(\theta_s)\mathbf{R}_{i+n}^{-1}\mathbf{a}(\theta_s)} \tag{5}$$

is also known as the Capon beamformer. Since $\mathbf{R}_{i+n}$ is unavailable in practice, it is usually replaced by the sample covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}(k)\mathbf{x}^H(k), \tag{6}$$

where $K$ denotes the number of snapshots.

Note that the proportion of the desired signal component becomes larger with the increase of signal-to-noise ratio (SNR), which will lead to severe signal self-nulling phenomenon especially at high SNRs. In addition, the performance of adaptive beamformer is also sensitive to the DOA estimation of desired signal.

## 3. THE PROPOSED ALGORITHM

In this section, a novel adaptive beamforming algorithm based on DOA support is proposed for the coprime array. Since the estimation resolution is proportional to the array aperture, we take advantage of large array aperture provided by coprime array for DOA estimation.

Considering the pair of sparse uniform linear subarrays separately, the received signals of the decomposed coprime subarrays are given by

$$\begin{aligned}
\mathbf{x}_M(k) &= \mathbf{a}_M(\theta_s)s(k) + \mathbf{i}_M(k) + \mathbf{n}_M(k) \\
\mathbf{x}_N(k) &= \mathbf{a}_N(\theta_s)s(k) + \mathbf{i}_N(k) + \mathbf{n}_N(k),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\mathbf{a}_M(\theta_s) &= [1, e^{-\jmath\pi N\sin(\theta_s)}, \cdots, e^{-\jmath\pi(M-1)N\sin(\theta_s)}]^T \\
\mathbf{a}_N(\theta_s) &= [1, e^{-\jmath\pi M\sin(\theta_s)}, \cdots, e^{-\jmath\pi(N-1)M\sin(\theta_s)}]^T
\end{aligned}$$

denote the steering vectors of the desired signal corresponding to the pair of decomposed coprime subarrays. Correspondingly, the sample covariance matrix of each subarray can be calculated as $\hat{\mathbf{R}}_M = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_M(k)\mathbf{x}_M^H(k)$ and $\hat{\mathbf{R}}_N = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_N(k)\mathbf{x}_N^H(k)$, respectively.

Due to its super-resolution property, the multiple signal classification (MUSIC) algorithm [21–23] is adopted to estimate the DOAs as the support information for the proposed adaptive beamforming algorithm. The MUSIC spatial pseudo-spectra of the pair of decomposed coprime subarrays are given by

$$\begin{aligned}
\mathbf{P}_M(\theta) &= \frac{1}{\mathbf{a}_M^H(\theta)\mathbf{E}_M\mathbf{E}_M^H\mathbf{a}_M(\theta)} \\
\mathbf{P}_N(\theta) &= \frac{1}{\mathbf{a}_N^H(\theta)\mathbf{E}_N\mathbf{E}_N^H\mathbf{a}_N(\theta)},
\end{aligned} \tag{8}$$

where $\theta \in [-\pi/2, \pi/2)$ denotes the hypothetical direction, $\mathbf{E}_M$ and $\mathbf{E}_N$ denote the noise subspaces corresponding to the pair of decomposed coprime subarrays, respectively.

Since the inter-element spaces of both decomposed coprime subarrays are much larger than a half-wavelength, phase ambiguity will appear in the MUSIC spectra $\mathbf{P}_M$ and $\mathbf{P}_N$. Hence, we can obtain the candidate DOA estimations $\{\theta_{M_i}, i = 1, 2, \cdots, D_M\}$ and $\{\theta_{N_j}, j = 1, 2, \cdots, D_N\}$ by searching for the peaks of the pair of spectra. Note that $D_M$ and $D_N$ denote the number of peaks, which include the DOAs of the desired signal, interferences, and their phase ambiguities. For the purpose of distinguishing the DOAs of these sources from their phase ambiguities, we proposed a

**Theorem** [19]: *Suppose $\phi$ is a DOA of the desired signal or interferences, there exists and uniquely exists a $\hat{\phi}$ that presents a peak in both MUSIC spectra $\boldsymbol{P}_M$ and $\boldsymbol{P}_N$, and $\hat{\phi}$ is the DOA estimation of decomposed coprime subarrays.*

The existence and uniqueness of the estimated DOA $\hat{\phi}$ has been proved in [19]. However, it is hard to get the perfect unique directions from $\{\theta_{M_i}\}$ and $\{\theta_{N_j}\}$ even under relatively

$$
\min_{\mathbf{\Lambda}} \left\| \begin{bmatrix} \hat{\mathbf{R}}_M & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_N \end{bmatrix} - \begin{bmatrix} \mathbf{A}_M(\hat{\boldsymbol{\theta}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_N(\hat{\boldsymbol{\theta}}) \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{bmatrix} \begin{bmatrix} \mathbf{A}_M^H(\hat{\boldsymbol{\theta}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_N^H(\hat{\boldsymbol{\theta}}) \end{bmatrix} - \begin{bmatrix} \hat{\sigma}_{nM}^2 \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \hat{\sigma}_{nN}^2 \mathbf{I}_N \end{bmatrix} \right\|_F^2 \tag{13}
$$

subject to $\mathbf{\Lambda} \succeq \mathbf{0}$,

ideal conditions. Hence, instead of searching for the completely overlapped peaks, we will estimate the DOA of desired signal by searching for the nearest peaks in $\mathbf{P}_M$ and $\mathbf{P}_N$ within the desired signal angular region $\mathbf{\Theta}$ as

$$
\min_{\theta_{M_i}, \theta_{N_j}} \left| \theta_{M_i} - \theta_{N_j} \right|, \quad \forall\, \theta_{M_i}, \theta_{N_j} \in \mathbf{\Theta}. \tag{9}
$$

Hence, the DOA of desired signal can be estimated as $\hat{\theta}_s = \frac{\theta_{M_i} + \theta_{N_i}}{2}$, namely, the mean of $\theta_{M_i}$ and $\theta_{N_j}$ minimizing the objective in (9). Based on this estimation, the steering vector of desired signal can be calculated as

$$
\mathbf{a}(\hat{\theta}_s) = [1, e^{-\jmath \pi u_2 \sin(\hat{\theta}_s)}, \cdots, e^{-\jmath \pi u_{M+N-1} \sin(\hat{\theta}_s)}]^T, \tag{10}
$$

where $u_i, i = 2, \cdots, M+N-1$ denotes the physical sensor positions in the coprime array.

Similarly, we can estimate the DOAs of interferences by

$$
\min_{\theta_{M_i}, \theta_{N_j}} \left| \theta_{M_i} - \theta_{N_j} \right| < \xi, \quad \forall\, \theta_{M_i}, \theta_{N_j} \in \bar{\mathbf{\Theta}}, \tag{11}
$$

where $\bar{\mathbf{\Theta}}$ is the complementary set of $\mathbf{\Theta}$, and hence, it covers the angular region containing all interferences. Here, $\xi$ denotes the resolution threshold that determines the estimation of interferences in $\bar{\mathbf{\Theta}}$ since the number of interferences is *a priori* unknown in practice. The estimated DOAs of interferences from (11) can be denoted as $\boldsymbol{\psi} = [\psi_1, \psi_2, \cdots, \psi_Q]^T$, where $Q$ denotes the number of estimated interferences.

When the interferences and noise are statistically independent, the interference-plus-noise covariance matrix $\mathbf{R}_{i+n}$ can be simplified as

$$
\mathbf{R}_{i+n} = \sum_{q=1}^{Q} \sigma_{\psi_q}^2 \mathbf{a}(\psi_q) \mathbf{a}^H(\psi_q) + \sigma_n^2 \mathbf{I}, \tag{12}
$$

where $\sigma_{\psi_q}^2$ and $\sigma_n^2$ denote the power of the interference with DOA $\psi_q$ and noise, respectively, and $\mathbf{I}$ denotes the identity matrix. By observing (12), we can see that the power of all interferences corresponding to the estimated DOAs in $\boldsymbol{\psi}$ is also required to reconstruct the interference-plus-noise covariance matrix. However, although it has a super-resolution capability, the MUSIC spectrum belongs to a kind of spatial pseudo-spectra. That is to say, we cannot obtain the power of each estimated source from the MUSIC spectrum directly.

To solve this problem, we will estimate the power of each estimated source based on the idea of covariance matrix joint estimation. Specifically, considering that the power of the desired signal and interferences received by the pair of decomposed coprime subarrays is identical, we can estimate their power by minimizing the difference of these two sample covariance matrices $\hat{\mathbf{R}}_M$ and $\hat{\mathbf{R}}_N$ with their corresponding theoretical covariance matrices jointly. The proposed covariance matrix joint estimation problem is formulated in (13), shown at the top of this page, where $\hat{\boldsymbol{\theta}} = [\hat{\theta}_s, \psi_1, \psi_2, \cdots, \psi_Q]^T$ contains the estimated DOAs of the sources, $\mathbf{A}_M(\hat{\boldsymbol{\theta}}) \in \mathbb{C}^{M \times (Q+1)}$ and $\mathbf{A}_N(\hat{\boldsymbol{\theta}}) \in \mathbb{C}^{N \times (Q+1)}$ denote the steering matrices of the pair of decomposed coprime subarrays, $\mathbf{\Lambda} = \mathrm{diag}\big([\sigma_s^2, \sigma_{\psi_1}^2, \sigma_{\psi_2}^2, \cdots, \sigma_{\psi_Q}^2]\big)$ contains the power of sources, $\hat{\sigma}_{nM}^2$ and $\hat{\sigma}_{nN}^2$ denote the noise power of the pair of subarrays approximately estimated by the minimum eigenvalue of $\hat{\mathbf{R}}_M$ and $\hat{\mathbf{R}}_N$, respectively, and $\mathbf{0}$ denotes the zero matrix of appropriate dimension.

The above covariance matrix joint estimation problem belongs to a least-squares problem, which can be efficiently solved by

$$
\mathrm{diag}(\hat{\mathbf{\Lambda}}) = \left[\mathbf{C}^H \mathbf{C}\right]^{-1} \mathbf{C}^H \mathbf{v}, \tag{14}
$$

where

$$
\begin{aligned}
\mathbf{C} \;&=\; \Bigg[\mathrm{vec}\left( \begin{bmatrix} \mathbf{a}_M(\hat{\theta}_s)\mathbf{a}_M^H(\hat{\theta}_s) & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_N(\hat{\theta}_s)\mathbf{a}_N^H(\hat{\theta}_s) \end{bmatrix} \right), \cdots, \\
&\qquad \mathrm{vec}\left( \begin{bmatrix} \mathbf{a}_M(\psi_Q)\mathbf{a}_M^H(\psi_Q) & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_N(\psi_Q)\mathbf{a}_N^H(\psi_Q) \end{bmatrix} \right)\Bigg], \\
\mathbf{v} \;&=\; \mathrm{vec}\left( \begin{bmatrix} \hat{\mathbf{R}}_M - \hat{\sigma}_{nM}^2 \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_N - \hat{\sigma}_{nN}^2 \mathbf{I}_N \end{bmatrix} \right).
\end{aligned}
$$

Based on the estimated DOAs of interferences $\boldsymbol{\psi}$ and their power $\mathrm{diag}(\hat{\mathbf{\Lambda}})$, the interference-plus-noise covariance matrix can be reconstructed as

$$
\tilde{\mathbf{R}}_{i+n} = \sum_{q=1}^{Q} \hat{\sigma}_{\psi_q}^2 \mathbf{a}(\psi_q) \mathbf{a}^H(\psi_q) + \hat{\sigma}_n^2 \mathbf{I}, \tag{15}
$$

where $\hat{\sigma}_n^2$ can be chosen as the mean of $\hat{\sigma}_{nM}^2$ and $\hat{\sigma}_{nN}^2$.

Substituting the estimated steering vector $\mathbf{a}(\hat{\theta}_s)$ (10) and the reconstructed interference-plus-noise covariance matrix $\tilde{\mathbf{R}}_{i+n}$ (15) into (5) together, the proposed adaptive beamforming weight vector can be calculated as

$$
\tilde{\mathbf{w}} = \frac{\tilde{\mathbf{R}}_{i+n}^{-1} \mathbf{a}(\hat{\theta}_s)}{\mathbf{a}^H(\hat{\theta}_s) \tilde{\mathbf{R}}_{i+n}^{-1} \mathbf{a}(\hat{\theta}_s)}. \tag{16}
$$

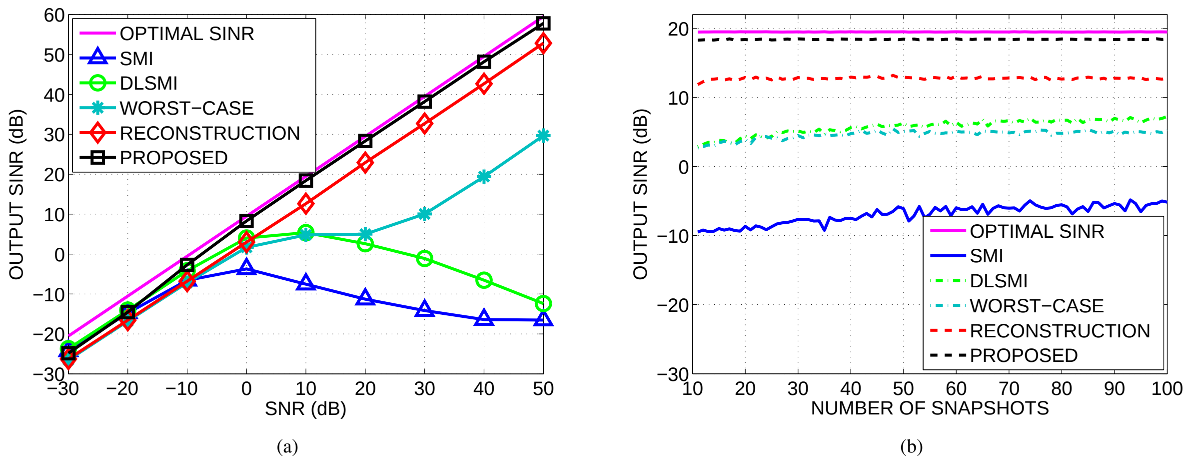


**Fig. 1**. Performance comparison. (a) Output SINR versus input SNR; (b) Output SINR versus number of snapshots.

Similarly to [12], instead of integrating over the whole interference angular region $\bar{\mathbf{\Theta}}$ as in [11], the proposed beamforming algorithm only calculates the interferences in the finite set $\psi$ for reconstructing the interference-plus-noise covariance matrix, and has the computational complexity $\mathcal{O}\big((M^2+N^2)L\big)$, where $L \gg M+N$ denotes the number of hypothetical directions in (8). Compared to [12] whose computational complexity is $\mathcal{O}\big((M+N)^2L\big)$, the proposed beamforming algorithm has lower computational complexity since the received signal is processed by a pair of decomposed coprime subarrays separately.

## 4. SIMULATION RESULTS

In our simulations, the coprime array structure is placed with the coprime factor $M=6$ and $N=5$, which means the coprime array actually consists of $M+N-1=10$ physical sensors. It is assumed that the desired signal is a narrowband plane-wave with DOA $\theta_s = 5°$, and two interferences are from $-10°$ and $20°$. The interference-to-noise ratio (INR) in each sensor is set to be 30 dB, and the noise is a zero-mean additive white Gaussian process. The influence of signal look direction mismatch is considered, where both the desired signal and interferences have a random DOA mismatch uniformly distributed in $[-4°, 4°]$ from trial to trial. For each scenario, 1,000 Monte-Carlo trials are run.

The proposed beamformer (16) is compared to the SMI beamformer [20], the DLSMI beamformer [24], the worst-case beamformer [9], and the reconstruction-based beamformer [11]. In the reconstruction-based beamformer and the proposed beamformer, we assume the angular region where the desired signal located in is $\mathbf{\Theta} = [0°, 10°]$, and hence, $\bar{\mathbf{\Theta}} = [-90°, 0°) \cup (10°, 90°]$. The grid of the hypothetical direction in (8) is chosen to be $0.1°$, and the resolution threshold $\xi$ in (11) is set to be double the grid. The norm upper bound of the steering vector mismatch is set to be $\varepsilon = 3$ in the worst-case beamformer, and the diagonal loading factor is chosen to be $10\sigma_n^2$ in the DLSMI beamformer.

The output SINR performance versus input SNR is depicted in Fig. 1(a) with the number of snapshots $K=50$. It is obvious that the performance of the proposed beamformer is better than the others, especially when SNR is high. In Fig. 1(b), the output SINR performance comparison is illustrated against the number of snapshots, where the input SNR is fixed at 10 dB. It is clear that the proposed beamforming algorithm converges faster than others. Benefit from the large array aperture of decomposed coprime subarrays, more accurate DOA estimation can be achieved while the phase ambiguities caused by the sparse sensor position can be effectively removed according to the property of coprime number. Moreover, power estimation is performed for each estimated source. Therefore, the proposed adaptive beamforming algorithm enjoys better output SINR performance than others.

## 5. CONCLUSION

Compared to the ULA, coprime array can provide the same array aperture with far fewer physical sensors. In this paper, we proposed a novel adaptive beamforming algorithm based on DOA support for the coprime array. By matching two MUSIC spatial spectra of the pair of decomposed coprime subarrays, we can estimate the source DOAs, which can be subsequently used to estimate their power by solving a covariance matrix joint estimation problem. With the reconstructed interference-plus-noise covariance matrix and the estimated DOA of desired signal, the MVDR-based adaptive beamformer is presented. Simulation results demonstrate the robustness of the proposed adaptive beamforming algorithm.

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