A SEMIDEFINITE RELAXATION APPROACH TO THE GEOLOCATION OF TWO UNKNOWN CO-CHANNEL EMITTERS BY A CLUSTER OF FORMATION-FLYING SATELLITES USING BOTH TDOA AND FDOA MEASUREMENTS

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ABSTRACT

We consider the problem of geolocating two unknown co-channel emitters by a cluster of formation-flying satellites using both time difference of arrival (TDOA) and frequency difference of arrival (FDOA) measurements. As the association between the TDOA/FDOA measurements obtained by each pair of satellites and the corresponding emitters is typically not known, the emitter-measurement association and the emitters' locations need to be jointly estimated. In this paper, we first formulate the joint estimation problem as a mixed integer nonlinear optimization problem. Then, we propose a semidefinite relaxation-based approach to tackle the problem and demonstrate its efficacy via simulations.

***Index Terms*—** Co-channel emitters geolocation, TDOA, FDOA, Emitter-measurement association, Semidefinite relaxation.

1. INTRODUCTION

The geolocation or localization of radiating emitters on the surface of the Earth by a cluster of formation-flying satellites has found many applications in practice. These applications arise in both military and civilian fields, such as reconnaissance, surveillance, navigation, and maritime search and rescue, etc. To perform the geolocation, time difference of arrival (TDOA) and frequency difference of arrival (FDOA) measurements are typically used. Although the problem of geolocation using TDOA/FDOA measurements has been extensively investigated in the literature, existing solution approaches mainly deal with non-co-channel sources [1–8]. However, due to the curvature of the Earth, signals from the emitters in different neighboring areas of the Earth would fall into the same frequency band used by different users or countries. Since the TDOA and FDOA measurements are often obtained by finding the correlation peaks through sliding cross-correlation of a period of the signals received at a pair of satellite receivers [9–14], it is very difficult to say one peak is associated with a given unknown emitter. This leads to the problem of data association in the geolocation of the multiple unknown co-channel emitters. In other words, the association between the TDOA/FDOA measurements and the corresponding emitters needs to be estimated as well. In recent years, there have been some works addressing the data association problem in time of arrival (TOA)-based geolocation; see, e.g., [15, 16]. However, the approaches proposed in those works cannot be easily extended to deal with TDOA/FDOA measurements. This calls for the development of new techniques, which is the main motivation of the current work.

In this paper, we consider the problem of geolocating two unknown co-channel emitters by a cluster of formation-flying satellites using both TDOA and FDOA measurements. The joint estimation of emitter-measurement association and the emitters' locations can be formulated as a mixed integer nonlinear optimization problem. Our contribution is twofold. First, we develop a semidefinite relaxation (SDR) [17] of this problem by performing a joint relaxation of the association and location variables. This allows us to achieve a higher accuracy in the solution. Then, by exploiting the problem structure, we show that the solution obtained by our proposed SDR can be further refined using a minimum weight perfect bipartite matching procedure [18] and a standard local serarch. Our simulation result demonstrates the efficacy of our proposed approach.

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2. PROBLEM FORMULATION

Consider the scenario where a cluster of formation-flying satellites with known satellite orbits collaborate to geolocate multiple static unknown emitters on the Earth, where these emitters fall into the same frequency channel. The following notations will be used in our formulation:

M : number of satellites in the cluster,

K : number of unknown emitters,

c : speed of light,

$\boldsymbol{s}_j$: j-th column of $\boldsymbol{s} \in \mathcal{R}^{3\times M}$, true location of the j-th satellite at given time corresponding to its true orbit,

$\dot{\boldsymbol{s}}_j$: j-th column of $\dot{\boldsymbol{s}} \in \mathcal{R}^{3\times M}$, true veolocity of the j-th satellite at given time,

$t_j^{(k)}$: signal propagation delay from the k-th unknown emitter to the j-th satellite,

$f_j^{(k)}$: the derivative of $t_j^{(k)}$ with respect to time t,

$\boldsymbol{x}_k \in \mathcal{R}^3$: location of the k-th unknown emitter to be estimated,

$\mathcal{I}$: index set $\mathcal{I} \triangleq \{1, 2, \cdots, M\}$, and

$\mathcal{K}$: index set $\mathcal{K} \triangleq \{1, 2, \cdots, K\}$.

By definition, we have

$$t_i^{(k)} = \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|, \;\; i \in \mathcal{I}, \;\; k \in \mathcal{K}, \tag{1}$$

$$f_i^{(k)} = \frac{dt_i^{(k)}}{dt} = \frac{1}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|}(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i. \tag{2}$$

Due to the line-of-sight electromagnetic wave propagation, the pairwise TDOA and FDOA measurements for the k-th unknown emitter can be expressed as

$$\begin{aligned}
\tau_{ij}^{(k)} &= t_i^{(k)} - t_j^{(k)} + n_{ij}^{(k)} \\
&= \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\| - \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_j\right\| + n_{ij}^{(k)}, \qquad (3)\\
\nu_{ij}^{(k)} &= f_i^{(k)} - f_j^{(k)} + v_{ij}^{(k)} \\
&= \frac{(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|} - \frac{(\boldsymbol{s}_j - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_j}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_j\right\|} + v_{ij}^{(k)}, \qquad (4)
\end{aligned}$$

respectively, where $i,j \in \mathcal{I}$, $i > j$, $k \in \mathcal{K}$, and $n_{ij}^{(k)}$ and $v_{ij}^{(k)}$ $(i,j \in \mathcal{I}, i > j)$ are independent Gaussian random variables with mean zero and variances σ_T^2 and σ_F^2, respectively.

Since the unknown emitters are assumed to share the same frequency band, the associations between them and the TDOA and FDOA measurements obtained from the pair (i,j) of satellites are not known. In other words, given an element from the TDOA measurement set $\{\tau_{ij}^{(k)} : k \in \mathcal{K}\}$ (resp. FDOA measurement set $\{\nu_{ij}^{(k)} : k \in \mathcal{K}\}$), one cannot determine which emitter it corresponds to. We are thus motivated to consider the following maximum likelihood formulation for estimating the locations of the unknown emitters using joint TDOA and FDOA measurements:

$$\begin{aligned}
\min_{\substack{\boldsymbol{x}_k, \boldsymbol{t}_i, \boldsymbol{f}_i, \boldsymbol{P}^{(ij)}, \boldsymbol{Q}^{(ij)} \\ i,j \in \mathcal{I},\, i>j,\, k \in \mathcal{K}}} \quad & \frac{1}{\sigma_T^2} \sum_{\substack{i,j \in \mathcal{I} \\ i>j}} \|\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{P}^{(ij)} \boldsymbol{\tau}_{ij}\|^2 \\
& + \frac{1}{\sigma_F^2} \sum_{\substack{i,j \in \mathcal{I} \\ i>j}} \|\boldsymbol{f}_i - \boldsymbol{f}_j - \boldsymbol{Q}^{(ij)} \boldsymbol{\nu}_{ij}\|^2 \\
\text{s.t.} \quad & t_i^{(k)} = \frac{1}{c}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|, \, i \in \mathcal{I}, \, k \in \mathcal{K}, \\
& f_i^{(k)} = \frac{(\boldsymbol{s}_i - \boldsymbol{x}_k)^T \dot{\boldsymbol{s}}_i}{c\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|}, \, i \in \mathcal{I}, \, k \in \mathcal{K}, \\
& \boldsymbol{t}_i = \left[t_i^{(1)}, \ldots, t_i^{(K)}\right]^T, \, i \in \mathcal{I}, \\
& \boldsymbol{f}_i = \left[f_i^{(1)}, \ldots, f_i^{(K)}\right]^T, \, i \in \mathcal{I}, \\
& \boldsymbol{P}^{(ij)}, \, \boldsymbol{Q}^{(ij)} \in \Pi_K, \, i,j \in \mathcal{I}, \, i > j.
\end{aligned} \tag{5}$$

Here, $\boldsymbol{\tau}_{ij} = [\tau_{ij}^{(1)}, \ldots, \tau_{ij}^{(K)}]^T$, $\boldsymbol{\nu}_{ij} = [\nu_{ij}^{(1)}, \ldots, \nu_{ij}^{(K)}]^T$, and Π_K is the set of $K \times K$ permutation matrices.

Observe that Problem (5) contains both integer (i.e., $\boldsymbol{P}^{(ij)}$ and $\boldsymbol{Q}^{(ij)}$) and continuous (i.e., $\boldsymbol{x}_k$, $\boldsymbol{t}_i$, and $\boldsymbol{f}_i$) decision variables. It is thus a mixed integer optimization problem, which is non-convex and difficult to solve in general. It should be noted, however, that if the variables $\{(\boldsymbol{x}_k, \boldsymbol{t}_i, \boldsymbol{f}_i) : i \in \mathcal{I}, k \in \mathcal{K}\}$ are fixed, then the optimal permutation matrices $\{(\boldsymbol{P}^{(ij)}, \boldsymbol{Q}^{(ij)}) : i,j \in \mathcal{I}, i > j\}$ can be found in polynomial time by solving $2\bar{M}$ minimum weight perfect bipartite matching problems, where $\bar{M} = M(M-1)/2$ [18]. Indeed, since the objective function of Problem (5) is separable in $\{(\boldsymbol{P}^{(ij)}, \boldsymbol{Q}^{(ij)}) : i,j \in \mathcal{I}, i > j\}$, it suffices to find $\boldsymbol{P}^{(ij)}$ (resp. $\boldsymbol{Q}^{(ij)}$) to minimize $\|\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{P}^{(ij)} \boldsymbol{\tau}_{ij}\|^2$ (resp. $\|\boldsymbol{f}_i - \boldsymbol{f}_j - \boldsymbol{Q}^{(ij)} \boldsymbol{\nu}_{ij}\|^2$), where $i,j \in \mathcal{I}$ and $i > j$. The former corresponds to finding a minimum weight perfect matching in the weighted complete bipartite graph $G^{(ij)} = (V, E, w^{(ij)})$ with bipartition $V = V_1 \cup V_2$, where $V_1 = V_2 = \mathcal{K}$ and

$$w^{(ij)}(k,k') = \left(t_i^{(k)} - t_j^{(k)} - \tau_{ij}^{(k)}\right)^2, \; k, k' \in \mathcal{K}.$$

The latter can be handled in a similar fashion. The above observation suggests that the key in tackling Problem (5) lies in finding a good estimate of $\{(\boldsymbol{x}_k, \boldsymbol{t}_i, \boldsymbol{f}_i) : i \in \mathcal{I}, k \in \mathcal{K}\}$. We propose to achieve this by employing the SDR technique [17] to relax the discrete and continuous variables *jointly*. As a result, we obtain an SDR of Problem (5). It is worth comparing our approach with that in [16], which *separately* relaxes the discrete and continuous variables and approximates the TOA measurements via Taylor expansion. The latter approach ignores the interactions between the discrete and continuous variables, which will significantly weaken the formulation. Moreover, since it uses Taylor expansion as an approximation tool, the resulting formulation is not necessarily a relaxation of the original problem. By contrast, our approach will always lead to a convex relaxation of the original problem, which makes it possible to evaluate the accuracy of the solution obtained from the relaxation.

3. SDR-BASED ALGORITHM FOR GEOLOCATING TWO UNKNOWN CO-CHANNEL EMITTERS

3.1. Deriving the Basic SDR

To fix ideas and avoid complications in notations, let us focus on the setting where there are only two unknown co-channel emitters (i.e., $K = 2$). The case of multiple unknown co-channel emitters will be discussed in detail in the full version of this paper. Define

$$\begin{aligned}
\tilde{\boldsymbol{t}} &= \left[(\boldsymbol{t}_1 - \boldsymbol{t}_2)^T, \ldots, (\boldsymbol{t}_1 - \boldsymbol{t}_M)^T, \ldots, (\boldsymbol{t}_{M-1} - \boldsymbol{t}_M)^T\right]^T \\
&= \bar{\boldsymbol{G}}\boldsymbol{t}, \\
\tilde{\boldsymbol{f}} &= \left[(\boldsymbol{f}_1 - \boldsymbol{f}_2)^T, \ldots, (\boldsymbol{f}_1 - \boldsymbol{f}_M)^T, \ldots, (\boldsymbol{f}_{M-1} - \boldsymbol{f}_M)^T\right]^T \\
&= \bar{\boldsymbol{G}}\boldsymbol{f}, \\
\boldsymbol{t} &= \left[\boldsymbol{t}_1^T, \ldots, \boldsymbol{t}_M^T\right]^T, \;\; \boldsymbol{f} = \left[\boldsymbol{f}_1^T, \ldots, \boldsymbol{f}_M^T\right]^T, \\
\boldsymbol{P} &= \text{blkdiag}\left(\boldsymbol{P}^{(12)}, \ldots, \boldsymbol{P}^{(1M)}, \ldots, \boldsymbol{P}^{(M-1,M)}\right), \\
\boldsymbol{Q} &= \text{blkdiag}\left(\boldsymbol{Q}^{(12)}, \ldots, \boldsymbol{Q}^{(1M)}, \ldots, \boldsymbol{Q}^{(M-1,M)}\right), \\
\boldsymbol{\tau} &= \left[\boldsymbol{\tau}_{(12)}^T, \ldots, \boldsymbol{\tau}_{(1M)}^T, \ldots, \boldsymbol{\tau}_{(M-1,M)}^T\right]^T, \\
\boldsymbol{\nu} &= \left[\boldsymbol{\nu}_{(12)}^T, \ldots, \boldsymbol{\nu}_{(1M)}^T, \ldots, \boldsymbol{\nu}_{(M-1,M)}^T\right]^T,
\end{aligned}$$

where $\bar{\boldsymbol{G}} = \boldsymbol{G} \otimes \boldsymbol{I}_K$ is a $KM(M-1)/2 \times KM$ matrix,[1] $\boldsymbol{I}_K$ denotes the $K \times K$ identity matrix, and $\boldsymbol{G}$ is defined in [19]. Then, the objective function of Problem (5) can be expressed as

$$\begin{aligned}
\theta &= \frac{1}{\sigma_T^2}\|\tilde{\boldsymbol{t}} - \boldsymbol{P}\boldsymbol{\tau}\|^2 + \frac{1}{\sigma_F^2}\|\tilde{\boldsymbol{f}} - \boldsymbol{Q}\boldsymbol{\nu}\|^2 \\
&= \frac{1}{\sigma_T^2}\boldsymbol{t}^T \bar{\boldsymbol{G}}^T \bar{\boldsymbol{G}} \boldsymbol{t} + \frac{1}{\sigma_T^2}\|\boldsymbol{P}\boldsymbol{\tau}\|^2 - \frac{2}{\sigma_T^2}(\boldsymbol{P}\boldsymbol{\tau})^T \bar{\boldsymbol{G}}\boldsymbol{t} \\
&\quad + \frac{1}{\sigma_F^2}\boldsymbol{f}^T \bar{\boldsymbol{G}}^T \bar{\boldsymbol{G}} \boldsymbol{f} + \frac{1}{\sigma_F^2}\|\boldsymbol{Q}\boldsymbol{\nu}\|^2 - \frac{2}{\sigma_F^2}(\boldsymbol{Q}\boldsymbol{\nu})^T \bar{\boldsymbol{G}}\boldsymbol{f},
\end{aligned} \tag{6}$$

[1] Here, $\otimes$ denotes the Kronecker product.

which is quadratic in the decision variables.

Recall that each element of a permutation matrix takes values in $\{0,1\}$, and that each row and each column sums to one. Thus, for the case where $K=2$, we can express $\boldsymbol{P}^{(ij)}$ and $\boldsymbol{Q}^{(ij)}$ as

$$\boldsymbol{P}^{(ij)} = \begin{bmatrix} 1-p^{(ij)} & p^{(ij)} \\ p^{(ij)} & 1-p^{(ij)} \end{bmatrix}, \tag{7}$$

$$\boldsymbol{Q}^{(ij)} = \begin{bmatrix} 1-q^{(ij)} & q^{(ij)} \\ q^{(ij)} & 1-q^{(ij)} \end{bmatrix}, \tag{8}$$

where $p^{(ij)}, q^{(ij)} \in \{0,1\}$. In particular, there exist matrices $\boldsymbol{B}_p$ and $\boldsymbol{B}_q$ such that

$$\boldsymbol{P\tau} = \boldsymbol{\tau} + \boldsymbol{B}_p\boldsymbol{p}, \;\; \boldsymbol{Q\nu} = \boldsymbol{\nu} + \boldsymbol{B}_q\boldsymbol{q}, \tag{9}$$

where

$$\begin{aligned} \boldsymbol{p} &= \left[p^{(12)},\dots,p^{(1M)},\dots,p^{(m-1,m)}\right]^T, \\ \boldsymbol{q} &= \left[q^{(12)},\dots,q^{(1M)},\dots,q^{(m-1,m)}\right]^T. \end{aligned}$$

It follows that

$$\|\boldsymbol{P\tau}\|^2 = \|\boldsymbol{\tau}\|^2 + 2\boldsymbol{\tau}^T\boldsymbol{B}_p\boldsymbol{p} + \boldsymbol{p}^T\boldsymbol{B}_p^T\boldsymbol{B}_p\boldsymbol{p}, \tag{10}$$

$$\|\boldsymbol{Q\nu}\|^2 = \|\boldsymbol{\nu}\|^2 + 2\boldsymbol{\nu}^T\boldsymbol{B}_q\boldsymbol{q} + \boldsymbol{q}^T\boldsymbol{B}_q^T\boldsymbol{B}_q\boldsymbol{q}. \tag{11}$$

Now, we are ready to derive the SDR of Problem (5). First, define the variable vectors

$$\boldsymbol{X} = [\boldsymbol{x}_1,\dots,\boldsymbol{x}_K], \;\; \boldsymbol{y} = \left[\boldsymbol{t}^T, \boldsymbol{f}^T, \boldsymbol{p}^T, \boldsymbol{q}^T\right]^T,$$

the cross terms

$$\begin{aligned} &\boldsymbol{T} = \boldsymbol{tt}^T, \; \boldsymbol{U} = \boldsymbol{tf}^T, \; \boldsymbol{T}_p = \boldsymbol{tp}^T, \; \boldsymbol{T}_q = \boldsymbol{tq}^T, \\ &\boldsymbol{F} = \boldsymbol{ff}^T, \; \boldsymbol{F}_p = \boldsymbol{fp}^T, \; \boldsymbol{F}_q = \boldsymbol{fq}^T, \\ &\tilde{\boldsymbol{P}} = \boldsymbol{pp}^T, \; \tilde{\boldsymbol{Q}} = \boldsymbol{qq}^T, \; \tilde{\boldsymbol{P}}_q = \boldsymbol{pq}^T, \end{aligned} \tag{12}$$

and the corresponding Gram matrices

$$\boldsymbol{Z} = \boldsymbol{X}^T\boldsymbol{X}, \tag{13}$$

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{T}_f & \boldsymbol{T}_p & \boldsymbol{T}_q \\ \boldsymbol{T}_f^T & \boldsymbol{F} & \boldsymbol{F}_p & \boldsymbol{F}_q \\ \boldsymbol{T}_p^T & \boldsymbol{F}_p^T & \tilde{\boldsymbol{P}} & \tilde{\boldsymbol{P}}_q \\ \boldsymbol{T}_q^T & \boldsymbol{F}_q^T & \tilde{\boldsymbol{P}}_q^T & \tilde{\boldsymbol{Q}} \end{bmatrix}, \;\; \bar{\boldsymbol{Y}} = \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{y} \\ \boldsymbol{y}^T & 1 \end{bmatrix}. \tag{14}$$

Then, it is elementary though tedious to verify that the objective function (6) can be rewritten as $\theta = \mathrm{tr}(\bar{\boldsymbol{E}}\bar{\boldsymbol{Y}})$, where

$$\bar{\boldsymbol{E}} = \begin{bmatrix} \bar{\boldsymbol{A}}_1 & \boldsymbol{0} & \boldsymbol{B}_t & \boldsymbol{0} & \bar{\boldsymbol{b}}_1 \\ \boldsymbol{0} & \bar{\boldsymbol{A}}_2 & \boldsymbol{0} & \boldsymbol{B}_f & \bar{\boldsymbol{b}}_2 \\ \boldsymbol{B}_t^T & \boldsymbol{0} & \bar{\boldsymbol{B}}_p & \boldsymbol{0} & \bar{\boldsymbol{d}}_1 \\ \boldsymbol{0} & \boldsymbol{B}_f^T & \boldsymbol{0} & \bar{\boldsymbol{B}}_q & \bar{\boldsymbol{d}}_2 \\ \bar{\boldsymbol{b}}_1^T & \bar{\boldsymbol{b}}_2^T & \bar{\boldsymbol{d}}_1^T & \bar{\boldsymbol{d}}_2^T & \bar{C}_1 + \bar{C}_2 \end{bmatrix}$$

and

$$\begin{aligned} &\bar{\boldsymbol{A}}_1 = \bar{\boldsymbol{G}}^T\bar{\boldsymbol{G}}/\sigma_T^2, \; \bar{\boldsymbol{b}}_1 = -\bar{\boldsymbol{G}}^T\boldsymbol{\tau}/\sigma_T^2, \; \bar{\boldsymbol{d}}_1 = \boldsymbol{B}_p^T\boldsymbol{\tau}/\sigma_T^2, \\ &\bar{C}_1 = \|\boldsymbol{\tau}\|^2/\sigma_T^2, \; \boldsymbol{B}_t = \bar{\boldsymbol{G}}^T\boldsymbol{B}_p/\sigma_T^2, \; \bar{\boldsymbol{B}}_p = \boldsymbol{B}_p^T\boldsymbol{B}_p/\sigma_T^2, \\ &\bar{\boldsymbol{A}}_2 = \bar{\boldsymbol{G}}^T\bar{\boldsymbol{G}}/\sigma_F^2, \; \bar{\boldsymbol{b}}_2 = -\bar{\boldsymbol{G}}^T\boldsymbol{\nu}/\sigma_F^2, \; \bar{\boldsymbol{d}}_2 = \boldsymbol{B}_q^T\boldsymbol{\nu}/\sigma_F^2, \\ &\bar{C}_2 = \|\boldsymbol{\nu}\|^2/\sigma_F^2, \; \boldsymbol{B}_f = \bar{\boldsymbol{G}}^T\boldsymbol{B}_q/\sigma_F^2, \; \bar{\boldsymbol{B}}_q = \boldsymbol{B}_q^T\boldsymbol{B}_q/\sigma_F^2. \end{aligned}$$

Next, we need to impose suitable constraints on the variable $\boldsymbol{y}$ and the cross terms in (12). Denote by $\boldsymbol{T}^{(kl)}$, $\boldsymbol{F}^{(kl)}$, and $\boldsymbol{U}^{(kl)}$ the (k,l)th block of $\boldsymbol{T}$, $\boldsymbol{F}$, and $\boldsymbol{U}$, respectively, where $k,l \in \mathcal{K}$. Observe that

$$\begin{aligned} T_{ii}^{(kk)} &= (t_i^{(k)})^2 = \frac{1}{c^2}\left\|\boldsymbol{x}_k - \boldsymbol{s}_i\right\|^2 \\ &= \frac{1}{c^2}\begin{bmatrix} \boldsymbol{s}_i \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{x}_k \\ \boldsymbol{x}_k^T & Z_{kk} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_i \\ -1 \end{bmatrix} \end{aligned} \tag{15}$$

and

$$U_{ii}^{(kk)} = f_i^{(k)} t_i^{(k)} = \frac{1}{c^2}(\boldsymbol{s}_i - \boldsymbol{x}_k)^T\dot{\boldsymbol{s}}_i. \tag{16}$$

Moreover, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} T_{ij}^{(kl)} &= t_i^{(k)} t_j^{(l)} = \frac{1}{c^2}\|\boldsymbol{x}_k - \boldsymbol{s}_i\|\|\boldsymbol{x}_l - \boldsymbol{s}_j\| \\ &\geq \frac{1}{c^2}\left|(\boldsymbol{x}_k - \boldsymbol{s}_i)^T(\boldsymbol{x}_l - \boldsymbol{s}_j)\right| \\ &= \frac{1}{c^2}\left|\begin{bmatrix} \boldsymbol{s}_i \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{x}_l \\ \boldsymbol{x}_k^T & Z_{kl} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_j \\ -1 \end{bmatrix}\right| \end{aligned} \tag{17}$$

and

$$\begin{aligned} &T_{ij}^{(kl)} + F_{ij}^{(kl)} \pm U_{ij}^{(kl)} \pm U_{ji}^{(lk)} \\ &\leq \left(T_{ii}^{(kk)} + F_{ii}^{(kk)} \pm 2U_{ii}^{(kk)} + T_{jj}^{(ll)} + F_{jj}^{(ll)} \pm 2U_{jj}^{(ll)}\right)/2 \\ &= \frac{1}{2c^2}\Bigg(\begin{bmatrix} \boldsymbol{s}_i \pm \dot{\boldsymbol{s}}_i \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{x}_k \\ \boldsymbol{x}_k^T & Z_{kk} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_i \pm \dot{\boldsymbol{s}}_i \\ -1 \end{bmatrix} \\ &\qquad + \begin{bmatrix} \boldsymbol{s}_j \pm \dot{\boldsymbol{s}}_j \\ -1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{x}_l \\ \boldsymbol{x}_l^T & Z_{ll} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_j \pm \dot{\boldsymbol{s}}_j \\ -1 \end{bmatrix} \\ &\qquad - \|\dot{\boldsymbol{s}}_i\|^2 - \|\dot{\boldsymbol{s}}_j\|^2 + c^2 F_{ii}^{(kk)} + c^2 F_{jj}^{(ll)}\Bigg). \end{aligned} \tag{18}$$

Lastly, we have the following bounds:

$$|f_i^{(k)}| \leq \|\dot{\boldsymbol{s}}_i\|/c, \tag{19}$$

$$U_{ij}^{(kl)} \leq \left(T_{ii}^{(kk)} + F_{jj}^{(ll)}\right)/2, \tag{20}$$

$$F_{ii}^{(kk)} \leq \|\dot{\boldsymbol{s}}_i\|^2/c^2, \tag{21}$$

$$\left|F_{ij}^{(kl)}\right| \leq \|\dot{\boldsymbol{s}}_i\|\|\dot{\boldsymbol{s}}_j\|/c^2. \tag{22}$$

Now, let us consider $\tilde{\boldsymbol{P}}$, $\tilde{\boldsymbol{Q}}$, and $\tilde{\boldsymbol{P}}_q$, which are related to the permutation matrices $\{(\boldsymbol{P}^{(ij)}, \boldsymbol{Q}^{(ij)}) : i,j \in \mathcal{I}, \, i > j\}$. It is evident that the entries of $\tilde{\boldsymbol{P}}$, $\tilde{\boldsymbol{Q}}$, and $\tilde{\boldsymbol{P}}_q$ are component-wise non-negative; i.e.,

$$\tilde{\boldsymbol{P}} \geq \boldsymbol{0}, \;\; \tilde{\boldsymbol{Q}} \geq \boldsymbol{0}, \;\; \tilde{\boldsymbol{P}}_q \geq \boldsymbol{0}. \tag{23}$$

Moreover, since the entries of $\boldsymbol{p}$ and $\boldsymbol{q}$ take values in $\{0,1\}$, we have

$$\boldsymbol{0} \leq \boldsymbol{p} \leq \boldsymbol{1}, \;\; \boldsymbol{0} \leq \boldsymbol{q} \leq \boldsymbol{1} \tag{24}$$

and

$$\mathrm{diag}(\tilde{\boldsymbol{P}}) = \boldsymbol{p}, \;\; \mathrm{diag}(\tilde{\boldsymbol{Q}}) = \boldsymbol{q}, \tag{25}$$

where $\boldsymbol{1}$ is the vector of all ones and $\mathrm{diag}(\boldsymbol{V})$ is the vector formed by the diagonal entries of the square matrix $\boldsymbol{V}$.

With the above preparations, we can relax the constraints in (13) and (14) via

$$\boldsymbol{Z} \succeq \boldsymbol{X}^T\boldsymbol{X} \iff \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{Z} \end{bmatrix} \succeq \boldsymbol{0}, \quad (26)$$

$$\boldsymbol{Y} \succeq \boldsymbol{y}\boldsymbol{y}^T \iff \bar{\boldsymbol{Y}} \succeq \boldsymbol{0}, \quad (27)$$

and

$$\boldsymbol{T} + \boldsymbol{F} \pm \boldsymbol{U} \pm \boldsymbol{U}^T \succeq (\boldsymbol{t} \pm \boldsymbol{f})(\boldsymbol{t} \pm \boldsymbol{f})^T$$
$$\iff \bar{\boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{T} + \boldsymbol{F} \pm \boldsymbol{U} \pm \boldsymbol{U}^T & \boldsymbol{t} \pm \boldsymbol{f} \\ (\boldsymbol{t} \pm \boldsymbol{f})^T & 1 \end{bmatrix} \succeq \boldsymbol{0}. \quad (28)$$

By putting all the pieces together, we finally arrive at the following SDR of Problem (5) when there are two unknown emitters:

$$\begin{aligned} \min \quad & \operatorname{tr}(\bar{\boldsymbol{E}}\bar{\boldsymbol{Y}}) + \delta_1 \sum_{\substack{i,j\in\mathcal{I} \\ k,l\in\mathcal{K}}} T_{ij}^{(kl)} + \delta_2 \sum_{\substack{i,j\in\mathcal{I} \\ k,l\in\mathcal{K}}} \left|F_{ij}^{(kl)}\right| \\ \text{s.t.} \quad & \text{(15)–(28) satisfied.} \end{aligned} \quad (29)$$

Here, $\delta_1, \delta_2 \geq 0$ are penalty parameters used to induce a good solution to the original problem (5).

3.2. Further Refinements

Since the SDR (29) contains many variables, in order to produce a good solution to the original problem (5), one could include more valid constraints in the relaxation. For instance, observe that the form of the objective function (6) and the relationships (9)–(11) suggest that once $\boldsymbol{t}$ and $\boldsymbol{f}$ are fixed, the relaxed variables $\boldsymbol{p}$ and $\boldsymbol{q}$ should satisfy the following optimality conditions:

$$\boldsymbol{B}_p^T\left(\boldsymbol{\tau} + \boldsymbol{B}_p\boldsymbol{p} - \bar{\boldsymbol{G}}\boldsymbol{t}\right) = \boldsymbol{0},$$
$$\boldsymbol{B}_q^T\left(\boldsymbol{\tau} + \boldsymbol{B}_q\boldsymbol{p} - \bar{\boldsymbol{G}}\boldsymbol{f}\right) = \boldsymbol{0}.$$

One can then incorporate these linear constraints into the SDR (29) to tighten the relaxation.

3.3. Completing the Description of the Proposed Algorithm

After solving the SDR (29), one can use the minimum weight perfect bipartite matching procedure mentioned in Section 2 and local search to refine the solution. This gives rise to the following algorithm for tackling the problem of geolocating two unknown emitters using joint TDOA and FDOA measurements:

Algorithm:

Step 1: Choose a pair (δ_1, δ_2) $(\delta_i \in [10^{-6}, 10^{-1}])$. *Use solver SeDuMi or SDPT3 in CVX [20] to solve the SDR (29) and obtain the location estimates* $\hat{\boldsymbol{X}}$ *of the two unknown emitters and the corresponding* $\boldsymbol{p}$ *and* $\boldsymbol{q}$.

Step 2: Apply any local optimization routine (e.g., Newton-type methods) to the objective function of Problem (5) *using* $(\hat{\boldsymbol{X}}, \boldsymbol{p}, \boldsymbol{q})$ *as the initial point.*

Step 3: Based on the result in Step 2, perform a minimum weight perfect bipartite matching as outlined in Section 2 to obtain permutation matrices $\{(\hat{\boldsymbol{P}}^{(ij)}, \hat{\boldsymbol{Q}}^{(ij)}) : i, j \in \mathcal{I},\ i > j\}$.

Step 4: Fix the permutation matrices obtained in Step 3. Then, Problem (5) *becomes a standard geolocation problem using joint TDOA and FDOA measurements, which can be solved by standard SDR techniques. The location estimates thus obtained can then be further refined by local search.*

4. NUMERICAL RESULTS

In the simulation, the geolocation is realized by three formation-flying satellites [21], whose true locations $(\times 10^6\ m)$ and velocities $(\times 10^3\ m/s)$ are obtained according to the routine in the Satellite Tool Kit (STK):

$$\boldsymbol{S} = \begin{bmatrix} -2.59693665 & -2.70482425 & -2.49791532 \\ 3.23460820 & 3.19406424 & 3.29006186 \\ 5.60250575 & 5.57715056 & 5.60777468 \end{bmatrix},$$

$$\dot{\boldsymbol{S}} = \begin{bmatrix} -7.017428 & -6.968662 & -7.061741 \\ -1.408503 & -1.457512 & -1.372917 \\ -2.439598 & -2.544959 & -2.340081 \end{bmatrix}.$$

The locations of the two unknown emitters are randomly chosen on the surface of the Earth, such as

$$\boldsymbol{x}_1 = [-2.53242580, 3.19230238, 4.91571459]^T \times 10^6 m,$$
$$\boldsymbol{x}_2 = [-2.53756142, 3.01653500, 5.02035731]^T \times 10^6 m,$$

which are within the coverage of a satellite formation (here we use S-type formation).

In the simulation, we generate measurement noise according to the truncated Gaussian distribution. Similar to that in [19], let υ be a random variable with zero mean Gaussian distribution truncated in the interval $|\upsilon| \leq \alpha\sigma$, its pdf is given by

$$p(\upsilon) = \begin{cases} \frac{b}{\sqrt{2\pi}\sigma}\left(e^{-\upsilon^2/2\sigma^2} - e^{-(\alpha\sigma)^2/2\sigma^2}\right) & \text{if } |\upsilon| \leq \alpha\sigma, \\ 0 & \text{if } |\upsilon| > \alpha\sigma, \end{cases}$$

where b is a normalizing constant, σ^2 is the variance, and α controls the size of the interval in which the variable υ lies. Here, α is set to 4. 50 Monte Carlo runs were performed using the above procedure for each given TDOA and FDOA measurement noise standard deviation, which are denoted by σ_t and σ_f, respectively. The RMSE of the geolocation error of the two unknown emitter versus the TDOA and FDOA measurement noise are plotted in Fig. 1. In Fig. 1, we assume that $\sigma_f = 0.1\sigma_t$ for narrowband signals since accurate FDOA measurements can be obtained for narrowband signals, while TDOA measurements can be accurately obtained for wideband signals [4]. It can be seen that the RMSE of each emitter geolocation can approximately approach the corresponding Cramer-Rao lower bound (CRLB) for all settings of σ_t, which demonstrates the efficacy of our proposed SDR-based approach.

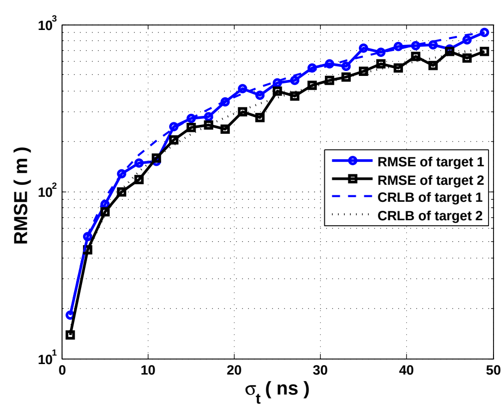

Fig. 1. RMSE Performance of the Two Co-Channel Geolocation

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