A LATTICE ALGORITHM FOR OPTIMAL PHASE UNWRAPPING IN NOISE

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ABSTRACT

Use of the phase of a signal to measure distance carries an inherent ambiguity. The problem is typically addressed by the use of several different frequencies and the Chinese Remainder Theorem or lattice methods, but these methods result in computational complexity issues. The difficulties are increased by the presence of noise. This paper presents a lattice-based algorithm to resolve phase ambiguity more efficiently and under more relaxed constraints than existing approaches. Simulations are presented to illustrate the performance of the proposed algorithm and compared with existing methods.

Index Terms— Lattice theory, Wrapped phase, Distance ambiguity, Chinese reminder theory, Lattice theory

1. INTRODUCTION

The (phase) ambiguity problem can arise in many engineering fields, such as radar system [1], sensor localization [2] and estimation [3]. In this problem, a sensor measurement, typically, the phase of a signal is only the remainder of ground true value divided by a known constant modulus plus additive noise. Such an ambiguity problem can be efficiently solved via either a closed-form CRT method [4] or a lattice [5] by using multiple measurements under different moduli, provided that the set of used moduli are co-prime integers. Without the co-prime constraint, an optimal solution may be achieved via the searching based algorithm [3]. But the implementation demands very high computational complexity because of the involvement of exhaustive searches over parameter space. In this paper, we present a lattice based algorithm to solve this general ambiguity problem without the co-prime constraint. Based on the work in [7], a modified iteration for closest point searching is derived so that the proposed algorithm is computationally more efficient than existing approaches.

The rest of the paper is organised as follows. In Section 2, signal phase ambiguity problem is described in the framework of Lattice Theory. We propose the lattice-based algorithm in Section 3. An adaptive searching algorithm for closest point searching in a lattice is presented in Section 4 and it is followed by a description of the computer simulations to highlight the performance of the proposed methods. Finally, we conclude the work in Section 6.

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2. PROBLEM FORMULATION IN LATTICE THEORY

Denoted by $r \in \mathbb{R}$, the distance being measured by the phase of signal at multiple wavelengths $\lambda_1, \cdots, \lambda_m$, the measurement corresponding to i-th wavelength can be written as

$$y_{i,0} = r \mod \lambda_i \Longleftrightarrow r = n_i\lambda_i + y_{i,0} \tag{1}$$

where $y_{i,0}$ is the i-th measurement in the absence of noise, and n_i is an unknown positive integer signifying the number of wavelengths involved in the transmission. To guarantee that the equations can be solved to provide a unique distance, we assume that $\lambda_1 > ... > \lambda_m$ and $r \leq \mathrm{LCM}(\lambda_1, ..., \lambda_m)$, where $\mathrm{LCM}(\cdot)$ is the least common multiple function and m is the number of used wavelengths. In this work, it is assumed that each of the set of phase measurements is corrupted with an additive noise of distribution $\omega_i \overset{iid}{\sim} \mathcal{N}(0, \delta^2\lambda_i^2)$ [5], where $\delta > 0$ is a small constant. (1) can be written as

$$c_i = n_i\lambda_i + y_{i,0} + \omega_i, \quad i = 1, \cdots, m. \tag{2}$$

where c_i are the possible values of r in the presence of noise. The problem devolves to one of jointly estimating the set of integers $\{n_1, \cdots, n_m\}$ and the underlying unique distance r.

Writing in vector forms $\boldsymbol{\omega} \sim \mathcal{N}(0, \mathbf{C})$, where $\boldsymbol{\omega} = [\omega_1, \cdots, \omega_m]$, $\mathbf{C} = \mathrm{diag}\{\delta^2\lambda_1^2, \cdots, \delta^2\lambda_m^2\}$, and $\mathbf{y} = [y_1, \cdots, y_m]$, $\mathbf{n} = [n_1, \cdots, n_m]$, $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_m]$, the likelihood $p(\mathbf{y}|\mathbf{n}, r)$ is proportional to

$$\exp\left\{-\frac{1}{2}(r\mathbf{1} - \mathbf{n}\cdot\boldsymbol{\lambda} - \mathbf{y})\mathbf{C}^{-1}(r\mathbf{1} - \mathbf{n}\cdot\boldsymbol{\lambda} - \mathbf{y})^T\right\}$$

The maximum likelihood solution is given by

$$(\hat{r}, \hat{\mathbf{N}}) = \arg \min_{(r,\mathbf{n})\in\mathbb{R}\times\mathbb{Z}^m} \sum_{i=1}^{m} \left(\frac{r}{\lambda_i} - n - \frac{y_i}{\lambda_i} \right)^2$$
$$= \arg \min_{(r,\mathbf{n})\in\mathbb{R}\times\mathbb{Z}^m} \|r\bar{\boldsymbol{\lambda}} - \mathbf{n} - \bar{\mathbf{y}}\| \quad (3)$$

where $\mathbf{1} = [1, \cdots, 1]$, $\bar{\boldsymbol{\lambda}} = [1/\lambda_1, \cdots, 1/\lambda_m]$ and $\bar{\mathbf{y}} = [y_1/\lambda_1, \cdots, y_m/\lambda_m]$.

This problem can be efficiently addressed using lattice methods. In lattice theory, a real lattice of dimension m is defined as the set of point in $\mathbb{R}^m$ satisfying

$$\boldsymbol{\Omega}(\mathbf{G}) \triangleq \{\mathbf{uG} : \mathbf{u} \in \mathbb{Z}^m\} \quad (4)$$

where $\mathbf{G}$ of dimension $m \times n$ is called generator matrix and $\mathbf{u}$ is a m dimension integer vector. Two fundamental problems involved in a lattice based are 1) find the nearest lattice point; and 2) basis reduction.

For any given point $\mathbf{x} \in \mathbb{R}^m$, the closest lattice point problem is to find a vector $\mathbf{c} \in \boldsymbol{\Omega}(\mathbf{G})$ such that $\|\mathbf{x} - \mathbf{c}\| \leq \|\mathbf{x} - \mathbf{c}'\|, \quad \forall \mathbf{c}' \in \boldsymbol{\Omega}(\mathbf{G})$, where $\|\cdot\|$ is the Euclidean norm.

The basis reduction problem is to find an alternative basis without changing the original lattice structure where the alternative basis is a set of shortest and "as orthogonal as possible" vectors. A standard approach to reduction is to take a unimodular matrix $\mathbf{U}$ ($|\det \mathbf{U}| = 1$) with integer entries and replace $\mathbf{G}$ by $\mathbf{T} = \mathbf{UG}$, i.e. these two basis generate same lattice space.

Another crucial conception in lattice theory is that of a *Voronoi cell*. The Voronoi cell of lattice $\boldsymbol{\Omega}(\mathbf{G})$, denoted by $\mathbf{V}(\mathbf{G})$, can be defined as the intersection of half spaces [7],

$$H_v = \left\{ \mathbf{x} \in \mathbb{R}^m | \mathbf{x} \cdot \mathbf{u} \leq \frac{1}{2}\mathbf{u} \cdot \mathbf{u},\ \mathbf{u} \in \boldsymbol{\Omega}(\mathbf{G}) \setminus \mathbf{0} \right\}$$

where $(\cdot)$ is dot product. The minimal set of lattice vectors such that $\mathbf{V}(\mathbf{G}) = \cap H_v$ is called the set of Voronoi relevant vectors, denoted by $\mathrm{Rel}(\mathbf{G})$. A method to find the Voronoi relevant vectors is described in [8].

The solution of the underlying problem (3) using lattice method involves two steps. First, we find the unknown integer set $\mathbf{N}$ based on measurements $\mathbf{y}$, then estimate r using MLE. Given $\mathbf{n}$, $\hat{r}$ is obtained by $\hat{r} = (\mathbf{n} + \bar{\mathbf{y}})\bar{\boldsymbol{\lambda}}^+$, where $\bar{\boldsymbol{\lambda}}^+$ is the Moore-Penrose pseudo inverse of $\bar{\boldsymbol{\lambda}}$: $\bar{\boldsymbol{\lambda}}^+ = (\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}})^{-1}\bar{\boldsymbol{\lambda}}^T$. Substituting $\hat{r}$ into (3), we have

$$\hat{\mathbf{N}} = \arg \min_{\mathbf{n}\in\mathbb{Z}^m} \|(\mathbf{n} + \bar{\mathbf{y}})(\mathbf{I}_m - \bar{\boldsymbol{\lambda}}\bar{\boldsymbol{\lambda}}^+)\| \quad (5)$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix. Let $\mathbf{A} = (\mathbf{I}_m - \bar{\boldsymbol{\lambda}}\bar{\boldsymbol{\lambda}}^+)$. (5) demonstrates that the estimation of the set of integers $\mathbf{n}$ is the problem of finding the nearest point $\mathbf{nA}$ in the lattice $\boldsymbol{\Omega}(\mathbf{A})$ with a basis $\mathbf{A}$ from a given point $\bar{\mathbf{y}}\mathbf{A}$.

$\mathbf{A}$ is able to be reduced to a diagonal matrix using Lemma 1. As a consequence, the closed-form lattice algorithm [5] can be used to find the estimate of $\mathbf{N}$.

Lemma 1. *Let $\lambda_i \in \mathbb{Z}$ and $g_{i,m} = \mathrm{GCD}(\lambda_i, \lambda_m)$ for $i = 1, \cdots, m-1$, where $\mathrm{GCD}(\cdot)$ is the great common divisor operation, then lattice basis $\mathbf{A}$ can be reduced to a diagonal basis with following form*

$$\mathrm{diag}\left\{ g_{1,m}\frac{\prod_{i=1}^{m-1}\lambda_i}{\lambda_1}, \cdots, g_{m-1,m}\frac{\prod_{i=1}^{m-1}\lambda_i}{\lambda_{m-1}}, 0 \right\},$$

if $\mathrm{GCD}\left(\frac{\lambda_i}{g_{i,m}}, \frac{\lambda_j}{g_{j,m}}\right) = 1, i, j = 1, \cdots, m-1, \ \ i \neq j.$

3. THE PROPOSED ALGORITHM

In this paper, we are interested in using moduli which do not satisfy the condition of Lemma 1 and therefore, the closed-form lattice algorithm is not applicable.

By the Hermite Normal Form(HNF) decomposition, $\mathbf{A}$ can be represented as the product of an upper-triangular matrix $\mathbf{T}$ and a unimodular matrix $\mathbf{U}$ [6], so that $\boldsymbol{\Omega}(\mathbf{A}) = \boldsymbol{\Omega}(\mathbf{T})$. The elements of the mth row and column of $\mathbf{T}$ are all zero since the rank of $\mathbf{A}$ is $m-1$.

Suppose that the closest lattice point to the given point $\bar{\mathbf{y}}\mathbf{A}$ and the associated integer vector corresponding to $\boldsymbol{\Omega}(\mathbf{T})$ are solved and denoted by $\mathbf{P} = [P_1, \cdots, P_{m-1}, 0]$ and $\mathbf{v} = [v_1, \cdots, v_{m-1}, v_m]$ respectively, where v_m is undetermined. Then we have $\mathbf{vT} = \hat{\mathbf{N}}\mathbf{A} = \mathbf{P}$. We aim to solve $\hat{\mathbf{N}}$ from these known values. Since $\mathbf{vT} = (\hat{\mathbf{N}}\mathbf{U}^{-1})\mathbf{T}$, therefore $\mathbf{vU} = \hat{\mathbf{N}}$ which can be written as follows, where $j = 1, \cdots, m$

$$\sum_{i=1}^{m-1} v_i u_{i,j} + v_m u_{m,j} = \hat{N}_j \quad (6)$$

All the values are known in these equations except $\hat{N}_j$ and the undetermined integer value v_m . $\hat{N}_j$ in (6) are integers and bounded by $0 < \hat{N}_j \leq \frac{\mathrm{LCM}(\lambda_1, \cdots, \lambda_m)}{\lambda_j}$, therefore

$$0 < v_m + \frac{\sum_{i=1}^{m-1} v_i u_{i,j}}{u_{m,j}} \leq \frac{\mathrm{LCM}(\lambda_1, \cdots, \lambda_m)}{\lambda_j u_{m,j}} \quad (7)$$

Lemma 2. *Let $\mathbf{U}$, $\mathbf{A}$ and $\mathbf{T}$ be integer matrices as defined in this section. Then there exist $\{u_{m,j} \in \mathbb{Z}, j = 1, \cdots, m\}$ satisfying following relation*

$$\frac{1}{u_{m,1}\lambda_1} = \cdots = \frac{1}{u_{m,m}\lambda_m} = \pm\frac{1}{LCM(\lambda_1, \cdots, \lambda_m)}$$

Proof. Since $\mathbf{UA} = \mathbf{T}$ and the entries of last row of $\mathbf{T}$ are all 0, then we have $u_{m,j}\prod_{i=1}^{m-1}\lambda_i - u_{m,m}\frac{\prod_{i=1}^{m}\lambda_i}{\lambda_j} = 0$ for $j = 1, \cdots, m$

Clearly, the solution of $\{u_{m,j}\}$ is $\frac{K}{\lambda_j}$. Consider $\{u_{m,j} \in \mathbb{Z}, j = 1, \cdots, m\}$, then let $K = k\mathrm{LCM}(\lambda_1, \cdots, \lambda_m)$ where

$k \in \mathbb{Z} \setminus 0$ ($\det \mathbf{U}$ will be 0 if $k = 0$). Substituting K into $u_{m,j} = \frac{K}{\lambda_j}$, we have

$$\frac{1}{u_{m,1}\lambda_1} = \cdots = \frac{1}{u_{m,m}\lambda_m} = \pm\frac{1}{k\text{LCM}(\lambda_1, \cdots, \lambda_m)}$$

Let $k = \pm 1$ and the lemma is proved. □

Let $u_{m,j} \in \mathbb{Z}^+$ and $j = 1$, from Lemma 2, (7) can be written as

$$-\frac{\sum_{i=1}^{m-1} v_i u_{i,1}}{u_{m,1}} < v_m \leq 1 - \frac{\sum_{i=1}^{m-1} v_i u_{i,1}}{u_{m,1}}$$

Lemma 3. *Let $X \in \mathbb{R}$. there exists an unique integer v_m satisfies $-X < v_m \leq 1 - X$.*

Consider that $v_m \in \mathbb{Z}$ and from Lemma 3, v_m can be uniquely determined by

$$\begin{cases} v_m = -\left\lfloor \frac{\sum_{i=1}^{m-1} v_i u_{i,1}}{u_{m,i}} \right\rfloor & \frac{\sum_{i=1}^{m-1} v_i u_{i,1}}{u_{m,i}} \notin \mathbb{Z} \\ v_m = 1 - \frac{\sum_{i=1}^{m-1} v_i u_{i,1}}{u_{m,i}} & \frac{\sum_{i=1}^{m-1} v_i u_{i,1}}{u_{m,i}} \in \mathbb{Z} \end{cases}$$

Therefore, all parameters in (6) are known, and $\hat{\mathbf{N}}$ can be uniquely determined.

4. FINDING THE CLOSEST LATTICE POINT BY RELEVANT VECTORS

Finding the lattice point in $\mathbf{\Omega}(\mathbf{G})$ which is closest to a given point $\mathbf{x} \in \mathbb{R}^m$ can be iteratively implemented as in [7]:

$$\mathbf{t}_{k+1} = \mathbf{t}_k + \mathbf{d}_k \tag{8}$$

$$\mathbf{d}_k = \arg \min_{\mathbf{d} \in \text{Rel}(\mathbf{G}) \cup \{\mathbf{0}\}} \|\mathbf{x} - \mathbf{t}_k - \mathbf{d}\| \tag{9}$$

with an initial guess on $\mathbf{t}_0$ which lies in $\mathbf{\Omega}(\mathbf{G})$. It can be proved that this algorithm converges to the closest point of $\mathbf{x}$ within a finite number of steps.

Inspired by [7] and [9], we may use an adaptive step size α_k rather a fixed step size in the iteration. Thus, the above iteration may be written as

$$\mathbf{t}_{k+1} = \mathbf{t}_k + \alpha_k \mathbf{d}_k \tag{10}$$

$$\{\mathbf{d}_k, \alpha_k\} = \arg \min_{\mathbf{c} \in \text{Rel}(\mathbf{G}),\ \alpha \in \mathbb{Z}^+ \cup \{0\}} \|\mathbf{x} - \mathbf{t}_k - \alpha\mathbf{c}\| \tag{11}$$

We have following proposition to optimally choose α_k:

Proposition 1. *The optimal choice of α_k satisfies $\alpha_k = Round\left(\left|\frac{\sum_{i=1}^{m-1} \beta_i c_i}{\sum_{i=1}^{m-1} c_i^2}\right|\right)$, where c_i and β_i is ith element of $\mathbf{c}$ and $\mathbf{x} - \mathbf{t}_k$ respectively and $\mathbf{c} \in Rel(\mathbf{G})$.*

Proof. Given $\mathbf{x}$ and $\mathbf{c}$, $\min_{\alpha \in \mathbb{R}} \|\mathbf{x} - \mathbf{t}_k - \alpha\mathbf{c}\|$ can be written into $\min_\alpha \sum_{i=1}^{m-1} (\beta_i - \alpha c_i)^2$

Taking derivative $\sum_{i=1}^{m-1} (\beta_i - \alpha c_i)^2$ w.r.t. α and setting it zero gives $\alpha = \frac{\sum_{i=1}^{m-1} \beta_i c_i}{\sum_{i=1}^{m-1} c_i^2}$.

Since α_k is either a positive integer or 0, therefore the optimal choice is $\alpha_k = Round\left(\left|\frac{\sum_{i=1}^{m-1} \beta_i c_i}{\sum_{i=1}^{m-1} c_i^2}\right|\right)$ □

Proposition 2. *The algorithm (10) will be convergent to closest point within finite steps.*

Proof. Denote the closest point of $\mathbf{x}$ by $\mathbf{P}$. If $\mathbf{t}_k$ lies in the Voronoi cell of $\mathbf{P}$, then from the definition of Voronoi region, we have $\sum_{i=1}^{m-1} \beta_i c_i < \frac{1}{2} \sum_{i=1}^{m-1} c_i^2$, where c_i and β_i is ith element of $\mathbf{c}$ and $\mathbf{x} - \mathbf{t}_k$, $\forall \mathbf{c} \in \text{Rel}(\mathbf{G})$, thus from Proposition 1, $\alpha_k = 0$.

If $\mathbf{t}_k$ does not lie in the Voronoi cell of $\mathbf{P}$, then from $\mathbf{t}_{k+1} = \mathbf{t}_k + \alpha_k \mathbf{d}_k$, we have

$$\|\mathbf{x} - \mathbf{t}_{k+1}\| \begin{cases} < \|\mathbf{x} - \mathbf{t}_k - \alpha_k \mathbf{c}\| & \alpha \neq 0 \\ = \|\mathbf{x} - \mathbf{t}_{k+1}\| & \alpha_k = 0 \end{cases}$$

where $\alpha \in \mathbb{Z}^+$,$\alpha \neq \alpha_k$, $\forall \mathbf{c} \in \text{Rel}(\mathbf{G})$ and $\mathbf{d}_k \neq \mathbf{c}$.

This implies that the distance between $\mathbf{x}$ and $\mathbf{t}_{k+1}$ will be decreased strictly until $\alpha_k = 0$, i.e. the algorithm is convergent. From the above analysis, $\alpha_k = 0$ means that the $\mathbf{t}_k$ lies in the the Voronoi cell of $\mathbf{P}$ and the closest point is attained. □

An illustrative example showing the searching path and number of number of iteration differences between the conventional searching algorithm and the proposed adaptive searching algorithm is given in Figure1.

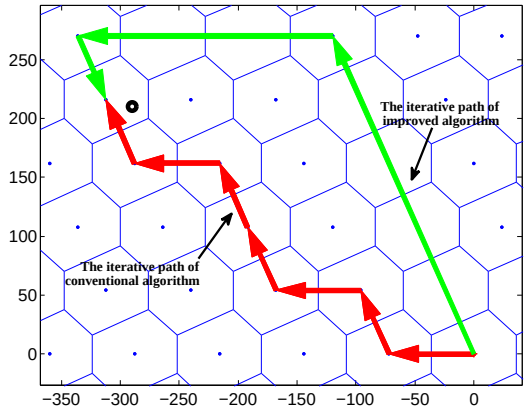


Fig. 1: An illustrative example for comparing the conventional searching and adaptive searching algorithms. The given point is indicated using a black circle and the lattice point is indicated by blue dot. Red arrowed-line shows the iteration path of conventional algorithm and the green arrowed-line shows the adaptive searching algorithm.

After obtaining the closest point to the given point $\mathbf{P}$, it is easy to find the associated vector $\mathbf{v}$ corresponding to lattice $\mathbf{T}$ via $\mathbf{vT} = \mathbf{P}$ since $\mathbf{T}$ is upper-triangular matrix.

5. SIMULATION

In this section, we compare the performance of proposed algorithm with the search-based CRT algorithm presented in [3]. The latter is regarded as an optimal algorithm.

Two sets of moduli, which do not satisfy the co-prime constraint, are used as below to demonstrate the efficiency

of the proposed algorithm.

$$\Lambda_1 = \{21, 22, 23, 24, 25, 26, 27, 29\}$$
$$\Lambda_2 = \{56, 57, 58, 59, 60, 61, 62, 63\}$$

The distance r is randomly selected between 0 and the LCM of the modulus set. The parameter δ in phase measurement noise variance $\delta^2\lambda_i^2$ is chosen such that $-20\log_{10}\delta = 30 + 2n,\ \ n = 0, 1, \cdots, 15$, which provides an indication for both noise level and signal to noise ratio in the simulation. All simulation results illustrated are averaged over 1000 Monte Carlo runs. Algorithm performance is measured in the probability of correctly estimating the set of integers $\mathbf{N}$ for a given measurement noise level, i.e., the probability of a correct signal phase reconstruction. In addition, computational complexity is also an important criterion.

Fig.2(a) shows the probabilities of correctly reconstructing signal phases for different measurement noise levels δ and different moduli sets Λ_1 and Λ_2, i.e. $Pr(\hat{\mathbf{N}} = \mathbf{N}|\delta)$. In both cases ($\Lambda_1$ and Λ_2), the proposed algorithm has an identical performance to the search-based CRT algorithm in the probabilities of correctly estimating both $\mathbf{N}$ and r. On the other hand, Fig.2(b) shows the ratio of required CPU time. It indicates that the required computational load by the proposed algorithm is significantly less than the searching-based CRT algorithm.

The closest point searching algorithm with an adaptive iteration step size plays a key role for the efficiency enhancement of the proposed algorithm. We compare the computational complexity of the new derived searching algorithm with that of the conventional algorithm presented in [7] in terms of CPU time in Fig. 3 versus the number of signal wavelengths used. It shows that the conventional searching algorithm increases its computational overhead much faster than the proposed one once the number of signal wavelengths exceed 8.

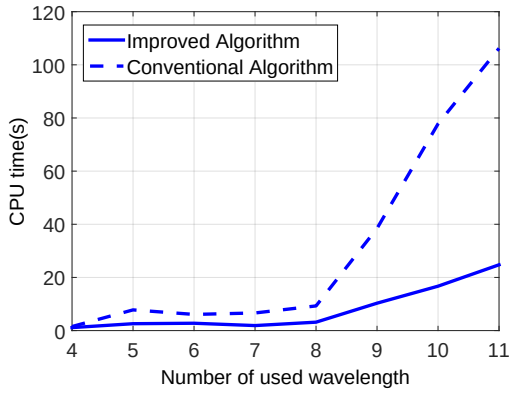
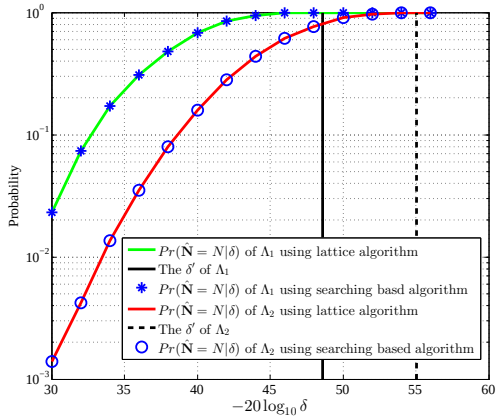
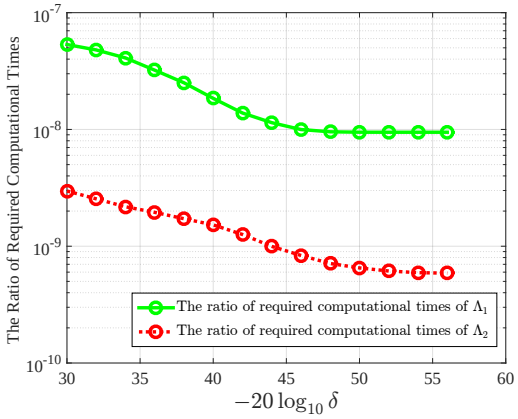


Fig. 3: The required CPU time of proposed closest point searching algorithm and conventional one versus different used number of moduli.

It is worth mentioning that if the underlying problem satisfies the co-prime constraint, the proposed algorithm is equivalent to the closed-form lattice algorithm presented in [5].



(a) Probability of correctly reconstruction.



(b) Ratio of required computation time.

Fig. 2: Comparison of the proposed algorithm and searching-based CRT algorithm vs. the amplitude of the noise using Λ_1 and Λ_2, respectively.

6. CONCLUSIONS

In this work, we present a lattice based estimator for estimating distances with phase wrapped signal measurements via multiple synchronized frequencies. The proposed algorithm addresses a more general situation where the co-prime constraint on signal wavelengths is relaxed. Furthermore, we propose an adaptive searching algorithm for finding the nearest lattice point for a given noisy measurement on a lattice, which greatly improves the efficiency of the proposed estimator.

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