FRI SAMPLING AND RECONSTRUCTION OF ASYMMETRIC PULSES

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ABSTRACT

We consider the problem of modelling asymmetric pulse trains as finite-rate-of-innovation (FRI) signals. In particular, we show that the sum of amplitude-scaled and time-shifted pulses with different asymmetry factors is an FRI signal. Such signals frequently arise in applications such as ultrasound and radio detection and ranging (RADAR) where the received signal has skewed pulses. In this paper, we model the asymmetric component of a pulse using its derivative. A sampling kernel with a sum-of-sincs frequency response is used to measure the samples, and a modified annihilating filter method is applied on the samples to estimate the parameters of the FRI signal. We show accurate reconstruction for signals containing asymmetric Gaussian, Cauchy-Lorentz, and sinc pulses. Analysis of the proposed scheme in the presence of noise shows that the error in the estimated parameters decreases by oversampling the signal.

Index Terms— Finite-rate-of-innovation (FRI), sampling, sum-of-sincs (SoS) kernel, asymmetric pulses, annihilating filter

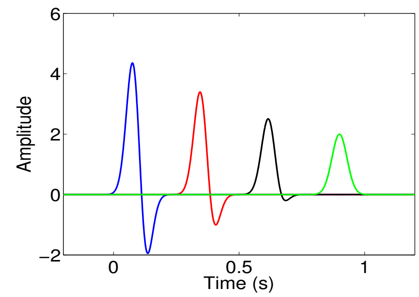


Fig. 1. A signal consisting of four pulses having different degrees of symmetry.

1. INTRODUCTION

Vetterli et al. [1] proposed a technique to sample and reconstruct a class of signals that are neither bandlimited nor belong to a shift-invariant space. This class of signals, termed finite-rate-of-innovation (FRI) signals, are completely specified by a finite number of parameters per interval. Vetterli et al. showed that a stream of Dirac impulses, nonuniform splines, and nonuniform polynomial splines belong to the class of FRI signals. Tur et al. [2] showed that signals that contain shifted and scaled pulses are FRI signals, which can be used to model ultrasound signals. In this paper, we address the following questions: (1) Can we model signals with varying pulse shape as FRI signals? (2) If yes, what should be the signal model? (3) Can the signal be perfectly reconstructed by sampling with the sum-of-sincs (SoS) sampling kernel?

Consider a signal $x(t)$ represented as

$$x(t) = \sum_{k=1}^{K} r_k h_k(t - t_k), \tag{1}$$

where r_k, t_k, are the amplitude, and time-delay, respectively, of the k^{th} pulse $h_k(t)$. $x(t)$ can be modeled effectively as an FRI signal if $\{h_k(t)\}_{k=1}^{K}$ are related to a known pulse shape template $h(t)$ by a finite set of parameters. One such parameter is the degree of asymmetry of $h(t)$. In Figure 1, four pulses with varying degree of asymmetry (decreasing from left to right) are shown.

In this paper, we address the change in the pulse shape resulting due to varying degrees of asymmetry. We show that the derivative operator can be used to model asymmetric pulse signals. We use a modified version of the popular annihilating filter method to estimate the parameters of the FRI signal.

This model can be used in pulse-based probing applications such as ultrasound and RADAR. Consider a setup where an ultrasound pulse $h(t)$, generated by an immersion transducer is used to probe a block made up of $K-1$ layers of uniform acoustic impedance immersed in water. The ultrasound pulse is reflected at the boundary of layers having different acoustic impedances. The reflections collected by the receiver are given by

$$x(t) = \sum_{k=1}^{K} r_k h(t - t_k), \tag{2}$$

where r_k and t_k are the amplitude and time-delay, respectively, of the k^{th} reflected pulse [2]. This model assumes that the pulse shape does not change at every reflection. Such reflections, called specular reflections [3], occur when the thickness of each of the layers is larger than the wavelength of ultrasound, each layer is homogeneous, and the boundaries of the layers are smooth. In many practical scenarios, these assumptions do not hold. For example, in practice, the typical frequency and wavelength of the ultrasound used are 10 MHz and 0.1 nm, respectively. A particle smaller than this wavelength scatters the ultrasound wave in all directions. Such reflections result in the skewing of the pulse shape $h(t)$, which cannot be modeled by (2). In contrast, (1) allows one to take into account skewing of the pulses. A specific model for signals containing pulses of varying degree of asymmetry and width was presented by Baechler et al. [4][5], who showed that the asymmetry and width of a Cauchy-Lorentz pulse can be estimated using the annihilating filter technique. However, the problem of modeling pulses of varying degree of asymmetry for a more generic pulse shape was not addressed.

1.1. Related literature

Vetterli et al. [1] showed that the problem of sampling and reconstruction of FRI signals can be reduced to one of finding the parameters of the signal from its frequency-domain samples. The spectral samples are estimated by measuring the signal through an ideal lowpass filter. Dragotti et al. [6] proposed that functions satisfying the Strang-Fix conditions can be used as sampling kernels to compute the moments of the signal. Dragotti et al. used the exponential reproducing kernel for recovering the exponential moments of a signal. Uriguen et al. [7] recently showed that by choosing the weighting coefficients optimally, arbitrary sampling kernels can be used to reproduce exponential signals. Seelamantula and Unser [8] designed sampling kernels using resistor-capacitor (RC) circuits to sample and reconstruct a stream of nonuniformly spaced Dirac impulses. An SoS sampling kernel was proposed by Tur et al. [2] to recover the Fourier measurements of periodic FRI signals. To handle aperiodic FRI signals, the sampling kernel was periodized. Mulleti et al. [9] showed that the sampling kernel need not be periodic if the sampling frequency is an integral multiple of the fundamental frequency of the sinc kernel. Due to the simplification offered by the technique of Mulleti et al., in this paper, we use the SoS kernel without periodization for sampling FRI signals. For reconstruction of the signal from its samples, the annihilating filter is used. Kusuma and Goyal [10] showed that the FRI methodology has a super-resolving property owing to the use of high-resolution-spectral-estimation (HRSE) techniques [11]. The FRI sampling framework has been applied to the estimation of time sequences of action potentials in neurophysiological data [12]. Variable pulse-width finite-rate-of-innovation (VPW-FRI) signals were introduced by Baechler et al. for compression of electrocardiogram (ECG) signals [4]. An ECG signal can be described by a set of parameters, the cardinality of which is of the order of the number of pulses, thus allowing for effective compression of the ECG signal. This algorithm has been extended for multichannel ECG data compression by Nair et al. [13]. The VPW-FRI sampling framework has also been used to estimate the heart rate from fetal ECG data [14].

1.2. Contributions of this paper

We propose a model for generating asymmetric pulse signals from a given pulse shape, such that the resulting signal takes the form of an FRI signal. We use a modified annihilating-filter technique to estimate the time-delays of the asymmetric pulses. We show results of applying the asymmetric FRI signal sampling technique on Gaussian, Cauchy-Lorentz, and sinc pulses. Performance of the reconstruction algorithm in the presence of noise is analyzed and the improvements obtained by oversampling the signal are reported.

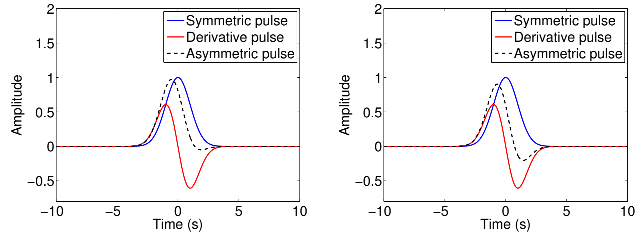
2. ASYMMETRIC PULSE FRI (AP-FRI)

2.1. Generating asymmetric pulses

Consider a symmetric pulse $h(t)$, which is used to generate pulses of varying degree of asymmetry. One way to achieve this is to use the derivative of the pulse. The pulse defined as

$$h_{\alpha,\beta}(t) = \alpha h(t) + \beta \frac{\mathrm{d}h(t)}{\mathrm{d}t}, \alpha, \beta \in \mathbb{R} - \{0\} \tag{3}$$

is asymmetric. Let us show this by working in the Fourier domain. As the pulse $h(t)$ is symmetric, its Fourier transform $H(\omega)$ is real



(a) (b)

Fig. 2. Asymmetric pulses $h_{\alpha,\beta}(t)$ generated using $h(t) = e^{-t^2/2}$, and $\theta = \tan^{-1}(\frac{\beta}{\alpha})$; (a) $\theta = 0.2\pi$, and (b) $\theta = 0.3\pi$.

[15]. Thus, the Fourier transform of its derivative given by

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} \overset{\mathrm{F}}{\longleftrightarrow} \mathrm{j}\omega H(\omega), \tag{4}$$

is imaginary. A signal having a purely imaginary Fourier transform is anti-symmetric in the time domain. Thus, the pulse $h_{\alpha,\beta}(t)$ containing non-zero symmetric and anti-symmetric components is asymmetric. The degree of asymmetry is quantified by $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$. By changing α and β, pulses of varying degrees of asymmetry can be generated. Using $h(t) = e^{-t^2/2\sigma^2}$ with $\sigma = 1$, the asymmetric pulses obtained are shown in Figure 2.

2.2. Asymmetric pulses and FRI

Consider a signal $x(t)$ consisting of K asymmetric pulses given by

$$x(t) = \sum_{k=1}^{K} h_{\alpha_k,\beta_k}(t - t_k), \tag{5}$$

where $h_{\alpha,\beta}(t)$ is defined in (3). The amplitudes of the symmetric and anti-symmetric components and location of the k^{th} pulse are given by α_k, β_k, and t_k, respectively. Let the time-delays $\{t_k\}_{k=1}^K$ be arranged in ascending order. $x(t)$ is an FRI signal because it is specified by a finite set of parameters. Defining a window $[0, \tau]$ such that $\{t_k\}_{k=1}^K \in [0, \tau]$, the rate of innovation of $x(t)$ is given by $\rho = 3K/\tau$. The Fourier transform of $x(t)$ is $X(\omega) = \sum_{k=1}^K (\alpha_k + \mathrm{j}\omega\beta_k) H(\omega) e^{-\mathrm{j}\omega t_k}$. For $\omega_o \in \mathbb{R}$ and a set of consecutive integers $\mathcal{K} = [\![-M, M]\!]$, chosen such that $H(m\omega_o) \neq 0, m \in \mathcal{K}$ we define,

$$\begin{aligned} Y(m\omega_o) &\triangleq X(m\omega_o)/H(m\omega_o), m \in \mathcal{K} \\ &= \sum_{k=1}^{K} (\alpha_k + \mathrm{j}m\omega_o\beta_k) e^{-\mathrm{j}m\omega_o t_k}. \end{aligned} \tag{6}$$

The condition on the cardinality of $\mathcal{K}$ and ω_o comes from the framework used to reconstruct the signal $x(t)$ from its samples, given in Section 2.3.

2.3. Modified annihilating filter

The annihilating filter is a spectral estimation tool that can be used to compute the time-delays $\{t_k\}_{k=1}^K$ from $Y(m\omega_o)|_{m\in\mathcal{K}}$ in (6). The problem of estimating the time-delays can be rephrased as finding the exponentials $\{u_k\}_{k=1}^K$ given $p[n]$ such that

$$p[n] \quad = \quad \sum_{k=1}^{K} c_k u_k^n + d_k n u_k^n, n \in [\![-2K, 2K]\!], \qquad (7)$$

where c_k and d_k are the unknown amplitudes. To solve for u_k from $p[n]$, we make the following proposition.

Proposition: The signal $p[n] = \sum_{k=1}^{K} c_k u_k^n + d_k n u_k^n$, where $c_k, d_k \in \mathcal{R}$ and $u_k \in \mathcal{C}$, is annihilated by the filter, $A(z) = \prod_{k=1}^{K}(1 - u_k z^{-1})^2$.

Proof. The filter $A(z) = \prod_{k=1}^{K}(1 - u_k z^{-1})^2$ can be expanded as, $A(z) = \sum_{n=0}^{2K} A[n] z^{-n}$, with the coefficients $A[n]$. The output on filtering $p[n]$ with $A[n]$, for $n = -2K, -2K+1, \cdots 2K$ is given by

$$\begin{aligned} z[n] &= \sum_{\ell=0}^{2K} A[\ell] p[n-\ell], \\ &= \sum_{k=1}^{K} c_k u_k^n \underbrace{\sum_{\ell=0}^{2K} A[\ell] u_k^{-\ell}}_{A(u_k)=0} + \sum_{k=1}^{K} d_k n u_k^n \underbrace{\sum_{\ell=0}^{2K} A[\ell] u_k^{-\ell}}_{A(u_k)=0} \\ &- \sum_{k=1}^{K} d_k u_k^n \underbrace{\sum_{\ell=0}^{2K} \ell A[\ell] u_k^{-\ell}}_{-z \frac{\mathrm{d}A(z)}{\mathrm{d}z}\big|_{z=u_k}=0} \\ &= 0. \end{aligned}$$

Thus, the filter $A(z) = \prod_{k=1}^{K}(1 - u_k z^{-1})^2$ annihilates the signal $p[n]$. ■

Using the above proposition in (6), we find that $A(z) = \prod_{k=1}^{K}(1 - e^{-\mathrm{j}\omega_o t_k} z^{-1})^2$ annihilates the sequence $Y(m\omega_o)|_{m \in \mathcal{K}}$ when $M \geq 2K$. The roots of the filter $A[n]$ are $\{e^{-\mathrm{j}\omega_o t_k}\}_{k=1}^K$ with multiplicity two. The condition on ω_o for unique identification of $\{t_k\}_{k=1}^K$ from $\{e^{-\mathrm{j}\omega_o t_k}\}_{k=1}^K$ is that $\omega_o \leq 2\pi / t_K$.

2.4. Amplitude of symmetric and anti-symmetric components

On computing the time instants $\{t_k\}_{k=1}^K$ in (6) using the annihilating filter, the amplitudes of the symmetric and asymmetric components of the pulse given by $\{\alpha_k, \beta_k\}_{k=1}^K$ can be estimated by solving the linear system of equations, $Y = Ar$, where Y, A, r are defined in equations (8)-(10). The amplitudes of the symmetric and antisymmetric components are given by $\alpha_k = r_k$, and $\beta_k = r_{k+K}$, for $k = 1, 2, \cdots, K$.

2.5. Sampling kernel

Knowing the frequency samples $X(m\omega_o)|_{m \in \mathcal{K}}$, the unknown parameters $\{\alpha_k, \beta_k, t_k\}_{k=1}^K$ are specified by using the modified annihilating filter technique followed by solving a system of linear equations. Thus, the goal is to estimate the frequency samples $X(m\omega_o)|_{m \in \mathcal{K}}$ from minimum number of samples of $x(t)$. An SoS sampling kernel, given in frequency domain by $S(\omega) = \sum_{m \in \mathcal{K}} \mathrm{sinc}\left(\frac{\omega}{\omega_o} - m\right)$, can be used to compute $X(m\omega_o)|_{m \in \mathcal{K}}$, when the sampling frequency $\omega_s = P\omega_o$ for $P \geq |\mathcal{K}| = 2M + 1$ and $P \in \mathbb{N}$ [9]. Using the minimal sampling frequency, we set $M = 2K$ and thus a sampling frequency of $\omega_s = (2M+1)\omega_o$. The corresponding sampling interval is $T_s = \frac{2\pi}{\omega_s}$.

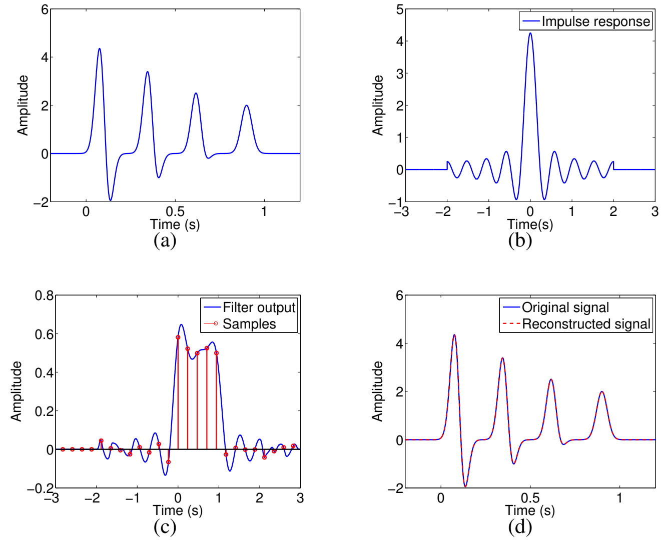


Fig. 3. (a) Stream of $K = 4$ Gaussian pulses, with $\sigma = 0.03$ and having different asymmetry factors; (b) Impulse response of SoS sampling kernel with $M = 2K$ and $\omega_0 = \pi/2$; (c) The output of the sampling filter and samples of the output at a frequency of 4.25 Hz; (d) Original and reconstructed asymmetric FRI signal.

3. SIMULATION RESULTS

3.1. Asymmetric Gaussian pulse

Using $h(t) = e^{-t^2/2\sigma^2}$ in (3) with $\sigma = 0.03$, the asymmetric Gaussian pulse is given by

$$h_{\alpha,\beta}(t) \quad = \quad \alpha e^{-t^2/2\sigma^2} - \beta \frac{t}{\sigma^2} e^{-t^2/2\sigma^2}, \alpha, \beta \in \mathbb{R} - \{0\}.$$

Using this pulse shape in (5), with $K = 4$ results in the signal shown in Figure 3(a). The impulse response of the SoS sampling kernel, with $M = 2K$ and $\omega_o = \pi/2$ is shown in Figure 3(b). The samples obtained at the critical sampling rate of $\omega_s = (2M+1)\omega_o$ are shown in Figure 3(c). On estimating the parameters of the signal from the samples using the modified-annihilating filter and the resulting linear system of equations, the signal reconstructed with an accuracy down to machine precision is shown in Figure 3(d).

3.2. Cauchy-Lorentz and sinc pulses

A sequence of asymmetric pulse shapes generated using Cauchy-Lorentz and sinc pulse shapes is shown in Figures 4(a) and 4(b). The Cauchy-Lorentz asymmetric pulse is given by $h_{\alpha,\beta}(t) = \alpha \frac{a}{a^2+t^2} + \beta \frac{-2at}{(a^2+t^2)^2}$, and the sinc asymmetric pulse shape is given by $h_{\alpha,\beta}(t) = \alpha \mathrm{sinc}(Bt) + \beta \frac{B\pi t \cos(B\pi t) - \sin(B\pi t)}{B\pi t^2}$. Applying the FRI sampling and reconstruction schemes on the asymmetric signals with $K = 4$, $M = 2K$, $\omega_o = \pi/2$ and $\omega_s = (2M+1)\omega_o$, the reconstructions to machine precision are shown in Figures 4(a) and 4(b).

$$
\begin{aligned}
Y &= [Y(-M\omega_o)\ \ Y(-(M-1)\omega_o)\ \ \cdots\ \ Y(M\omega_o)]^T && (8)\\
r &= [\alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_K\ \ \beta_1\ \ \beta_2\ \ \cdots\ \ \beta_K]^T && (9)\\
A &= \begin{bmatrix} e^{jM\omega_o t_1} & e^{jM\omega_o t_2} & \cdots & -M\omega_o e^{jM\omega_o t_1} & -M\omega_o e^{jM\omega_o t_1} & \cdots & -M\omega_o e^{jM\omega_o t_K} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e^{-jM\omega_o t_1} & e^{-jM\omega_o t_2} & \cdots & M\omega_o e^{-jM\omega_o t_1} & M\omega_o e^{-jM\omega_o t_1} & \cdots & M\omega_o e^{-jM\omega_o t_K} \end{bmatrix} && (10)
\end{aligned}
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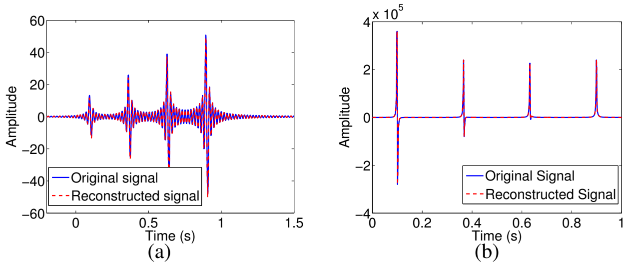

Fig. 4. (a) Stream of $K = 4$ sinc pulses, with $B = 100$ and different asymmetry factors; (b) Stream of $K = 4$ Cauchy-Lorentz pulses, with $a = 0.002$ having different asymmetry factors. In both cases, an SoS sampling kernel with $M = 2K$, $\omega_o = \pi/2$ and sampling frequency of 4.25 Hz is used.

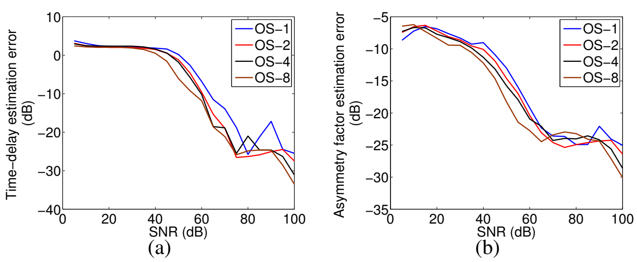

Fig. 5. (a) Average error in the estimated time-delay, and (b) average error in the asymmetry factor of two asymmetric Gaussian pulses for samples corrupted by additive white Gaussian noise. The effect of oversampling (OS) on the accuracy of the parameters for oversampling factors of $1, 2, 4, 8$.

3.3. Noise analysis of asymmetric FRI pulse signals

In this section, we analyze the performance of the FRI reconstruction technique when the samples of the asymmetric signal are corrupted by additive white Gaussian noise (AWGN). Consider the FRI signal given in (5), with two asymmetric Gaussian pulses. Let the time-delays, amplitudes of symmetric and asymmetric Gaussian components be given by $\mathbf{t} = [1/4\ 3/4]^T$, $\boldsymbol{\alpha} = [2\ 2]^T$, and $\boldsymbol{\beta} = [0.1\ 0.05]^T$, respectively. Therefore, the degree of asymmetry is $\boldsymbol{\theta} = [0.05\ 0.025]$. The samples obtained using the SoS kernel, with $M = 2K$, $\omega_0 = \pi/2$ and $\omega_s = (2M+1)\omega_o$ are corrupted by AWGN. Let the estimated parameters, time-delay and degree of asymmetry be denoted by $\hat{\mathbf{t}}$ and $\hat{\boldsymbol{\theta}}$, respectively. It has been shown in literature that, in general, Cadzow denoising [16] and oversampling techniques can be used to improve the noise performance of the FRI algorithm [2]. In this paper, we study the effect of oversampling on the reconstruction of asymmetric FRI sampling and reconstruction algorithms. The errors $\|\mathbf{t} - \hat{\mathbf{t}}\|^2$ and $\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2$ averaged over 1000 realizations of AWGN are shown in Figure 5. The maximum improvement in the error of the estimated time-delay and degree-of-asymmetry parameters with an oversampling (OS) factor, OS$=8$, over those estimated from samples with OS$=1$ are 8.02 dB and 5.44 dB, respectively. In Table 1, we compare the estimated time-delays of the FRI reconstruction algorithm [2] applied to a signal with unchanging pulse shape with our proposed asymmetric FRI reconstruction algorithm for the signals given in (5). We observe that, in the presence of noise, the asymmetric FRI algorithm is more susceptible to noise in comparison to the FRI algorithm for unchanging pulse shape. This is attributed to the fact that there are multiple roots in the asymmetric FRI framework, which makes the annihilating filter more sensitive to noise. The issue of making the annihilating filter noise-robust requires more investigation.

SNR (dB)	Gaussian		Cauchy-Lorentz	
	(a)	(b)	(a)	(b)
10	-1.49	2.77	-1.39	2.96
20	-27.51	2.17	-27.53	1.99
30	-38.28	2.12	-38.31	1.92

Table 1. Error in the estimated time-delays of pulses whose shape is unchanging (a), and of pulses whose degree of asymmetry changes with time (b), when the samples of the signal are corrupted by AWGN. The errors in the estimated time-delays of Gaussian and Cauchy-Lorentz pulses are reported in decibels.

4. CONCLUSIONS

We have addressed the problem of modeling asymmetric pulse trains as FRI signals. We have proposed to use the derivative of the pulse shape to model the asymmetric component of the pulse. A modified annihilating filter was used to estimate the time-delays of the asymmetric pulses. Experimental results show that, using the proposed technique, exact sampling and reconstruction of asymmetric signals generated using Gaussian, Cauchy-Lorentz and sinc pulse shape templates is achieved. Performance analysis in the presence of noise showed that multiple roots arising in the asymmetric FRI problem makes the annihilating filter more sensitive to noise. Experimental results show that oversampling the signal decreases the error in the estimated parameters.

5. ACKNOWLEDGEMENT

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