PERIODIC NON-UNIFORM SAMPLING FOR FRI SIGNALS

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ABSTRACT

A typical finite-rate-of-innovation (FRI) signal reconstruction scheme is based on the measurement of uniform samples in time/frequency domain, and the application of the annihilating filter on the measured samples. We propose a continuous-time annihilation framework for a class of FRI signals. In particular, we show that FRI signals of sum-of-weighted exponential form can be annihilated by a composition of translation operators and show that the parameters of the signal can be estimated in the periodic non-uniform sampling (PNU) scenario. We discuss the advantages of PNU sampling over uniform sampling and extend it for general FRI signal sampling and reconstruction. Simulations are performed and the results are compared with state-of-the-art methods for signal-to-noise ratios ranging from -20 to 100 dB. An improvement in the estimation accuracy of 15–55 dB in terms of bias and mean-square error is achieved over conventional methods by a rearrangement of uniform samples to follow PNU sampling.

Index Terms— Finite rate of innovation, periodic non-uniform sampling, continuous-time annihilation, annihilating filter.

1. INTRODUCTION

In the past decade, extensive research has gone into sampling and reconstruction of structured signals that are not necessarily bandlimited. The structure could be the sparsity of the signal in certain bases representation, parametric representation of the signal or a multiband structure. Such signals have finite degrees of freedom (DOF) and hence can be reconstructed from a finite number of measurements.

Vetterli et al. [1] introduced the notion of finite-rate-of-innovation (FRI) sampling and showed that certain signals can be specified by a finite number of free parameters per unit time interval. They considered signals such as a stream of Dirac impulses, piecewise polynomials, a stream of differential Dirac pulses, non-uniform splines, etc. In a typical FRI framework, one estimates the parameters of the signal by designing a suitable sampling kernel that converts the sampled sequence to a power-sum sequence [2]. The parameters of the FRI signal are estimated from the samples by applying high-resolution spectral estimation techniques such as the annihilating filter (AF) [3], multiple signal classification (MUSIC) algorithm [4], estimation of signal parameters via rotational invariance technique (ESPRIT) [5], minimum-norm method [6], and their numerous variants [7, 8]. Vetterli et al. proposed infinite duration Gaussian and sinc sampling kernels in conjunction with the AF method for signal reconstruction. To have practically realizable sampling kernels, Dragotti et al. [9] proposed a class of functions that includes polynomial reproducing kernels, exponential reproducing kernels, and kernels with rational transfer functions. Tur et al. extended the class of FRI signals to include signals of the form of sum-of-weighted and time-shifted (SWTS) copies of a known pulse [10]. They proposed a finite duration kernel with repetitions and sum-of-sincs (SoS) frequency response, and demonstrated application to reconstruction of ultrasound signals in the FRI framework. Recently, Mulleti et al. [11] showed that SWTS-FRI signals can be sampled using SoS sampling kernels without repetitions. Bernet et al. [12] extended FRI reconstruction framework for piecewise sinusoids with polynomials and without polynomials. Matusiak and Eldar [13] addressed the sampling and reconstruction of SWTS-FRI signal with unknown pulse shapes. Uriguen et al. [14] proposed an FRI signal sampling strategy using arbitrary sampling kernels. Multichannel sampling strategies for a stream of Dirac impulses were proposed by Seelamantula and Unser [15], and Olkkonen and Olkkonen [16] using causal exponential sampling kernels. These methods do not require the AF for reconstruction and give a closed-form expression for the estimated parameters. Alternative multichannel FRI sampling methods were reported by Gedalyahu et al. [17], Kusuma and Goyal [18], and Asl et al. [19]. In particular, Asl et al. considered the multicahannel framework with unknown delays and gains in each channel, and show the reconstruction of FRI signals and estimation of delays and gains in each channel by using kernels with polynomial reproducing property. Most of the FRI sampling methods rely on uniform sampling of filtered FRI signals. Sun [20] and Wei et al. [21] proposed non-uniform sampling methods for FRI signals.

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Another class of structured signals with finite DOF consists of signals with sparse representation in a known bases. Sampling and reconstruction methods for such signals are developed within the framework of compressive sensing [22–26]. Multiband signals constitute a class of structured signals with effective bandwidth (cumulative spectral support) much lesser than the maximum frequency component. A periodic non-uniform sampling scheme called multi-coset sampling was proposed by Venkataramani and Bresler [27] for sampling multiband signals at sub-Nyquist rates. Mishali and Eldar [28] proposed a compressive sensing framework to sample and reconstruct multiband signals.

1.1. Our contribution

We consider the problem of sampling and reconstruction of FRI signals of sum-of-weighted exponentials (SWE) form given by

$$f(t) = \sum_{\ell=1}^{L} a_\ell \, e^{\alpha_\ell \, t}, \tag{1}$$

where all $a_\ell, \alpha_\ell \in \mathbb{C}$. The goal is to estimate the parameters $\{a_\ell, \alpha_\ell\}_{\ell=1}^{L}$ from $f(t)$. We develop a continuous-time annihilation method for the FRI signal by applying shift/translation operators and show that the parameters of SWE-FRI signals can be estimated by more generalized periodic non-uniform (PNU) sampling as opposed to conventional uniform sampling. We extend the PNU sampling framework to the reconstruction of generalized FRI signals that consist of a sum-of-weighted and time-shifted pulses. Simulations are performed for various noise levels and bias and mean-square error (MSE) are computed and the MSE is compared with the Cramér-Rao lower bound (CRLB).

2. THE KEY IDEA

2.1. Annihilation of FRI signals on a PNU sampling grid

Consider a single exponential signal $x(t) = e^{\alpha\, t}$, where $\alpha \in \mathbb{C}$, and the shifted version $\mathcal{S}_T\{x\}(t) \triangleq x(t-T) = e^{\alpha\,(t-T)} = e^{-\alpha\, T}\, x(t)$. Clearly, $(I - e^{\alpha\, T}\mathcal{S}_T)\{x\}(t) = 0, \forall t \in \mathbb{R}$. Thus, the operator $(I - e^{\alpha\, T}\mathcal{S}_T)$ is an annihilator of the exponential $x(t) = e^{\alpha\, t}$. Given a function $x(t)$ with an unknown parameter α, with the goal of estimating α, we could construct an operator $(I - e^{\beta\, T}\mathcal{S}_T)$ and determine the value of β for which $(I - e^{\beta\, T}\mathcal{S}_T)\{x\}(t) = 0$. This value is unique and is given by $\beta = \alpha$.

Since the shift operator is linear, it is straightforward to verify that the SWE function $f(t) = \sum_{\ell=1}^{L} a_\ell \, e^{\alpha_\ell \, t}$, is annihilated for all time by the operator $\mathcal{M} \triangleq \prod_{\ell=1}^{L} \underbrace{(I - e^{\alpha_\ell \, T}\mathcal{S}_T)}_{\mathcal{M}_\ell}$. The operator $\mathcal{M}$ can be expressed in additive form as $\mathcal{M} = \sum_{\ell=0}^{L} \gamma_\ell \, \mathcal{S}_{\ell T}$, where $\gamma_0 = 1$, and $\mathcal{S}_{\ell T}$ is the ℓ-times dilated version of the translation operator $\mathcal{S}_T$ and γ_ℓs are dependent on the quantum of shift T and the exponential parameters $\{\alpha_\ell\}_{\ell=1}^{L}$.

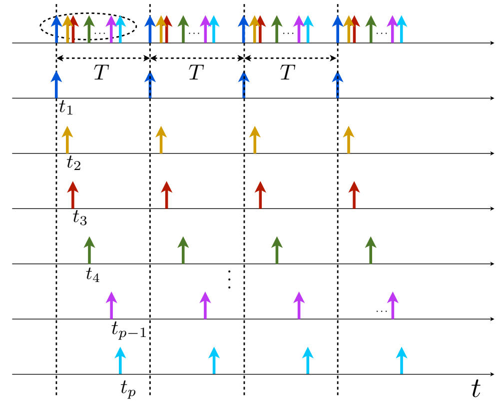

Fig. 1. Illustration of interleaved sampling grids. The non-uniform grid in the top most plot is actually comprised of various uniform sampling grids as shown.

Since $f(t) = \sum_{\ell=1}^{L} a_\ell \, e^{\alpha_\ell \, t}$ is an FRI signal with the free-variables $\{a_\ell, \alpha_\ell\}_{\ell=1}^{L}$, reconstruction of $f(t)$ is equivalent to estimating the free variables. To address the question of estimating the parameters a_ℓs and α_ℓs, we start with the annihilation equation using the operator $\mathcal{M}$, which yields

$$\sum_{\ell=0}^{L} \gamma_\ell \, f(t - \ell\, T) = 0, \;\; \forall t, \tag{2}$$

and setup a system of equations

$$\sum_{\ell=0}^{L} \gamma_\ell \, f(t_k - \ell\, T) = 0, \;\; k = 1, 2, 3, \cdots, M, \tag{3}$$

corresponding to a set of random sampling instants t_ks, and $\gamma_0 = 1$ with $M \geq L$. In (3), starting at each randomly chosen initial grid time instant t_k, a uniform sampling scheme with sampling interval T is applied to measure L samples at $t_k - \ell\, T$, $\ell = 0, 1, 2, \cdots, L$. With this sampling strategy, the sampling indices given by $t_k - \ell T$ for $k = 1, 2, 3, \cdots, M$ and $\ell = 0, 1, 2, \cdots, L$ constitute a periodic non-uniform sampling grid as shown in Fig. 1.

The corresponding matrix equation for annihilation is given by (4) where a minimum of $L(L+1)$ measurements of f are required for annihilation with $M = L$. The vector γ can be estimated from (4) as the null-space vector of the matrix $\mathcal{M}\mathbf{f}$ with the condition $\gamma_0 = 1$. The roots of the polynomial with the coefficients $\{\gamma_\ell\}_{\ell=0}^{L}$ are given as $\left\{e^{\alpha_\ell T}\right\}_{\ell=1}^{L}$, from which α_ℓ can be computed (T is known). Once α_ℓs are

$$
\underbrace{\begin{pmatrix} f(t_1) & f(t_1-T) & \cdots & f(t_1-LT) \\ f(t_2) & f(t_2-T) & \cdots & f(t_2-LT) \\ f(t_3) & f(t_3-T) & \cdots & f(t_3-LT) \\ \vdots & \vdots & \ddots & \vdots \\ f(t_M) & f(t_M-T) & \cdots & f(t_M-LT) \end{pmatrix}}_{\mathcal{M}\mathbf{f}} \underbrace{\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_L \end{pmatrix}}_{\boldsymbol{\gamma}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbf{0}}, \tag{4}
$$

estimated, the amplitudes a_ℓs can be estimated from measurements of the signal $f(t)$ by least-squares (LS) regression.

It can be shown that the matrix $\mathcal{M}\mathbf{f}$ in (4) has rank L for randomly chosen t_ks and $M \geq L$. Hence, the estimated vector $\boldsymbol{\gamma}$ is unique up to a scaling factor, and parameters α_ℓs can be determined uniquely from PNU sampling of FRI signals $f(t)$. In the presence of noise, the matrix $\mathcal{M}\mathbf{f}$ has full column rank and $\boldsymbol{\gamma}$ is estimated as a solution to the optimization problem: $\underset{\boldsymbol{\gamma}}{\text{minimize}}\, \|\mathcal{M}\mathbf{f}\boldsymbol{\gamma}\|^2$ such that $\|\boldsymbol{\gamma}\|^2 = 1$. Larger the number of rows M in (4), more the number of samples used for estimation of $\boldsymbol{\gamma}$; consequently, the parameter estimation accuracy is increased in the presence of noise.

2.2. PNU sampling of generalized FRI signals

In the preceding analysis, the PNU sampling and reconstruction method has been derived for SWE-FRI signals. In this section, we show that the sampling method can be extended to a larger class of FRI signals. Consider SWTS-FRI signal $g(t) = \sum_{\ell=1}^{L} a_\ell\, h(t-\tau_\ell)$, where a_ℓs and τ_ℓs are unknown parameters to be estimated by assuming that the pulse $h(t)$ is known. The Fourier transform of $g(t)$ is $\hat{g}(\omega) = \hat{h}(\omega) \sum_{\ell=1}^{L} a_\ell\, e^{-\mathrm{j}\,\omega\,\tau_\ell}$, where $\hat{h}(\omega)$ is the frequency spectrum of the pulse $h(t)$. A typical reconstruction method for SWTS-FRI signals consists of estimating *uniform* samples of $\hat{x}(\omega) = \frac{\hat{g}(\omega)}{\hat{h}(\omega)}$, by assuming that $\hat{h}(\omega)$ is non-zero at those sample locations, and then applying the AF method and LS method to estimate the parameters. The function $\hat{x}(\omega)$ is an FRI signal of the SWE form and instead of sampling it uniformly, one can apply the proposed PNU sampling and perform annihilation to estimate its parameters.

2.3. Advantages of PNU sampling over uniform sampling

The SWE signal reconstruction from uniform samples requires a minimum of $2L$ contiguous samples, whereas with PNU sampling, one requires a minimum of $L(L+1)$ samples with $L+1$ contiguous samples at each grid. The main advantage with interleaved sampling is that the set of $L+1$ contiguous samples can be measured from different segments of the SWE-FRI signal. In the presence of noise, based on the distribution of noise over time/frequency, segments with higher signal-to-noise ratio (SNR) can be used to measure the samples. In contrast, for the uniform sampling scheme, it is not feasible to obtain samples from different segments of the signal due to the requirement of contiguous sample locations.

The advantage of taking measurements from high-SNR segments is more relevant in sampling and reconstruction of SWTS pulses from their Fourier transform measurements. In the presence of noise, the division operation in computing $\hat{x}(\omega) = \frac{\hat{g}(\omega)}{\hat{h}(\omega)}$, which is effectively a deconvolution operation, enhances the effect of noise in the frequency band for which the magnitude of $\hat{h}(\omega)$ is close to zero. By using the PNU sampling grid, we have the freedom to measure $L+1$ samples from the frequency bands where $|\hat{h}(\omega)|$ is significant. Moreover, in the presence of colored noise, the samples can be measured from frequency bands with low noise floors.

The proposed continuous-time annihilation-based sampling and reconstruction methods require non-uniform samples, whereas most practical systems employ uniform sampling. The PNU sampling method can be easily extended to systems where uniform samples are available without altering the sampling mechanism. Suppose we are given uniform samples (with sampling interval T_s) of an SWE-FRI signal, we can construct (4) by choosing t_ks randomly from integer multiples of T_s and selecting the delay T to be an integer multiple of T_s. Hence, a rearrangement of uniform samples will result in PNU sampling and reconstruction.

3. SIMULATION RESULTS

We consider an FRI signal of the form shown in (1) with $L = 3$, amplitudes $[a_1, a_2, a_3]$ = $[2.0, 1.5, 1.0]$ and exponents $[\alpha_1, \alpha_2, \alpha_3]$ = j$[0.20, 0.37, 0.65]$, respectively. We measured $N = 128$ uniform samples of the signal with the sampling interval $T_s = 1$. The samples are contaminated by independent and identically distributed additive white Gaussian noise (AWGN) with zero mean and variance σ^2. To estimate the parameters $\{\alpha_\ell, a_\ell\}_{\ell=1}^{L}$ of the FRI signal, the following methods are applied: (1) AF method, (2) AF method with Cadzow denoising [29], which is denoted as Cadzow-AF, (3) ESPRIT, (4) proposed PNU sampling based method, and (5) proposed method with Cadzow denoising on uniform samples, denoted as Cadzow-PNU. The estimation performances of these methods are compared in terms of average bias and average MSE computed over 1000 independent realizations for each noise levels. The SNR is varied from -20 dB to 100 dB in steps of 5 dB. In PNU sampling, t_ks are selected randomly over $[1, N]$ for $M = 25$ with a delay $T = 8$. The CRLB in estimation of α_ℓ (given by Variance$(\alpha_\ell) = \frac{6\sigma^2}{N^3 a_\ell^2}$ [7]) is calculated and MSEs in estimation of α_ℓ are compared against it. The plots in the first columns of Fig. 2 and Fig. 3 show the squared bias in the estimation of α_ℓs and a_ℓs, respectively. The squared bias in estimation of α_ℓs is lesser by 20–50 dB with PNU sampling method as compared with uniform sampling following

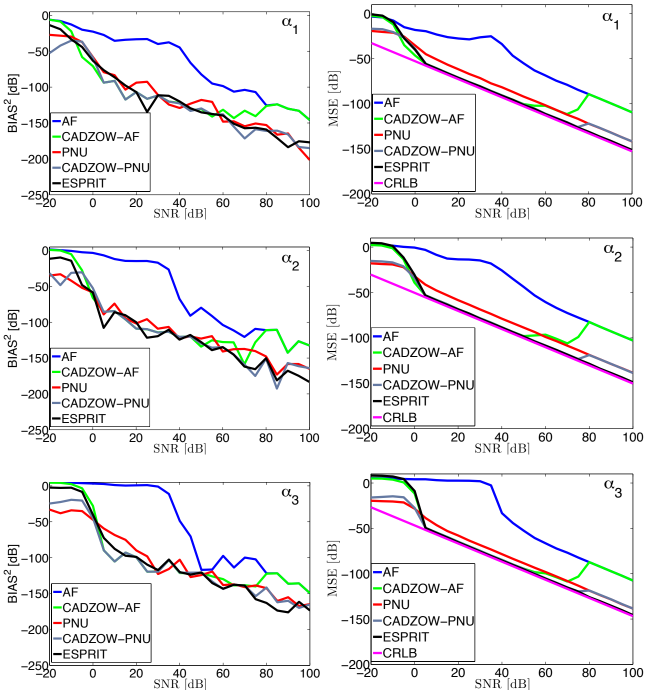

Fig. 2. Performance comparison in the estimation of α_ℓs for N=128 uniform samples. The parameters of the FRI signal are L = 3, $[\alpha_1, \alpha_2, \alpha_3]$ = j$[0.20, 0.37, 0.65]$ and $[a_1, a_2, a_3]$ = $[2.0, 1.5, 1.0]$. In the PNU sampling scheme, T=8 and M=25.

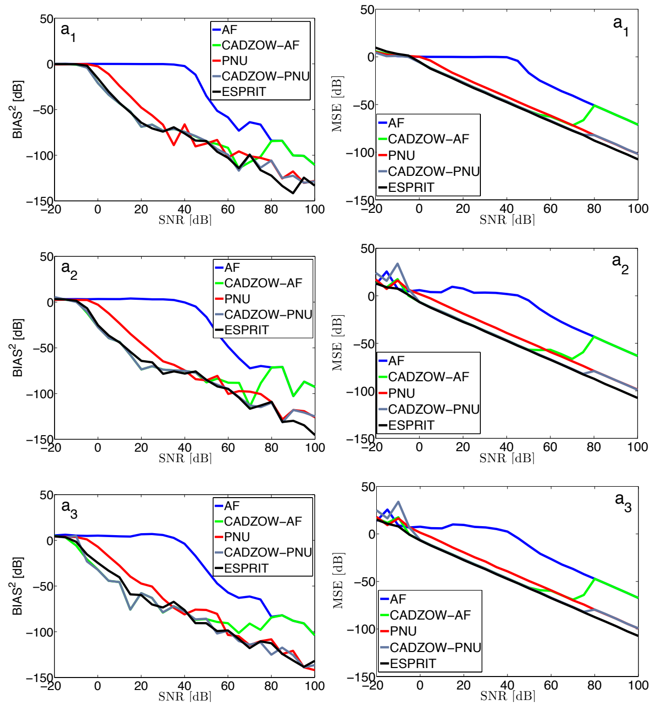

Fig. 3. Performance comparison in the estimation of a_ℓs for N=128 uniform samples. The parameters of the FRI signal are L = 3, $[\alpha_1, \alpha_2, \alpha_3]$ = j$[0.20, 0.37, 0.65]$ and $[a_1, a_2, a_3]$ = $[2.0, 1.5, 1.0]$. In the PNU sampling scheme, T=8 and M=25.

by AF method for the SNR range of -20 to 100 dB, whereas the improvement in bias performance in estimation of a_ℓs is 5–50 dB for SNR $\geq$ 0 dB. After applying the Cadzow denoising method, the error due to bias with AF based method is reduced by 5–50 dB over the SNR range -20 to 100 dB for both α_ℓs and a_ℓs; however, it has not changed significantly in PNU sampling method whose performance with Cadzow denoising is very similar to that of ESPRIT method.

The second column of Fig. 2 shows that the MSE error in estimation of α_ℓs is lesser by 20–55 dB with the proposed method as compared with the AF based method, as SNR is increased from -20 to 100 dB. Hence, by rearrangement of uniform samples (cf. (4)) one can improve the estimation performance significantly. With the proposed method, the MSE is lesser by 7 dB compared with CRLB for SNR $\geq$ 0 dB. An improvement of 5–45 dB in MSE is noted in estimation of a_ℓs for positive SNRs. By applying Cadzow denoising, the MSE in the estimation of α_ℓ by AF method is decreased by 50 dB, whereas the MSE improvement in the proposed method with denoising is 7 dB for SNR$\geq$0 dB. For SNR lesser than 0 dB, the PNU and Cadzow-PNU methods perform better in terms of MSE compared to Cadzow-AF and ESPRIT methods, whereas, for high SNR, the performance of the methods with Cadzow denoising and performance of ESPRIT method is closer to the CRLB. As the SNR goes beyond 80 dB, the improvement due to denoising ceases and the methods perform similarly to their counterparts without denoising. In the second column of Fig. 3, an improvement of 5–45 dB in MSE in the estimation of a_ℓs is observed with PNU sampling as compared to AF method for SNR $\geq$ 0 dB. With Cadzow denoising, the performance of AF based method is increased by 5–45 dB, whereas the MSE is improved by 7 dB with the proposed method for positive SNRs.

4. CONCLUSIONS

We proposed a continuous-time, shift-based annihilation approach with PNU sampling for reconstruction of a class of FRI signals. The proposed method is more generic than uniform sampling method in terms of selecting samples from different segments of a signal. We showed that the PNU sampling method can also be extended to uniform samples by rearranging them. Simulation results show a significant improvement in the estimation performance by applying the proposed method as compared with the conventional uniform sampling based AF (with or without Cadzow denoising) and ESPRIT with a simple adaptation to PNU sampling. The enhancement in performance is due to the delay factor T, which improves the resolution in exponent estimation. We are working towards deriving the performance bounds in parameter estimation as a function of T and noise statistics, and applications to real data.

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