ASYMPTOTIC ANALYSIS OF LINEAR SPECTRAL STATISTICS OF THE SAMPLE COHERENCE MATRIX

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ABSTRACT

Correlation tests of multiple Gaussian signals are typically formulated as linear spectral statistics on the eigenvalues of the sample coherence matrix. This is the case of the Generalized Likelihood Ratio Test (GLRT), which is formulated as the determinant of the sample coherence matrix, or the locally most powerful invariant test (LMPIT), which is formulated as the Frobenius norm of this matrix. In this paper, the asymptotic behavior of general linear spectral statistics is analyzed assuming that both the sample size and the observation dimension increase without bound at the same rate. More specifically, almost sure convergence of a general class of linear spectral statistics is established, and an associated central limit theorem is formulated. These asymptotic results are shown to provide an accurate statistical description of the behavior of the GLRT and the LMPIT in situations where the sample size and the observation dimension are both large but comparable in magnitude.

Index Terms— Coherence matrix, correlation test, random matrix theory, central limit theorem.

1. INTRODUCTION

The problem of testing the structure of the covariance matrix of a set of multivariate Gaussian observations has applications in multiple fields, such as sensor networks, radar, radioastronomy, finance or cognitive radio. Typically, one needs to test whether the covariance matrix of the observations is proportional to the identity matrix (sphericity test [1, 2]), or whether it is a positive diagonal matrix (correlation test). The first problem is equivalent to testing whether the constituent scalar random variables are independent and identically distributed (i.i.d.) whereas the second problem is equivalent to whether the constituent signals are mutually independent but not necessarily identically distributed. In this paper, we will focus on this second test, namely the correlation test for Gaussian distributed observations.

Let $\mathbf{y}_n$, $n = 1 \dots N$, denote a collection of $M \times 1$ i.i.d. random vectors following a zero-mean Gaussian distribution with covariance matrix $\mathbf{R}_M$, and consider the corresponding sample covariance matrix

$$\hat{\mathbf{R}}_M = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}_n \mathbf{y}_n^H.$$

Let $\mathbf{D}_M$ and $\hat{\mathbf{D}}_M$ denote the diagonal matrices constructed from the diagonal entries of $\mathbf{R}_M$ and $\hat{\mathbf{R}}_M$ respectively, i.e. $\mathbf{D}_M = \mathrm{dg}\left(\mathbf{R}_M\right) \equiv \mathbf{R}_M \odot \mathbf{I}_M$ and $\hat{\mathbf{D}}_M = \mathrm{dg}\left(\hat{\mathbf{R}}_M\right)$, where $\odot$ is the Hadamard (element-wise) product and $\mathbf{I}_M$ is the $M \times M$ identity matrix. The main focus of this paper is on the following binary hypothesis test:

$$\mathcal{H}_0 : \mathbf{y}_n \sim \mathcal{CN}(0, \mathbf{R}_M), \mathbf{R}_M = \mathbf{D}_M$$
$$\mathcal{H}_1 : \mathbf{y}_n \sim \mathcal{CN}(0, \mathbf{R}_M), \mathbf{R}_M \neq \mathbf{D}_M.$$

One of the first contributions addressing this problem can be traced back to work in [3], which derived the Generalized Likelihood Ratio Test (GLRT) for this problem assuming that the observations are real-valued. It was shown in [3] that, assuming $N > M$, the GLRT rejects the null hypothesis for large enough values of the following statistic

$$\hat{\eta}_M^{GLRT} = \frac{-1}{M} \log\det\left(\hat{\mathbf{C}}_M\right) \tag{1}$$

where $\hat{\mathbf{C}}_M$ is the sample coherence matrix, defined as

$$\hat{\mathbf{C}}_M = \hat{\mathbf{D}}_M^{-1/2} \hat{\mathbf{R}}_M \hat{\mathbf{D}}_M^{-1/2}$$

and where $(\cdot)^{1/2}$ denotes the positive square root. It was more recently shown [4, 5] that the above GLRT also holds for complex-valued Gaussian random variables. Other extensions of the GLRT to more general statistical observation structures can be found in [6, 7, 8].

The GLRT is known to be asymptotically optimal when the number of samples tends to infinity $N \to \infty$, but its performance may degrade considerably in situations characterized by small sample sizes (low N) or by close hypotheses

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(low cross-correlations under $\mathcal{H}_1$). It is well known that under these situations other tests can significantly outperform the GLRT. One popular choice is the Frobenius Norm Test, which accepts the null hypothesis for sufficiently small values of

$$\hat{\eta}_M^{FNT} = \frac{1}{M}\left\|\hat{\mathbf{C}}_M\right\|_F^2 = \frac{1}{M}\operatorname{tr}\left[\hat{\mathbf{C}}_M^2\right].$$

This test was proposed in [4] as an approximation of the GLRT for low values of the cross-correlation coefficients under $\mathcal{H}_1$ (low SNR in signal detection applications), and was recently shown to be the locally most powerful invariant test (LMPIT) for the correlation detection problem [9].

All the above statistics can be seen as particular cases of a broad class of random variables constructed from the eigenvalues of the sample coherence matrix $\hat{\mathbf{C}}_M$, typically referred to as linear spectral statistics (LSS). Let $\hat{\lambda}_1 \leq \ldots \leq \hat{\lambda}_M$ denote the eigenvalues of the sample coherence matrix $\hat{\mathbf{C}}_M$. LSS as random variables that can be written in the form

$$\hat{\eta}_M = \frac{1}{M}\sum_{m=1}^{M} f\left(\hat{\lambda}_m\right) \tag{2}$$

for a (generally complex) function f, well defined on the positive real axis. Obviously, the two statistics $\hat{\eta}_M^{GLRT}$ and $\hat{\eta}_M^{FNT}$ are particular instances of LSS, since they can both be expressed as in (2) with $f(z) = -\log z$ for the GLRT and $f(z) = z^2$ for the Frobenius Norm Test.

The objective of this paper is to characterize the statistical behavior of general LSS constructed from the eigenvalues of the sample coherence matrix $\hat{\mathbf{C}}_M$, i.e. random variables that are constructed as in (2) for some generic complex function $f(z)$. It should be pointed out that some particularizations of this problem have been well studied in the literature, especially the GLRT statistic $\hat{\eta}_M^{GLRT}$. For example, it was recently shown [10] that under the null hypothesis $\hat{\eta}_M^{GLRT}$ can be represented as a product of independent beta-distributed random variables, a fact that can be used to numerically compute the threshold of the test to guarantee a certain probability of false alarm. Apart from this work, the vast majority of published work relies on classical large sample size approximations, assuming $N \to \infty$ for fixed M. For example, the asymptotic distribution of $\hat{\eta}_M^{GLRT}$ was established in [11, 12, 13], whereas the power of the test was extensively studied in [12, 13], in both cases under large sample size asymptotics.

These classical asymptotic studies are accurate for moderately high sample volume N and relatively low observation dimension M. However, it is well known [14] that when M becomes large and tends to be comparable in magnitude to the sample size N, classical asymptotic approximations are no longer valid. For this reason, in this paper we propose to assume that both M and N are large but comparable in magnitude. In mathematical terms, we will assume that both $M, N \to \infty$ in a way that, if we define $c_M = M/N$,

$$0 < \liminf(c_M) \leq \limsup(c_M) < \infty. \tag{3}$$

We will first study the almost sure asymptotic behavior of the LSS in (2), and then we will characterize the asymptotic fluctuations of these statistics by establishing a central limit theorem.

2. CONVERGENCE OF THE LSS

We will make the following technical assumptions:

(**As1**) The set of M-dimensional complex observations $\mathbf{y}_n$, $n = 1, \ldots, N$, are i.i.d. random vectors distributed according to a complex circularly symmetric Gaussian law of zero mean and covariance $\mathbf{R}_M$.

(**As2**) The observation dimension M is a function of N and (3) holds.

(**As3**) If $\lambda_{\min}(\mathbf{R}_M)$ and $\lambda_{\max}(\mathbf{R}_M)$ denote the minimum and maximum eigenvalues of the Hermitian matrix $\mathbf{R}_M$, $\sup_M \lambda_{\max}(\mathbf{R}_M) < \infty$ and[1] $\inf_M \lambda_{\min}(\mathbf{R}_M) > 0$.

To study the asymptotic behavior of the LSS under the above assumptions, we will consider the Stieltjes transform of the empirical eigenvalue distribution of $\hat{\mathbf{C}}_M$. This is a complex function defined for $z \in \mathbb{C}^+$ (upper complex semiplane) as

$$\hat{m}_M(z) = \frac{1}{M}\operatorname{tr}\left[\left(\hat{\mathbf{C}}_M - z\mathbf{I}_M\right)^{-1}\right] = \frac{1}{M}\sum_{m=1}^{M}\frac{1}{\hat{\lambda}_m - z}.$$

Using the results in [15] one can show that for, all M, N sufficiently high, all the positive eigenvalues $\hat{\mathbf{C}}_M$ are located inside the interval $\mathcal{S} = \left(s^{-1}, s\right)$ with probability one, where

$$s = \sup_M \frac{\left(1+\sqrt{c_M}\right)^2 \lambda_{\max}(\mathbf{R}_M)}{\left(1-\sqrt{c_M}\right)^2 \lambda_{\min}(\mathbf{R}_M)}.$$

Using this fact, one can analytically extend the definition of $\hat{m}_M(z)$ to $\mathbb{C}\backslash\mathcal{S}\cup\{0\}$ for all large M, N using the Schwarz reflection principle [16]. Now, let $\mathcal{C}^-$ denote a clockwise oriented simple contour that encloses $\mathcal{S}$ and only intersects $\mathbb{R}^+$. By simple application of the Cauchy integral formula, we can express the LSS in (2) as

$$\hat{\eta}_M = \frac{1}{2\pi \mathrm{j}}\oint_{\mathcal{C}^-} f(z)\hat{m}_M(z)dz$$

almost surely for all large M, N (so that all the eigenvalues are enclosed by $\mathcal{C}$). Consequently, we can essentially investigate the asymptotic behavior of $\hat{\eta}_M$ by studying the asymptotic behavior of $\hat{m}_M(z)$. The following result establishes almost sure convergence of this function in $\mathbb{C}^+$. The proof can be established following the arguments in e.g. [17] and is therefore omitted.

[1] In fact, only the first condition (bounded spectral norm) is needed for Theorem 1.

Theorem 1. *Under* $(\mathbf{As1})-(\mathbf{As3})$, $|\hat{m}_M(z)-\bar{m}_M(z)| \to 0$ *almost surely for all* $z \in \mathbb{C}^+$ *as* $M, N \to \infty$, *where*

$$\bar{m}_M(z) = \frac{N-M}{Mz} - \frac{N}{M}\frac{1}{\omega_M(z)}$$

with $\omega_M(z)$ *denoting the unique solution in* $\mathbb{C}^+$ *to the following polynomial equation in* ω

$$z = \omega\left(1 - \frac{1}{N}\operatorname{tr}\left[\mathbf{C}_M(\mathbf{C}_M - \omega\mathbf{I}_M)^{-1}\right]\right) \tag{4}$$

and $\mathbf{C}_M = \mathbf{D}_M^{-1/2}\mathbf{R}_M\mathbf{D}_M^{-1/2}$ *being the true coherence matrix.*

By comparing this result with the analogous convergence for the sample covariance matrix [17], one can conclude that the asymptotic behavior of the eigenvalues of $\hat{\mathbf{C}}_M$ essentially coincides with that of the eigenvalues of $\mathbf{D}_M^{-1/2}\hat{\mathbf{R}}_M\mathbf{D}_M^{-1/2}$. In other words, one can replace $\hat{\mathbf{D}}_M$ by $\mathbf{D}_M$ in the definition of $\hat{\mathbf{C}}_M$ without affecting the global asymptotic behavior of its eigenvalues.

This result can now be used in order to establish the almost sure convergence of $\hat{\eta}_M$ as defined in (2). We will assume that:

$(\mathbf{As4})$ The complex function $f(z)$ is holomorphic on an open subset including $\mathcal{S}$.

By analytically extending $\bar{m}_M(z)$ and $\omega_M(z)$ to the whole $\mathbb{C}\backslash\mathcal{S}\cup\{0\}$ with the Schwarz reflexion principle and using the dominated convergence theorem, one can see that under $(\mathbf{As1})-(\mathbf{As4})$, $|\hat{\eta}_M - \bar{\eta}_M| \to 0$ where

$$\bar{\eta}_M = \frac{1}{2\pi\,\mathrm{j}}\oint_{\mathcal{C}^-} f(z)\bar{m}_M(z)dz. \tag{5}$$

For some practical values of $f(z)$, this integral can be computed in closed form by a simple change of variable. More specifically, using the integration technique developed in [18] one can show that

$$\bar{\eta}_M^{FNT} = \frac{1}{M}\operatorname{tr}\left[\mathbf{C}_M^2\right] + \frac{M}{N}.$$

whereas, assuming $\inf(N/M) > 1$,

$$\bar{\eta}_M^{GLRT} = \frac{-1}{M}\log\det\mathbf{C}_M + 1 - \frac{N-M}{M}\log\left(\frac{N}{N-M}\right).$$

Observe that the situation here is radically different from the classical asymptotic case ($N \to \infty$ for fixed M), where $\hat{\eta}_M^{GLRT}$ and $\hat{\eta}_M^{FNT}$ are consistent estimators of

$$\frac{-1}{M}\log\det\mathbf{C}_M \text{ and } \frac{1}{M}\operatorname{tr}\left[\mathbf{C}_M^2\right]$$

respectively. When we allow the observation dimension M to increase with the sample size, these two estimators become clearly inconsistent. The asymptotic bias terms disappear when $c_N \to 0$, agreeing with the fact that $\hat{\eta}_M \to \eta_M$ when $N \to \infty$ for a fixed M. We next provide a more interesting result that characterizes the asymptotic fluctuations of $\hat{\eta}_M$ around $\bar{\eta}_M$ in this asymptotic regime.

3. ASYMPTOTIC FLUCTUATIONS OF THE LSS

In this section, we will establish a central limit theorem on LSS constructed from the eigenvalues of $\hat{\mathbf{C}}_M$. In order to introduce this result, we need to define some complex functions that are related to the asymptotic mean and variance of the LSS. To simplify the notation, we will drop here the dependence on M in all matrices. Let us define

$$\begin{aligned}\mu_M(\omega) &= \frac{1}{N}\operatorname{tr}\left[\mathbf{C}(\mathbf{C}-\omega\mathbf{I})^{-1}\operatorname{dg}\left(\mathbf{C}(\mathbf{C}-\omega\mathbf{I})^{-1}\right)\right] \\ &\quad - 2\frac{1}{N}\operatorname{tr}\left[\mathbf{C}^2(\mathbf{C}-\omega\mathbf{I})^{-2}\operatorname{dg}\left(\mathbf{C}(\mathbf{C}-\omega\mathbf{I})^{-1}\right)\right] \\ &\quad + \omega\frac{1}{N}\operatorname{tr}\left[\left(\mathbf{C}(\mathbf{C}-\omega\mathbf{I})^{-2}\odot\mathbf{C}\right)\left((\mathbf{C}-\omega\mathbf{I})^{-1}\odot\mathbf{C}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\sigma_M^2(\omega_1,\omega_2) &= \frac{1}{N}\operatorname{tr}\left[\mathbf{C}\boldsymbol{\Delta}(\omega_1)\mathbf{C}\boldsymbol{\Delta}(\omega_2)\right] + \\ &\quad + \frac{1}{(\omega_2-\omega_1)^2} - \frac{1}{N}\operatorname{tr}\left[\mathbf{C}^2(\mathbf{C}-\omega_1\mathbf{I})^{-2}(\mathbf{C}-\omega_2\mathbf{I})^{-2}\right]\end{aligned}$$

where $\boldsymbol{\Delta}(\omega) = (\mathbf{C}-\omega\mathbf{I})^{-2} - \operatorname{dg}\left(\mathbf{C}(\mathbf{C}-\omega\mathbf{I})^{-2}\right)$.

Theorem 2. *Let* μ_M *and* σ_M^2 *be defined as*

$$\mu_M = \frac{1}{2\pi\,\mathrm{j}}\oint_{\mathcal{C}_\omega^-} F(\omega)\mu_M(\omega)\,d\omega \tag{6}$$

$$\sigma_M^2 = \frac{-1}{4\pi^2}\oint_{\mathcal{C}_\omega^-}\oint_{\mathcal{C}_\omega^-} F(\omega_1)F(\omega_2)\sigma_M^2(\omega_1,\omega_2)\,d\omega_1 d\omega_2 \tag{7}$$

where the functions $\mu_M(\omega)$ *and* $\sigma_M^2(\omega_1,\omega_2)$ *are specified above,* $\mathcal{C}_\omega$ *is a contour obtained as* $\mathcal{C}_\omega = \omega_M(\mathcal{C})$, *and where* $F(\omega)$ *is equal to* $f(z)$ *after replacing* z *with the right hand side of (4). Assume that* $\sup_M|\mu_M| < \infty$ *and that* $0 < \inf_M\left|\sigma_M^2\right| \leq \sup_M\left|\sigma_M^2\right| < \infty$. *Then, under* $(\mathbf{As1})-(\mathbf{As4})$,

$$\sigma_M^{-1}\left(M\left(\hat{\eta}_M - \bar{\eta}_M\right) - \mu_M\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{8}$$

where $\bar{\eta}_M$ *is defined in (5).*

This result can be proven by following the approach established in [19] (we omit the details here due to space constraints). We can particularize this theorem to the two statistics that are of interest here, by simply carrying out the integrals in (6)-(7) and then checking that these quantities are bounded according to the statement of the Theorem 2. We summarize the results in the following corollary:

Corollary 1. *Under* $(\mathbf{As1})-(\mathbf{As4})$, *the convergence in (8) holds for* $\hat{\eta}_M^{FNT}$ *by respectively replacing* μ_M *and* σ_M^2 *by*

$$\mu_M^{FNT} = \frac{1}{N}\operatorname{tr}\left[(\mathbf{C}\odot\mathbf{C})^2\right] - 2\frac{1}{N}\operatorname{tr}\left[\mathbf{C}^2\right] \text{ and}$$

$$\begin{aligned}\sigma_{M,FNT}^2 &= 2\left(\frac{1}{N}\operatorname{tr}\left[\mathbf{C}^2\right]\right)^2 \\ &\quad + 4\frac{1}{N}\operatorname{tr}\left[\mathbf{C}\left(\mathbf{C}-\operatorname{dg}\left(\mathbf{C}^2\right)\right)\mathbf{C}\left[\mathbf{C}-\operatorname{dg}\left(\mathbf{C}^2\right)\right]\right].\end{aligned}$$

If, in addition, $\inf N/M > 1$*, then (8) also holds for* $\hat{\eta}_M^{GLRT}$ *by respectively replacing* μ_M *and* σ_M^2 *by*

$$\mu_M^{GLRT} = -\frac{1}{2}\frac{M}{N} \text{ and}$$
$$\sigma^2_{M,GLRT} = \log\left(\frac{N}{N-M}\right) - 2\frac{M}{N} + \frac{1}{N}\operatorname{tr}\left[\mathbf{C}^2\right].$$

According to this corollary, under either one of the two hypotheses, both the GLRT and the Frobenius Norm Test statistics asymptotically fluctuate around $\bar{\eta}_M$ like a Gaussian random variable, with mean and variance given by the above expressions. Hence, in practical applications one might approximate the asymptotic law of $\hat{\eta}_M$ under both hypothesis as Gaussian random variable $\mathcal{N}\left(\bar{\eta}_M + \mu_M/M, \sigma_M^2/M^2\right)$, where the actual form of $\bar{\eta}_M$, μ_M and σ_M^2 will depend on the test and the considered hypothesis. We will next see that this provides a very good approximation of the statistics when both M, N are large.

4. NUMERICAL RESULTS

We considered here a scenario with a variable number of sensors that collected complex Gaussian signals as described in $(\mathbf{As1})$. We adopted here the simulation setting in [9], where the true covariance matrix was fixed to the identity under $\mathcal{H}_0$, and to $\mathbf{R}_M = \mathbf{F}_M \Lambda \mathbf{F}_M^H$ under the alternative $\mathcal{H}_1$, where $\mathbf{F}_M$ is the $M \times M$ Fourier matrix and Λ is a diagonal matrix that contains uniformly distributed values between 0.5 and 1.5. A total of 10^5 independent realizations of the two statistics $\hat{\eta}_M^{GLRT}$ and $\hat{\eta}_M^{FNT}$ were obtained for each hypothesis.

Figure 1 compares the simulated and asymptotic false alarm and detection probabilities for different values of M, N under both $\mathcal{H}_0$ and $\mathcal{H}_1$. First of all, we observe that the asymptotic distribution of the two statistics provides an accurate description of the actual performance for a wide range of values of M, N, even in situations where M, N are comparable in magnitude. On the other hand, it can be noticed that the Frobenius norm test provides uniformly better performance than the GLRT. This can also be illustrated in Figure 2, which compares the empirical and the asymptotic Receiver Operating Characteristic (ROC) of the two statistics curves for different values of M, N. Here again, we see that the asymptotic curves obtained with the proposed approximations are quite close to the empirical ones, although a higher degree of accuracy is obtained for the GLRT case.

5. CONCLUSIONS

This paper has presented an asymptotic analysis of LSS of the sample coherence matrix. It has been shown that the global asymptotic behavior of the eigenvalues essentially coincides with that of a sample covariance matrix constructed from observations that are correlated according to the true coherence matrix. This result has been used to establish the almost sure convergence of general LSS, and in particular the statistics of the GLRT and the LMPIT for the correlation test problem. A central limit theorem has been presented that establishes the Gaussianity of the fluctuations of these LSS around their asymptotic equivalents. Simulations for the GLRT and LMPIT statistics indicate that the proposed asymptotic description provides a accurate approximation even in situations where the sample volume is comparable to the observation dimension.

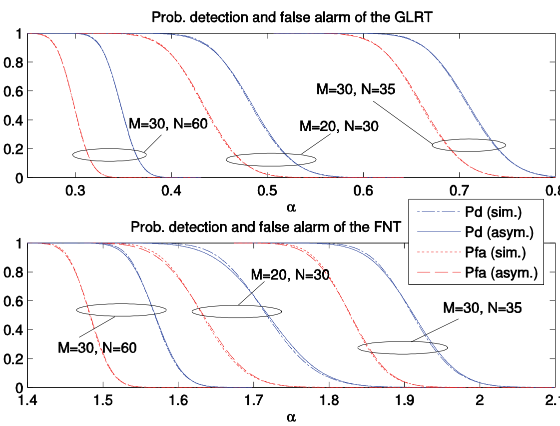

Fig. 1. Asymptotic and simulated probabilities of false alarm and detection for the GLRT (upper plot) and the FNT (lower plot) as a function of the threshold α.

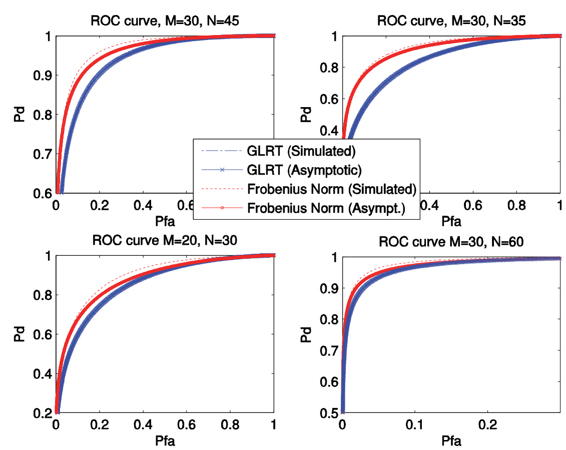

Fig. 2. Simulated vs. asymptotic ROC curves for different values of M, N. The Gaussian approximation tends to be more accurate for the GLRT than for the Frobenius norm test for moderate values of M, N.

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