TIKHONOV-GALERKIN STOCHASTIC SYSTEM IDENTIFICATION IN SO(3)

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ABSTRACT

We consider estimation of the angular velocity of a satellite undergoing noisy process dynamics. Modelling the attitude as a diffusion process evolving in $\mathrm{SO}(3)$, we propose an offline nonparametric estimator of the angular velocity over an interval based on noise free observations of the attitude. This is an ill-posed problem, and Tikhonov regularization is employed along with a Fourier-Galerkin method to solve the resulting stochastic Euler equation. A geometry preserving numerical scheme is proposed to simulate the attitude dynamics. Simulations are given which provide heuristics for choosing the regularization parameter in different SNR scenarios.

***Index Terms*—** Attitude estimation, Lie group integrator, Tikhonov regularization.

1. INTRODUCTION

In this paper we consider the problem of estimating the angular velocity of a rigid body whose attitude evolves in $\mathrm{SO}(3)$. Problems of this type arise for example in aerospace engineering [1], computer vision [12] and robotics [16].

Quaternion attitude representations are commonly used, but these suffer from important difficulties [3]. Unfortunately, those papers which employ an $\mathrm{SO}(3)$ attitude representation tend to also use deterministic kinematics [4], [13], [17]. Indeed there are very few papers which employ both stochastic kinematics and an $\mathrm{SO}(3)$ attitude representation, and those works don't consider angular velocity estimation [11], [15].

Often, angular velocity is not explicitly estimated as filtering approaches can be used to obtain attitude estimates from noisy measurements directly [5]. But these approaches have drawbacks. They employ quaternion attitude representations and they do not guarantee that the manifold structure is preserved in the presence of stochastic kinematics. One must consider continuous time kinematics and propose geometry preserving stochastic integrators for discrete time implementation.

Some have considered angular velocity estimation directly. An online approach using star tracker measurements was introduced in [4] using an $\mathrm{SO}(3)$ representation and deterministic kinematics. In [17] the second author developed an adaptive algorithm to estimate the attitude via the angular velocity, but the algorithm also employed deterministic kinematics. Recently, optical flow methods have also found application [6].

In this work, we use stochastic kinematics and model the attitude as a stochastic diffusion process in $\mathrm{SO}(3)$. Given noise free observations of the attitude over $[0, T]$, we aim to estimate the angular velocity over the interval. This is an ill-conditioned inverse problem, and so we apply Tikhonov regularization. We solve the stochastic Euler equation numerically using a classical Galerkin method. Such an approach should not be confused with the well known stochastic Galerkin method [7].

Testing the proposed estimator requires one to simulate the attitude diffusion numerically. In doing so it is vital that each simulated trajectory lies in the manifold. In this respect, the simulation problem is much harder than classical SDE simulation [10]. This area is very much stll under development, and we propose a novel scheme which can be interpreted in the framework of [14].

The remainder of the paper is organized as follows. Section 2 introduces the problem and establishes some notation. In Section 3 the Tikhonov problem is formulated and the Euler equation is derived, while Section 4 contains the Galerkin method for its practical solution. Geometric simulations of the SDE are provided in Section 5.1, and the performance of the Tikhonov-Galerkin estimator is studied in Section 5.2. Conclusions are drawn in Section 6.

2. SYSTEM IDENTIFICATION IN SO(3)

Starting by injecting Brownian motion noise increments with covariance $\sigma^2 I dt$ into well known deterministic kinematics $\dot{R}(t) = R(t) S(\omega_0(t))$, where $\omega_0(t)$ represents the angular velocity in radians per second, one obtains the following Ito equation for the attitude $R(t)$ [2], [11].

$$dR(t) = -\sigma^2 R(t) dt + R(t) S(\omega_0(t)) dt + \sigma R(t) S(dB(t)) \tag{1}$$

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where we have defined

$$S(a) := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

and $B(t) = (B_1(t), B_2(t), B_3(t))^T$ is a standard Brownian motion in $\mathbb{R}^3$.

The problem is to estimate the time varying parameter $\omega_0(t)$ and the process noise parameter σ from observations on $[0, T]$ given by $y(t) = R^T(t)\, y(0)$. The observation SDE is therefore

$$\begin{aligned} dy(t) &= -\sigma^2 y(t)\, dt + S(y(t))\, \omega_0(t)\, dt \\ &\quad + \sigma S(y(t))\, dB(t) \end{aligned} \tag{2}$$

We concentrate on the estimation of $\omega_0(t)$ on $[0, T]$ since σ^2 has a simple expression in terms of quadratic variation as follows. Using stochastic calculus, e.g. [9], and letting $[y_i, y_i]_t$ denote the quadratic variation at t of the i^{th} component of y, we find

$$\begin{aligned} \sum_{i=1}^{3} d[y_i, y_i]_t &= dy^T(t)\, dy(t) \\ &= -\sigma^2 dB^T(t)\, S^2(y(t))\, dB(t) \\ &= \sigma^2 dB^T(t)\, dB(t) - \sigma^2 \left(dB^T(t)\, y(t)\right)^2 \\ &= 2\sigma^2 dt \end{aligned} \tag{3}$$

where we have used the identity $S^2(y(t)) = y(t)\, y^T(t) - I$. Integrating (3) from 0 to T one obtains

$$\sigma^2 = \frac{1}{2T} \sum_{i=1}^{3} [y_i, y_i]_T \tag{4}$$

which can be simply approximated pathwise.

3. TIKHONOV PENALIZED PROBLEM

In this section we formally propose a Tikhonov penalized least squares problem to estimate $\omega_0(t)$ from observations $y(t), t \in [0, T]$, and derive the associated Euler equation. In light of (4) we treat σ as known. The Tikhonov penalty is necessary since ω_0 is infinite dimensional, see e.g. [8] for an introduction to inverse problems.

Throughout, let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ the associated inner product. We introduce the penalized least squares functional[1] based on (2), with derivative penalty controlled by a parameter $\alpha > 0$.

$$\begin{aligned} J(\omega) &= \frac{1}{T} \int_0^T \left\| \dot{y}(t) + \sigma^2 y(t) - S(y(t))\, \omega(t) \right\|^2 dt \\ &\quad + \frac{\alpha}{T} \int_0^T \|\dot{\omega}(t)\|^2 dt \end{aligned}$$

[1] The paths of y are not differentiable, however this heuristic definition allows us to derive an estimator from a weak formulation involving Itô integrals only.

Expanding and dropping constant terms,

$$\begin{aligned} J(\omega) &= \frac{1}{T} \int_0^T \left\| \sigma^2 y(t) - S(y(t))\, \omega(t) \right\|^2 dt \\ &\quad + \frac{2}{T} \int_0^T \left\langle dy(t), \sigma^2 y(t) - S(y(t))\, \omega(t) \right\rangle \\ &\quad + \frac{\alpha}{T} \int_0^T \|\dot{\omega}(t)\|^2 dt \end{aligned}$$

Now consider a perturbing function $\phi(t)$, and let $\epsilon > 0$. Proceeding formally, at a minimizer ω_α one has

$$\frac{d}{d\epsilon} J(\omega_\alpha + \epsilon\phi) \Big|_{\epsilon=0} = 0$$

Performing this computation gives

$$\begin{aligned} &\int_0^T \left\langle \sigma^2 y(t)\, dt - S(y(t))\, \omega_\alpha(t)\, dt + dy(t), S(y(t))\, \phi(t) \right\rangle \\ &= \alpha \int_0^T \left\langle \dot{\omega}_\alpha(t), \dot{\phi}(t) \right\rangle dt \end{aligned}$$

Since the adjoint of $S(y(t))$ is $-S(y(t))$ in this setting, and $S(y(t))\, y(t) = 0$, we obtain

$$\begin{aligned} &\int_0^T \left\langle S^2(y(t))\, \omega_\alpha(t)\, dt - S(y(t))\, dy(t), \phi(t) \right\rangle \\ &= \alpha \int_0^T \left\langle \dot{\omega}_\alpha(t), \dot{\phi}(t) \right\rangle dt \end{aligned}$$

Integrating by parts and assuming $\dot{\omega}_\alpha(0) = \dot{\omega}_\alpha(T) = 0$,

$$\begin{aligned} &\int_0^T \left\langle \alpha \ddot{\omega}_\alpha(t)\, dt + S^2(y(t))\, \omega_\alpha(t)\, dt, \phi(t) \right\rangle \\ &= \int_0^T \langle S(y(t))\, dy(t), \phi(t) \rangle \end{aligned}$$

We therefore arrive at the Euler equation

$$\alpha \ddot{\omega}_\alpha(t) + S^2(y(t))\, \omega_\alpha(t) = S(y(t))\, \dot{y}(t) \tag{5}$$

with Neumann boundary conditions $\dot{\omega}_\alpha(0) = \dot{\omega}_\alpha(T) = 0$.

4. NUMERICAL SOLUTION USING THE GALERKIN METHOD

To approximate the solution of (5), consider the weak form

$$\begin{aligned} &\int_0^T \left\langle \alpha \ddot{\omega}_\alpha(u) + S^2(y(u))\, \omega_\alpha(u), \Psi_j(u) \right\rangle du \\ &= \int_0^T \langle S(y(u))\, dy(u), \Psi_j(u) \rangle \end{aligned}$$

for an appropriate basis $\{\Psi_j\}_{j=0}^{\infty}$. Integration by parts yields

$$-\alpha\int_0^T\left\langle\dot{\omega}_\alpha(u),\dot{\Psi}_j(u)\right\rangle du + \int_0^T\left\langle S^2(y(u))\,\omega_\alpha(u),\Psi_j(u)\right\rangle du = \int_0^T\left\langle S(y(u))\,dy(u),\Psi_j(u)\right\rangle \quad (6)$$

To proceed we introduce a concrete basis, namely

$$\Psi_i(t)=\begin{cases}[\nu_i(t),0,0]^T & \text{if } 0\le i\le n-1\\ [0,\nu_{i-n}(t),0]^T & \text{if } n\le i\le 2n-1\\ [0,0,\nu_{i-2n}(t)]^T & \text{if } 2n\le i\le 3n-1\end{cases}$$

and $\{\nu_i\}_{i=0}^{\infty}$ is the Fourier cosine basis, i.e. $\nu_i(t)=\cos\left(\frac{\pi i t}{T}\right)$.

In this paper we consider estimators of ω_α of the form

$$\omega_\alpha^{(n)}(t)=\sum_{i=0}^{3n-1}c_i\Psi_i(t) \quad (7)$$

where $c_i\in\mathbb{R}$. Note that $\dot{\omega}_\alpha^{(n)}(0)=\dot{\omega}_\alpha^{(n)}(T)=0$.

Substituting (7) for ω_α in (6) and using the identity $S^2(y)=yy^T-I$ on the sphere yields

$$-\alpha\sum_{i=0}^{3n-1}c_i\int_0^T\left\langle\dot{\Psi}_i(u),\dot{\Psi}_j(u)\right\rangle du + \sum_{i=0}^{3n-1}c_i\int_0^T y^T(u)\,\Psi_i(u)\,y^T(u)\,\Psi_j(u)\,du - \sum_{i=0}^{3n-1}c_i\int_0^T\left\langle\Psi_i(u),\Psi_j(u)\right\rangle du = \int_0^T\left\langle S(y(u))\,dy(u),\Psi_j(u)\right\rangle,\quad j=0,\dots,3n-1$$

Scaling by $-\frac{1}{T}$ we obtain a $3n$ dimensional symmetric linear system for the coefficients,

$$Kc=F \quad (8)$$

where for $i,j\in\{0,\dots,3n-1\}$,

$$K_{i,j}=\begin{cases}a_{i,j} & \text{if } i\ne j\\ a_{i,j}+1 & \text{if } i=j=0 \bmod n\\ a_{i,j}+\frac{1}{2}+\alpha\frac{(j \bmod n)^2\pi^2}{2T^2} & \text{else}\end{cases}$$

with

$$a_{i,j}=-\frac{1}{T}\int_0^T y^T(u)\,\Psi_i(u)\,y^T(u)\,\Psi_j(u)\,du$$

and finally,

$$F_i=-\frac{1}{T}\int_0^T\left\langle S(y(u))\,dy(u),\Psi_i(u)\right\rangle$$

The form of K is a consequence of the following orthogonality relations,

$$\frac{1}{T}\int_0^T\left\langle\dot{\Psi}_i(u),\dot{\Psi}_j(u)\right\rangle du=\begin{cases}\frac{(j \bmod n)^2\pi^2}{2T^2} & \text{if } i=j\\ 0 & \text{if } i\ne j\end{cases}$$

Also we have

$$\frac{1}{T}\int_0^T\left\langle\Psi_i(u),\Psi_j(u)\right\rangle du=\begin{cases}1 & \text{if } i=j=0 \bmod n\\ \frac{1}{2} & \text{if } i=j\ne 0 \bmod n\\ 0 & \text{if } i\ne j\end{cases}$$

Experimentally the system matrix K is diagonally dominant, so we apply a diagonal preconditioner M on the left of (8) with entries

$$M_{i,i}=\frac{1}{K_{i,i}} \quad (9)$$

Experimentally this improves conditioning dramatically. For large n one might apply an iterative method such as the classical conjugate gradient method to the preconditioned system as it is symmetric and positive definite.

5. SIMULATIONS

In this section we first present an algorithm for the simulation of the SDE (2), before examining the performance of the Tikhonov-Galerkin estimator.

5.1. Geometric SDE Simulation

We propose the following algorithm for the simulation of (2), for $1\le k\le N$,

$$y_{k+1}=\exp\left(M_k^T\right)y_k \quad (10)$$

$$\exp\left(M_k^T\right)=I-\frac{\sin(\theta_k)}{\theta_k}M_k+\frac{1-\cos(\theta_k)}{\theta_k^2}M_k^2 \quad (11)$$

$$\theta_k=\|M_k\| \quad (12)$$

$$M_k=S_k+J_k \quad (13)$$

$$S_k=\sigma S(\Delta W_k)=\sigma\sqrt{\delta}S(\epsilon_k) \quad (14)$$

$$J_k=\frac{\delta}{2}\left(S(\omega(k\delta))+S(\omega(k\delta+\delta))\right) \quad (15)$$

where $\epsilon_k\overset{i.i.d.}{\sim}\mathcal{N}(0,I)$ and $\delta=\frac{T}{N}$ is a fixed step size which we choose to be $\delta=10^{-4}$ in all simulations to follow.

The algorithm (10) - (15) can be interpreted as a version of the stochastic Runge-Kutta Munthe-Kaas method developed in [14]. We use an Euler-Maruyama scheme combined with the trapezoidal rule to approximate the integral of $S(\omega(t))$ arising in the associated SDE in $\mathfrak{so}(3)$ [14]. Each iterate is then mapped to $SO(3)$ using the Rodriguez formula for the matrix exponential, so that the Lie group structure is preserved at each iteration. This implies for example that $\|y_k\|=1$ for all k.

5.2. Tikhonov-Galerkin Simulations

To evaluate the performance of the Tikhonov-Galerkin method, we first make some relevant definitions. Throughout, we take

$$\omega_0(t) := [7, 1, -5]^T + 5\cos\left(\frac{2\pi\rho t}{T}\right)[1, 1, 1]^T \qquad (16)$$

where ρ is the number of periods of $\omega_0(t)$ available. The relative mean integrated squared error measure (RMISE) is

$$\text{RMISE} = \frac{\int_0^T \mathbb{E}\left[\left\|\omega_\alpha^{(n)}(t) - \omega_0(t)\right\|^2\right] dt}{\int_0^T \|\omega_0(t)\|^2\, dt}$$

RMISE is estimated using sample averages over 100 trajectories, with $\delta = 10^{-4}$ and an $N = \frac{T}{\delta}$ point Riemann sum approximation for the integral.

A dimensional analysis of (2) and (5), noting $\omega_0(t)$ has dimensions of inverse time, reveals that a scale free regularization parameter is

$$\gamma := \alpha\sigma^4$$

With that in mind, the signal to noise ratio (in dB) in this setting is given by

$$SNR := 10\log_{10}\left(\frac{1}{T\sigma^4}\int_0^T \|\omega_0(t)\|^2\, dt\right)$$

Figure 1 demonstrates for $\rho = 1, n = 25$ the relationship between RMISE and γ for several SNR. Further studies are required to determine the optimal choice of n. Figures 2 and 3 demonstrate the pathwise approximation of our estimator under optimal γ (according to Figure 1) for 20dB SNR and 40dB SNR respectively.

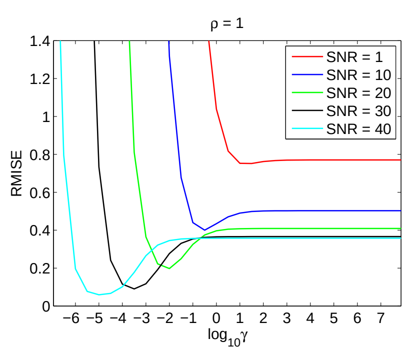

Fig. 1. RMISE against $\log_{10}(\gamma)$ for $\rho = 1$. $T = 1, \delta = 10^{-4}, n = 25$.

6. CONCLUSION

In this paper we developed an offline nonparametric estimator for the angular velocity of a rigid body evolving according to a diffusion process in SO(3). We introduced a Tikhonov penalized least squares problem and solved the resulting stochastic Euler equation using the Galerkin method. Diagonal preconditioning of the resulting system was found to be effective. A novel manifold-preserving numerical integrator was proposed for the SDE in order to examine the performance of the estimator.

We found that increasing the SNR resulted in a smaller optimal regularization parameter, with respect to the relative mean integrated squared error criterion. Additionally, a higher SNR resulted in better RMISE performance when the regularization parameter was chosen optimally. Pathwise approximation results for different SNR regimes were also shown.

Further investigation into the theoretical properties, including principled approaches for regularization parameter selection, as well as the effect of n and ρ will be the subject of future work.

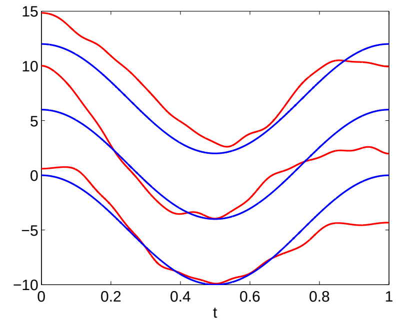

Fig. 2. One realization of $\omega_\alpha^{(25)}(t)$ against $\omega_0(t)$ for $\rho = 1$, $\delta = 10^{-4}$, SNR = 20dB and $\gamma = 10^{-2}$.

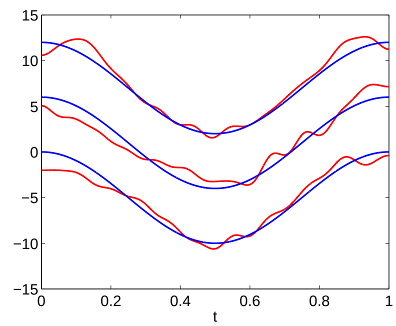

Fig. 3. One realization of $\omega_\alpha^{(25)}(t)$ against $\omega_0(t)$ for $\rho = 1$, $\delta = 10^{-4}$, SNR = 40dB and $\gamma = 10^{-5}$.

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