BLIND SIGNAL SEPARATION OF RATIONAL FUNCTIONS USING LÖWNER-BASED TENSORIZATION

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ABSTRACT

A novel deterministic blind signal separation technique for separating signals into rational functions is proposed, applicable in various situations. This new technique is based on a tensorization of the observed data matrix into a set of Löwner matrices. The obtained tensor can then be decomposed with a block tensor decomposition, resulting in a unique separation into rational functions under mild conditions. This approach provides a viable alternative to independent component analysis (ICA) in cases where the independence assumption is not valid or where the sources can be modeled well by rational functions, such as frequency spectra. In contrast to ICA, this technique is deterministic and not based on statistics, and therefore works well even with a small number of samples.

***Index Terms*—** Blind Signal Separation, higher-order tensor, Block Term Decomposition, rational functions, Independent Component Analysis

1. INTRODUCTION

The basic problem of blind signal separation (BSS) consists of the decomposition of the observed data matrix in an unknown linear combination of unknown source signals. The common way to solve BSS is through independent component analysis (ICA), starting with a hypothesis of independence of the source signals [1]. ICA has been widely applied, for example in biomedical sciences, image processing, telecommunications and finance [1, 2, 3, 4]. However, when the sources are not mutually statistically independent, one cannot justify the use of ICA and must resort to other techniques.

In [5] a new deterministic framework of block component analysis (BCA) was proposed. It is based on the use of the block term decomposition [6], as a low multilinear rank is a very natural structure for a lot of real-life data. In [7] a separation technique for exponential polynomials was given which can be used for smooth and periodic functions. We present a variant of this method for the separation of rational functions using Löwner matrices. These matrices have mainly been used for rational interpolation in system identification [8]. The developed technique in this paper is able to open up possibilities for applications in new domains.

A large class of functions and signals can be well approximated by rational functions such as the Gaussian distribution function or frequency spectra which represent a typical pole-like behavior. In such cases the assumption of rationality is not restrictive at all and very pertinent. Proof-of-concepts are given with simulations in the last section, together with an elaboration on possible applications.

1.1. Notation and basic definitions

Tensors can be seen as higher-order generalizations of vectors (denoted by a bold, lowercase letter, e.g., $\mathbf{a}$) and matrices (denoted by a bold, uppercase letter, e.g., $\mathbf{A}$). We denote a general N-th order tensor of size $I_1 \times I_2 \times \cdots \times I_N$ by a calligraphic letter as $\mathcal{A} \in \mathbb{K}^{I_1 \times I_2 \times \cdots \times I_N}$ ($\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$); it is a multi-way array with numerical values $a_{i_1 i_2 \cdots i_N} = \mathcal{A}(i_1, i_2, \ldots, i_N)$. Two main products are used in this paper. The mode-n tensor–matrix product between a tensor $\mathcal{A} \in \mathbb{K}^{I_1 \times I_2 \times \cdots \times I_N}$ and a matrix $\mathbf{B} \in \mathbb{K}^{J \times I_n}$ is defined as

$$(\mathcal{A} \cdot_n \mathbf{B})_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 i_2 \cdots i_N} b_{j i_n}.$$

The outer product of two tensors $\mathcal{A} \in \mathbb{K}^{I_1 \times I_2 \times \cdots \times I_N}$ and $\mathcal{B} \in \mathbb{K}^{J_1 \times J_2 \times \cdots \times J_M}$ is given as

$$(\mathcal{A} \otimes \mathcal{B})_{i_1 i_2 \cdots i_N j_1 j_2 \cdots j_M} = a_{i_1 i_2 \cdots i_N} b_{j_1 j_2 \cdots j_M}.$$

This research is funded by (1) a Ph.D. grant of the Agency for Innovation by Science and Technology (IWT), (2) Research Council KU Leuven: GOA/10/09 MaNet, CoE PFV/10/002 (OPTEC), (3) F.W.O.: project G.0427.10, G.0830.14N, G.0881.14N, (4) the Belgian Federal Science Policy Office: IUAP P7 (DYSCO II, Dynamical systems, control and optimization, 2012-2017), (5) EU: The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Advanced Grant: BIOTENSORS (no. 339804). This paper reflects only the authors' views and the Union is not liable for any use that may be made of the contained information.

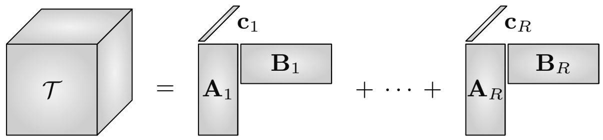


Fig. 1. A block term decomposition in rank-$(L_r, L_r, 1)$ terms

The Frobenius norm of a tensor $\mathcal{A}$ is denoted by $\|\mathcal{A}\|$, being the root of the sum of the squares of the tensor entries.

1.2. Basic tensor decompositions

A polyadic decomposition (PD) of a tensor $\mathcal{T}$ is given by a sum of R rank-1 tensors:

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \otimes \mathbf{a}_r^{(2)} \otimes \cdots \otimes \mathbf{a}_r^{(N)} \triangleq \left[\!\left[\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(N)} \right]\!\right]$$

It is called canonical (CPD) when R is the minimum number of terms for the decomposition to be exact, and this R is defined as the rank of the tensor. The mode-n rank of a tensor $\mathcal{T}$ is the dimension of the subspace spanned by its mode-n vectors. These vectors are constructed by fixing all but one index, e.g., $\mathbf{a} = \mathcal{A}(i_1, \ldots, i_{n-1}, :, i_{n+1}, \ldots, i_N)$. If the mode-1 rank, mode-2 rank and mode-3 rank of a third-order tensor are equal to L, M and N respectively, it is said to have trilinear rank (L, M, N). This becomes the multilinear rank when generalized to arbitrary order. The proposed technique in this paper is based on a specific instance of the general rank-(L_r, M_r, N_r) block term decomposition (BTD), namely the decomposition of a third order tensor $\mathcal{T} \in \mathbb{K}^{I_1 \times I_2 \times I_3}$ into a sum of rank-$(L_r, L_r, 1)$ terms:

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{E}_r \otimes \mathbf{c}_r, \tag{1}$$

with the matrix $\mathbf{E}_r \in \mathbb{K}^{I_1 \times I_2}$ having rank L_r and vector $\mathbf{c}_r \in \mathbb{K}^{I_3}$ being nonzero. We can verify that the multilinear rank of term r is $(L_r, L_r, 1)$, while each $\mathbf{E}_r$ can be factorized to give

$$\mathcal{T} = \sum_{r=1}^{R} (\mathbf{A}_r \mathbf{B}_r^{\mathrm{T}}) \otimes \mathbf{c}_r, \tag{2}$$

with $\mathbf{A}_r \in \mathbb{K}^{I_1 \times L_r}$ and $\mathbf{B}_r \in \mathbb{K}^{I_2 \times L_r}$; illustrated in Figure 1.

One main feature of tensor decompositions is that they are unique under mild conditions. Results for the CPD can be found in [9, 10, 11] and references therein, while [6, 7] discuss uniqueness for the general BTD and the rank-$(L_r, L_r, 1)$ BTD, respectively. These results can be used to deliver a unique separation of a mixture of rational sources. For a more elaborate survey on tensor decompositions we refer to [12, 13]. An example of the use of the rank-$(L_r, L_r, 1)$ BTD is found in [**?**].

2. LÖWNER-BASED BLIND SIGNAL SEPARATION

We propose a new technique for blind signal separation where the source signals are represented as rational functions. We first briefly review Löwner matrices and discuss their application to BSS. We then introduce the tensorization, decomposition and reconstruction, in the spirit of the technique for separating exponential polynomials in [7].

2.1. Löwner matrices

A Löwner matrix is a type of low displacement rank matrix, being a matrix depending on only $\mathcal{O}(n)$ parameters [14]:

Definition 1 (Löwner)**.** *Given a function $f(t)$ sampled on N different points t_i. We partition the point set $T = \{t_1, t_2, \ldots, t_N\}$ into two point sets $X = \{x_1, x_2, \ldots, x_\alpha\}$ and $Y = \{y_1, y_2, \ldots, y_{N-\alpha}\}$, and define the Löwner matrix* $\mathbf{L}$ *with*

$$l_{i,j} = \frac{f(x_i) - f(y_j)}{x_i - y_j}. \tag{3}$$

If N is even and $\alpha = N/2$, then $\mathbf{L}$ is square. The following theorem gives the connection between Löwner matrices and rational functions [15], with the degree of a rational function defined as the maximum of the degrees of the polynomial in its numerator and the polynomial in its denominator:

Theorem 1. *Given a Löwner matrix $\mathbf{L}$ of size $I \times J$ associated to a function $f(t)$ on point set T with $N = I + J$. If $f(t)$ is a rational function with degree δ and if $I, J \geq \delta$, then $\mathbf{L}$ has rank δ.*

The proof can be found in [16, 17], however it is easy to verify for $\delta = 1$: substituting $f(t) = \gamma \frac{t+a}{t+b}$ into (3) gives $l_{i,j} = \gamma(\beta - \alpha) \cdot \frac{1}{x_i + \beta} \cdot \frac{1}{y_j + \beta}$ which is a rank-one matrix. A further detailed discussion on Löwner matrices is found in [18].

2.2. Blind Signal Separation

We consider the following data model in linear blind signal separation (BSS):

$$\mathbf{X} = \mathbf{M} \cdot \mathbf{S} + \mathbf{N}, \tag{4}$$

with $\mathbf{X} \in \mathbb{K}^{K \times N}$ containing the observed data with K known sensor signals, $\mathbf{S} \in \mathbb{K}^{R \times N}$ containing the R unknown source signals, $\mathbf{M} \in \mathbb{K}^{K \times R}$ the mixing matrix and $\mathbf{N} \in \mathbb{K}^{K \times N}$ the additive noise. The general goal in BSS is to recover the unknown sources in $\mathbf{S}$ and/or the unknown mixing vectors in $\mathbf{M}$, given only the sensor data $\mathbf{X}$. Let us now assume that the sources are rational functions and describe a technique to obtain a unique separation of a linear mixture of them. Each row in $\mathbf{S}$ is represented by a rational function of degree L_r. We assume single poles for simplicity (coinciding poles can be used too) and use a partial fraction representation:

$$s_r(t) = \sum_{l_r=1}^{L_r} \frac{\alpha_{l_r,r}}{t + \beta_{l_r,r}}. \tag{5}$$

We assume that the numerator degree is strictly smaller than the denominator degree. Otherwise, while Theorem 1 would still hold, the decomposition (explained in the next section) would not be unique.

2.3. Löwnerization and decomposition

Let us now describe the technique to separate the mixed rational functions. Each row of $\mathbf{X}$ is transformed into a $(I \times J)$ Löwner matrix with $I + J = N$ and then stacked into a tensor $\mathcal{X}$. We call this transformation a *Löwnerization* and the resulting tensor $\mathcal{X}$ with size $I \times J \times K$ is called the Löwner tensor. The BSS-model in (4) is linear, so the K Löwner matrices of the sensors become linear combinations of the Löwner representations of the sources. One can write

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{L}_r \otimes \mathbf{m}_r,$$

in which each $\mathbf{L}_r$ is the Löwner matrix for source r and each $\mathbf{m}_r$ is the mixing vector for source r. As described above, the matrices $\mathbf{L}_r$ have low rank if the associated functions are rational functions of limited degree L_r. We assume $\min(I, J) \geq \max_r L_r$. Factorizing the matrices $\mathbf{L}_r$ of ranks L_r into $\mathbf{A}_r\mathbf{B}_r^{\mathrm{T}}$ for $1 \leq r \leq R$, we obtain the following decomposition of $\mathcal{X}$:

$$\mathcal{X} = \sum_{r=1}^{R} (\mathbf{A}_r\mathbf{B}_r^{\mathrm{T}}) \otimes \mathbf{m}_r, \tag{6}$$

which is precisely the decomposition in rank-$(L_r, L_r, 1)$ terms from equations (1) and (2). The partial fractions from (5) reappear in $\mathbf{A}_r$ and $\mathbf{B}_r$, and one can prove that the decomposition in (6) is unique under reasonably mild conditions [7], even for coinciding poles. In applications, the block term decomposition of the tensorized dataset can be carried out for example with Tensorlab [19].

2.4. Reconstruction of the sources

From the decomposition one directly recovers an estimate of $\mathbf{M}$ in the third mode (called $\hat{\mathbf{M}}$). In order to also recover $\mathbf{S}$, one can invert $\hat{\mathbf{M}}$ to obtain $\hat{\mathbf{S}} = \hat{\mathbf{M}}^{-1}\mathbf{M}\mathbf{S} + \hat{\mathbf{M}}^{-1}\mathbf{N}$.

We can also use the mode-1 and mode-2 information from the decomposition, i.e., the reconstructed $\hat{\mathbf{L}}_r$. In [7] it was shown how it is possible to average along the antidiagonals in the Hankel matrices to recover the exponential polynomial source signals. In the case of Löwner matrices, we can project the reconstructed $\hat{\mathbf{L}}_r$ onto the nearest Löwner matrix in a least-squares sense. From equation (3), changing f into $\mathbf{s}_r$ and $\mathbf{L}$ into $\hat{\mathbf{L}}_r$, one obtains a linear system with unknown $\mathbf{s}_r$ which can be solved in a least-squares sense:

$$\mathbf{s}_r = \arg\min_{\mathbf{s}_r} \frac{1}{2} \left\| \mathrm{vec}(\hat{\mathbf{L}}_r) - \mathbf{F}\mathbf{s}_r \right\|^2 \quad \text{for } 1 \leq r \leq R,$$

in which $\mathbf{F} \in \mathbb{K}^{IJ \times N}$ is constructed from equation (3). With the point set $T = \{x_1, y_1, x_2, y_2\}$ for example, the linear system becomes the following:

$$\begin{bmatrix} (\hat{\mathbf{L}}_r)_{1,1} \\ (\hat{\mathbf{L}}_r)_{2,1} \\ (\hat{\mathbf{L}}_r)_{1,2} \\ (\hat{\mathbf{L}}_r)_{2,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{(x_1-y_1)} & \frac{-1}{(x_1-y_1)} & 0 & 0 \\ 0 & \frac{-1}{(x_2-y_1)} & \frac{1}{(x_2-y_1)} & 0 \\ \frac{1}{x_1-y_2} & 0 & 0 & \frac{-1}{x_1-y_2} \\ 0 & 0 & \frac{1}{x_2-y_2} & \frac{-1}{x_2-y_2} \end{bmatrix} \begin{bmatrix} s_r(x_1) \\ s_r(y_1) \\ s_r(x_2) \\ s_r(y_2) \end{bmatrix}$$

The reconstructed signals are determined up to a permutation, scaling and a constant vector $\boldsymbol{\mu}$. The latter can be recovered with the following optimization:

$$\boldsymbol{\mu} = \arg\min_{\boldsymbol{\mu}} \frac{1}{2} \left\| \mathbf{X} - \hat{\mathbf{M}} \left(\hat{\mathbf{S}} - \boldsymbol{\mu}\mathbf{e}^{\mathrm{T}} \right) \right\|^2$$

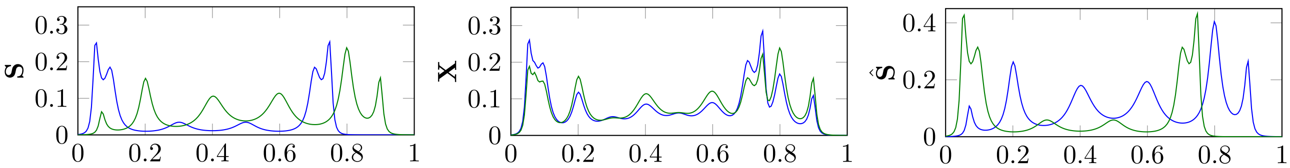
with $\mathbf{e} = [1, \ldots, 1]$, which again yields a linear system of equations. Note that this solution method enables the recovery of sources in the case of less sensors than sources, while the first solution method is not applicable when $K < R$ or when the mixing matrix does not have full column rank.

3. RESULTS AND DISCUSSION

3.1. Simulations

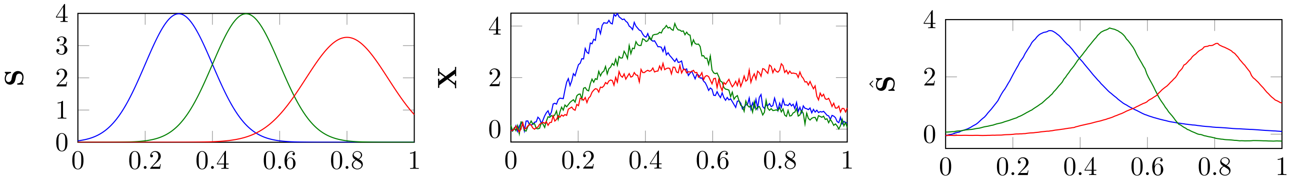
We provide two experiments. Each signal is sampled in $t \in [0, 1]$ with $N = 200$ data points. The first experiment considers the separation of two rational functions without measurement noise. Each function has six complex conjugated pole pairs with the real parts in $[0, 1]$, see Figure 2, and is thus of degree 12. The mixing matrix $\mathbf{M} = [1, 0.7; 0.7, 1]$ is used. For the second experiment, we show that Gaussian signals can be well approximated. We separate three Gaussians mixed with $\mathbf{M} = [1, 0.5, 0.3; 0.5, 0.9, 0.2; 0.4, 0.5, 0.7]$. The means of the Gaussian functions are 0.3, 0.5 and 0.8 and the variances are 0.01, 0.01 and 0.015. The observations are perturbed by noise to obtain a signal-to-noise ratio of 25 dB.

The technique is applied using a Löwnerization with an interleaved partitioning of the sample points, i.e. $T = \{x_1, y_1, x_2, y_2, \ldots, x_{100}, y_{100}\}$. The tensor is decomposed using `btd_nls` from Tensorlab. For the first experiment, $L_1 = L_2 = 12$ is used. For the second experiment, $L_1 = L_2 = L_3 = 2$ is used, i.e., the Gaussians are approximated by rational functions of degree 2. In practical cases, a trial-and-error method can be used to deduct L_r, knowing that the multilinear rank of $\mathcal{X}$ is bounded by $(\sum_{r=1}^{R} L_r, \sum_{r=1}^{R} L_r, 1)$; the choice of L_r need not be very precise [7]. The sources are found from the reconstructed $\mathbf{L}_r$ (the second method in subsection 2.4). The sources can only be uniquely reconstructed up to scaling and permutation, being the standard indeterminacies in BSS. In the first experiment a perfect reconstruction is found, while in the second, we have a relative error on the mixing matrix (defined as the relative difference in Frobenius norm after optimal scaling and permutation) of 0.1491; see Figures 2 and 3. Note that ICA does not work for these experiments: FastICA returns a relative error of 0.35 and 0.68, respectively. We also include an experiment in Figure 4 for different signal-to-noise-ratios with Gaussian i.i.d. noise for a mixture of two rational sources with poles $0.2 \pm 0.05j$ and $0.8 \pm 0.05j$, respectively. The median across 100 experiments of the relative difference in Frobenius norm between the real and reconstructed $\mathbf{M}$ and $\mathbf{S}$ is shown.



(a) The original sources $\mathbf{S}$ (b) The observed data $\mathbf{X}$ (c) The reconstructed sources $\hat{\mathbf{S}}$

Fig. 2. Example of a separation of a mixture of two rational functions, determined up to scaling and permutation



(a) The original sources $\mathbf{S}$ (b) The observed data $\mathbf{X}$ (c) The reconstructed sources $\hat{\mathbf{S}}$

Fig. 3. Example of separating three Gaussian functions, while approximating them by rational functions of degree 2

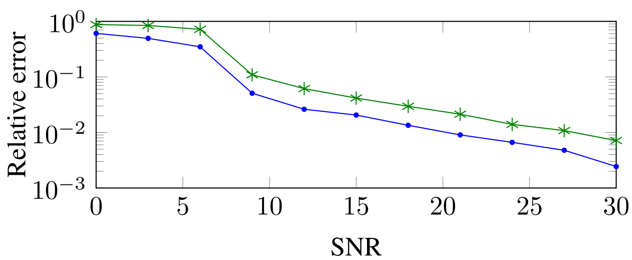


Fig. 4. The relative error for the mixing matrix ( ) and the source signals ( ). It is defined as the relative difference in Frobenius norm after optimal scaling and permutation.

Note that this technique also works for $K > R$ and even for $K < R$. In the former, a preprocessing step using SVD can be used [7]. In the latter, uniqueness ensures a correct solution even when there are less sensors than sources.

3.2. Application examples

We highlight a specific application in the domain of chemometrics. In excitation-emission spectroscopy [20, 21], different chemical components (e.g. tyrosine, tryptophane and phenylalanine) contribute to the measurements in a linear combination due to Beer-Lambert's law. Their excitation and emission spectra can be separated from the spectroscopy measurements with a canonical polyadic decomposition based on many different samples (for different concentrations) stacked into a tensor. The Löwnerization technique can reduce the number of samples needed to only a single sample as the excitation (or emission) spectra can be seen as the sources of a blind signal separation problem and well approximated by rational functions. Second, the technique enables the analysis when one only has a single sample at his or her disposal.

As the example indicates, rational functions are perfectly suited to model frequency spectra because of their smooth character and pole-like behavior. Biomedical signals can be processed too, as electrocardiography signals for example contain a low intrinsic structure. The separation of mother and fetal electrocardiograms [22] could then be done with Löwnerization.

3.3. Discussion and future research

In this paper we limited the discussion to single poles. Rational functions with coinciding poles can be separated too, while uniqueness still holds. A decomposition equivalent to the Vandermonde decomposition of Hankel matrices [23] is expected to be possible too for Löwner matrices. Our technique has similarities to the one described in [7]; future research is needed to extract the relations between the different techniques (a major advantage of our technique is that it does not need equidistant sample points in contrast to the technique in [7]) and to reveal other deterministic BSS-algorithms applicable as valid alternatives to the ICA-algorithm.

4. CONCLUSION

We have proposed a technique for separating a mixture into rational source functions based on the Löwnerization of the observed data matrix, as a new method for blind signal separation. This is an alternative to the popular independent component analysis technique when mutual statistical independence in the sources is not present or when the sources are smooth and have a pole-like behavior such as frequency spectra. The technique makes use of the tensor block term decomposition of which uniqueness results provide unique reconstructability of the rational sources. This deterministic technique works well even with a limited number of samples. Derivation details and real-life illustrations will be given in a full paper.

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