A QUATERNION FREQUENCY ESTIMATOR FOR THREE-PHASE POWER SYSTEMS

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ABSTRACT

Motivated by the need for accurate frequency estimation in power systems, a novel algorithm for estimating the fundamental frequency of both balanced and unbalanced three-phase power systems, which is robust to noise and distortions, is developed. This is achieved through the use of quaternions in order to provide a unified framework for joint modeling of voltage measurements from all the phases of a three-phase system. Next, the recently introduced $\mathbb{HR}$-calculus is employed to derive a state space estimator based on the quaternion extended Kalman filter (QEKF). The proposed algorithm is validated over a variety of case studies using both synthetic and real-world data.

Index Terms— Three-phase power systems, frequency estimation, quaternion state space modeling, quaternion-valued signal processing, quaternion Kalman filtering.

1. INTRODUCTION

Deviations from the nominal system frequency adversely affect the components of the power grid [1], such as compensators and loads, resulting in harmful operating conditions that can propagate throughout the network. Frequency stability is therefore one of the most important factors in power quality [2]. Accurate frequency tracking is a prerequisite to ensuring frequency stability of the grid; furthermore, real-time frequency tracking reveals essential information about the dynamics of the power grid, such as power generation-consumption mismatch. Future envisioned smart grid technologies will incorporate distributed power generation based on renewable energy sources, where the wide-area grid can be divided into a number of smaller self-contained sections named micro-grids, with some micro-grids disconnecting from the wide-area grid for prolonged lengths of time, referred to as islanding. Perfect synchrony is needed to dynamically manage these micro-grids, which requires robust frequency estimators [3]-[5].

The importance of frequency estimation in power grids has motivated the development of a variety of algorithms dedicated to this cause. Early approaches were based on the use of the voltage measurements from a single phase [6]-[7]. Although all phases of a three-phase system have the same frequency, approaches based on a single phase cannot fully characterize three-phase power systems and can lead to non-unique solutions [8]; for instance, when one or two of the phases that are selected for frequency estimation experience a sudden reduction in the voltage or a short circuit, referred to as voltages sags [9]. To deal with all three phases simultaneously, the Clarke transform maps the measured three-phase voltages onto the complex domain [10]-[13]; however, this approach is optimal only when the three-phase system is balanced [3]-[5].

Quaternions provide a natural framework for processing three- and four-dimensional signals and are gaining increasing popularity in engineering applications [14]-[19]. One major development that has lead to the resurgence of quaternions in signal processing is the introduction of the $\mathbb{HR}$-calculus [20] which allows the derivatives of both analytic and non-analytic quaternion-valued functions to be calculated directly in the quaternion domain. Based on the $\mathbb{HR}$-calculus, a class of novel quaternion Kalman filtering algorithms, including the strictly linear QEKF, have been presented in [21].

In this work, a novel frequency estimation algorithm suitable for both balanced and unbalanced three-phase power systems is developed. This is achieved by exploiting the multi-dimensional nature of quaternions to make possible the full characterization of three-phase power systems in the three-dimensional domain, where they naturally reside. This both eliminates the need for the Clarke transform and makes possible the use of the QEKF to estimate the frequency of the system. The proposed approach is verified over a range of simulations in practical power system scenarios using both synthetic and real-world data.

2. BACKGROUND

The instantaneous voltages of each phase in a three-phase power system are given by [22]

$$\begin{aligned} v_{a,n} =& V_{a,n}\sin\left(2\pi f\Delta T n + \phi_{a,n}\right) \\ v_{b,n} =& V_{b,n}\sin\Big(2\pi f\Delta T n + \phi_{b,n} + \frac{2\pi}{3}\Big) \\ v_{c,n} =& V_{c,n}\sin\Big(2\pi f\Delta T n + \phi_{c,n} + \frac{4\pi}{3}\Big) \end{aligned} \tag{1}$$

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where $V_{a,n}$, $V_{b,n}$, and $V_{c,n}$ are the instantaneous amplitudes, $\phi_{a,n}$, $\phi_{b,n}$, and $\phi_{c,n}$ represent the instantaneous phases, and f is the system frequency, while $\Delta T = 1/f_s$ is the sampling interval with f_s denoting the sampling frequency. The Clarke transform, given by [22]

$$\begin{bmatrix} v_{0,n} \\ v_{\alpha,n} \\ v_{\beta,n} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} v_{a,n} \\ v_{b,n} \\ v_{c,n} \end{bmatrix} \tag{2}$$

maps the three-phase voltages onto a new domain where they are represented by the complex number $v_n = v_{\alpha,n} + iv_{\beta,n}$, while $v_{0,n}$ is ignored in most practical applications. In a balanced three-phase system, $V_n = V_{a,n} = V_{b,n} = V_{c,n}$ and $\phi_n = \phi_{a,n} = \phi_{b,n} = \phi_{c,n}$, resulting in $v_{0,n} = 0$ and $v_n = \sqrt{\frac{3}{2}} V_n e^{i(2\pi f \Delta T n + \phi_n)}$, which can be expressed by the first order regression

$$v_n = e^{i2\pi\Delta T} v_{n-1} \tag{3}$$

where we refer to $e^{i2\pi\Delta T}$ as the phase incrementing element. The frequency of the system can then be estimated using standard linear complex Kalman filters employing the state space model given in Algorithm 1, where x_n is the phase incrementing element, $\mathbf{u}_n$ the state evolution noise, and w_n the observation noise [5].

Algorithm 1 Complex-valued state space model (C-SS)

State evolution equation: $\begin{bmatrix} x_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_n v_n \end{bmatrix} + \mathbf{u}_n$

Observation equation: $v_n = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ v_n \end{bmatrix} + w_n$

Estimate of frequency: $\hat{f}_n = \frac{1}{2\pi\Delta T} \Im\left(\ln\left(x_n\right)\right)$

This approach is proven to be optimal when the three-phase system is balanced. When the system is unbalanced, $v_{0,n} \neq 0$ and contains information regarding the system frequency; however, due to the two-dimensional nature of complex numbers, it is excluded from the analysis. Moreover, for unbalanced three-phase systems

$$v_n = V_{1,n} e^{i(2\pi f \Delta T n)} + V_{2,n} e^{-i(2\pi f \Delta T n)}$$

where $V_{1,n}, V_{2,n} \in \mathbb{C}$, so that the linear fist order regression in (3) is no-longer adequate to express v_n. To this end, we employ quaternions in order to model the voltages of thee-phase power systems directly in the three-dimensional space without the use of the Clarke transform.

Quaternions are an associative, non-commutative, division algebra denoted by $\mathbb{H}$. A quaternion variable $q \in \mathbb{H}$ consists of a real part $\Re(q)$ and a three-dimensional imaginary part or pure quaternion, $\Im(q)$, which comprises three imaginary components $\Im_i(q)$, $\Im_j(q)$, and $\Im_k(q)$; therefore, q can be expressed as

$$\begin{aligned} q = \Re(q) + \Im(q) &= \Re(q) + \Im_i(q) + \Im_j(q) + \Im_k(g) \\ &= q_r + iq_i + jq_j + kq_k \end{aligned}$$

where $q_r, q_i, q_j, q_k \in \mathbb{R}$, while i, j, and k are imaginary units obeying the following product rules

$$\begin{gathered} ij = k, jk = i, ki = j, \\ i^2 = j^2 = k^2 = ijk = -1. \end{gathered}$$

The involution of $q \in \mathbb{H}$ around $\eta \in \mathbb{H}$ is defined as $q^\eta \triangleq \eta q \eta^{-1}$ and can be seen as the quaternion equivalent of the complex conjugate, as it can be used to express the real-valued components of a quaternion number, $q \in \mathbb{H}$, as [20]-[21],[23]

$$\begin{aligned} q_r &= \frac{1}{4}\left(q + q^i + q^j + q^k\right) \quad & q_i &= \frac{1}{4i}\left(q + q^i - q^j - q^k\right) \\ q_j &= \frac{1}{4j}\left(q - q^i + q^j - q^k\right) \quad & q_k &= \frac{1}{4k}\left(q - q^i - q^j + q^k\right). \end{aligned} \tag{4}$$

The quaternion conjugate is also an involution and is defined as

$$q^* = \Re(q) - \Im(q) = \frac{1}{2}\left(q^i + q^j + q^k - q\right) \tag{5}$$

while the norm of $q \in \mathbb{H}$ is given by

$$|q| = \sqrt{qq^*} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}.$$

The expressions in (4) establish a relation between the augmented quaternion variable, $[\mathbf{q}, \mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k]^T \in \mathbb{H}^4$, and the real-valued vector $[\mathbf{q}_r, \mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k]^T \in \mathbb{R}^4$, which has been instrumental in the development of the quaternion augmented statistics [24]-[25] and the $\mathbb{HR}$-calculus [20]. The augmented quaternion statistics in conjunction with the $\mathbb{HR}$-calculus have led to the development of a class of quaternion Kalman filters including the strictly linear QEKF [21] that operates akin to its complex-valued counterpart, with the difference that the Jacobian of the state evolution function is calculated by the $\mathbb{HR}$-calculus. For example, $\frac{\partial q^*}{\partial q} = -0.5$, which is a consequence of (5) and is in contrast with the results in the complex domain.

A quaternion $q \in \mathbb{H}$ can alternatively be expressed by its polar presentation given by [16]

$$q = |q| e^{\xi\theta} = |q|\big(\cos(\theta) + \xi \sin(\theta)\big)$$

where

$$\xi = \frac{\Im(q)}{|\Im(q)|} \text{ and } \theta = \text{atan}\left(\frac{|\Im(q)|}{\Re(q)}\right).$$

Moreover, it is straightforward to prove that the $\sin(\cdot)$ and $\cos(\cdot)$ functions can be expressed as

$$\sin(\theta) = \frac{1}{2\xi}\left(e^{\xi\theta} - e^{-\xi\theta}\right), \ \cos(\theta) = \frac{1}{2}\left(e^{\xi\theta} + e^{-\xi\theta}\right) \tag{6}$$

where $\xi^2 = -1$.

3. QUATERNION FREQUENCY ESTIMATOR

The three phase voltages in (1) are now combined together to generate the pure quaternion signal

$$q_n = iv_{a,n} + jv_{b,n} + kv_{c,n} \tag{7}$$

where all the elements of q_n have the same frequency; therefore, analytical geometry dictates that q_n traces an ellipse in a subspace (one plane) of the three-dimensional imaginary subspace of $\mathbb{H}$ [26]. This is shown in Figure 1, where the system voltages of a balanced and an unbalanced three-phase system are presented.

Without loss of generality and in order to simplify our analysis we define a new set of imaginary units, $\{\zeta, \zeta', \zeta''\}$ such that

$$\zeta\zeta' = \zeta'', \;\; \zeta'\zeta'' = \zeta, \;\; \zeta''\zeta = \zeta'. \tag{8}$$

The ζ and ζ' imaginary units are designed to reside in the same plane as q_n, which results in ζ'' being normal to this plane. An arbitrary ellipse in the $\zeta - \zeta'$ plane can then be expressed as

$$q_n = \zeta A_n \sin(2\pi f \Delta T n + \phi_{\zeta,n}) + \zeta' B_n \sin(2\pi f \Delta T n + \phi_{\zeta',n})$$

where $A_n, B_n \in \mathbb{R}$, are instantaneous amplitudes and $\phi_{\zeta,n}, \phi_{\zeta',n}$ are instantaneous phases. The expression above can be rearranged using (8) to give

$$\begin{aligned} q_n =& \big(A_n \sin(2\pi f \Delta T n + \phi_{\zeta,n}) \\ &+ \zeta'' B_n \sin(2\pi f \Delta T n + \phi_{\zeta',n})\big)\zeta. \end{aligned} \tag{9}$$

Given that $\zeta''^2 = -1$, upon replacing the $\sin(\cdot)$ and $\cos(\cdot)$ functions with their polar representations from (6), the expression in (9) yields

$$\begin{aligned} q_n =& \Big(\frac{A_n}{2\zeta''}\Big)\big(e^{\zeta''(2\pi f \Delta T n + \phi_{\zeta,n})} - e^{-\zeta''(2\pi f \Delta T n + \phi_{\zeta,n})}\big)\zeta \\ &+ \Big(\frac{B_n}{2}\Big)\big(e^{\zeta''(2\pi f \Delta T n + \phi_{\zeta',n})} - e^{-\zeta''(2\pi f \Delta T n + \phi_{\zeta',n})}\big)\zeta. \end{aligned}$$

Factoring out the terms $e^{\zeta''(2\pi \; f \Delta T n)}$ and $e^{-\zeta'' 2\pi f \Delta T n)}$, the expression above can be rearranged to give

$$\begin{aligned} q_n =& \underbrace{\Big(\frac{A_n e^{\zeta''(\phi_{\zeta,n})}}{2\zeta''} + \frac{B_n e^{\zeta''(\phi_{\zeta',n})}}{2}\Big) e^{\zeta''(2\pi f \Delta T n)} \zeta}_{q_n^+} \\ & - \underbrace{\Big(\frac{A_n e^{-\zeta''(\phi_{\zeta,n})}}{2\zeta''} + \frac{B_n e^{-\zeta''(\phi_{\zeta',n})}}{2}\Big) e^{-\zeta''(2\pi f \Delta T n)} \zeta}_{q_n^-} \end{aligned}$$

where q_n has been divided into the two counter-rotating signals q_n^+ and q_n^-, which can be expressed by the corresponding fist order quaternion linear regressions

$$q_n^+ = e^{\zeta''(2\pi f \Delta T)} q_{n-1}^+ \;\text{ and }\; q_n^- = e^{-\zeta''(2\pi f \Delta T)} q_{n-1}^-. \tag{10}$$

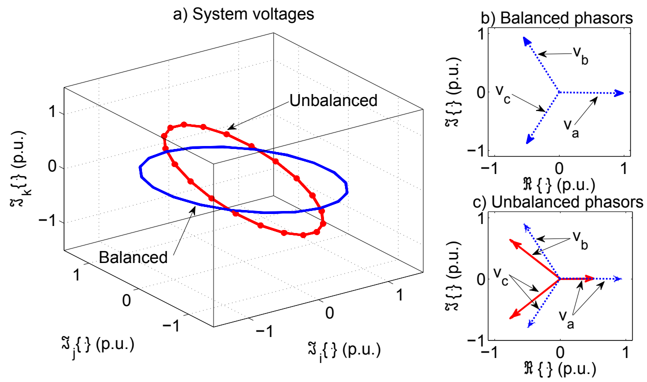

Fig. 1. Geometric view of the system voltages, q_n, and the corresponding phasor diagrams of a balanced and an unbalanced three-phase system: a) system voltages, b) phasor representation of a balanced three-phase system, c) phasor representation of an unbalanced three-phase system. Solid red lines represent an unbalanced system, while dashed blue lines lines represent a balanced system.

Taking into account the linear regressions in (10), where the phase incrementing element of q_n^+ is the quaternion conjugate of the phase incrementing element of q_n^-, a state space model for q_n is proposed in Algorithm 2, where $\varphi_n = e^{\zeta'' 2\pi f \Delta T}$, $\mathbf{g}_n$ is the state evolution noise, and r_n is the observation noise. Note that Algorithm 2 can be implemented using the strictly linear QEKF presented in [21].

Algorithm 2 Quaternion-valued state space model (Q-SS)

State evolution equation: $\begin{bmatrix} \varphi_{n+1} \\ q_{n+1}^+ \\ q_{n+1}^- \end{bmatrix} = \begin{bmatrix} \varphi_n \\ \varphi_n q_n^+ \\ \varphi_n^* q_n^- \end{bmatrix} + \mathbf{g}_n$

Observation equation: $q_n = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_n \\ q_n^+ \\ q_n^- \end{bmatrix} + r_n$

Estimate of frequency: $\hat{f}_n = \frac{1}{2\pi \Delta T} \Im \left(\ln \left(\varphi_n \right) \right)$

Remark 1. For a balanced three-phase system it can be shown that

$$\begin{aligned} q_n =& \underbrace{\Big(i + j\cos\big(\frac{2\pi}{3}\big) + k\cos\big(\frac{4\pi}{3}\big)\Big)}_{\sqrt{1.5}\zeta} V_n \sin(2\pi f \Delta T n + \phi_n) \\ &+ \underbrace{\Big(j\sin\big(\frac{2\pi}{3}\big) + k\sin\big(\frac{4\pi}{3}\big)\Big)}_{\sqrt{1.5}\zeta'} V_n \cos(2\pi f \Delta T n + \phi_n) \end{aligned}$$

which represents a circle in the $\zeta - \zeta'$ plane and describes only one rotating element (q_n^+ or q_n^-, dependent on the three-phase being positive or negative sequence); therefore q_n^+ and q_n^- can be used to indicate faults in the system.

4. SIMULATIONS

The performance of the proposed frequency estimator was validated over a range of practical power grid scenarios. In all the simulations, the sampling frequency was $f_s = 1$ kHz, the system frequency was $f = 50$ Hz, the system voltages were normalized, and the voltage measurements were assumed to be corrupted by white Gaussian noise with signal-to-noise ratio (SNR) of 50dB.

In the first set of simulations, a system operating under balanced conditions suddenly suffered a voltage sag characterized by a 50% reduction in the amplitude of $v_{a,n}$ and a 20 degree shift in the phases of $v_{b,n}$ and $v_{c,n}$; furthermore, the frequency of the system experienced a step increase of 2 Hz. The voltages of the three-phase system and the estimates obtained by the Q-SS and C-SS algorithms are shown in Figure 2. Notice that the proposed Q-SS algorithm accurately estimates the frequency of the system under both balanced and unbalanced operating conditions, while under unbalanced operating conditions the C-SS algorithm experienced oscillatory errors at twice the system frequency.

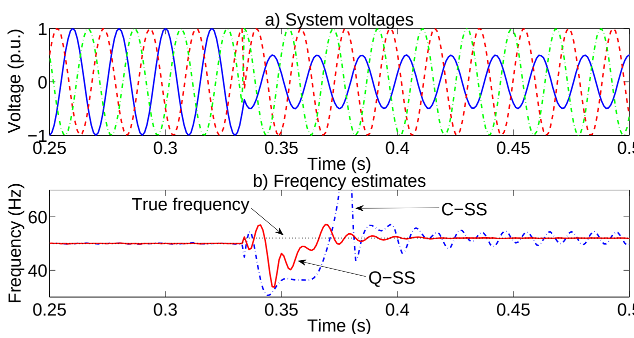

Fig. 2. Frequency estimation under balanced and unbalanced operating conditions: a) system voltages, b) estimates of the system frequency.

In the second set of simulations, the unbalanced three-phase system in the first simulation experienced a frequency rise (*cf.* decay) at the rate of 5 Hz/s, a case encountered when power generation is higher (*cf.* lower) than power consumption. The estimates of the system frequency are shown in Figure 3. Observe that the proposed Q-SS algorithm was able to track the frequency of the system with a very small steady-state error; however, its complex-valued counterpart, the C-SS, suffered from large oscillatory errors.

In the third set of simulations, real-world data recoded at a $110/20/10$ kV transformer station were considered. The recording showed the “phase-to-ground” voltages of the system, which suffered a fault 5 seconds into the recording, that only lasted for 80 milliseconds. The convergence and steady-state behaviors of the C-SS and Q-SS algorithms are compared in Figure 4, where it is noticeable that although the two algorithms converge at the same point in time, the steady-state behavior of the quaternion-valued Q-SS algorithm is far

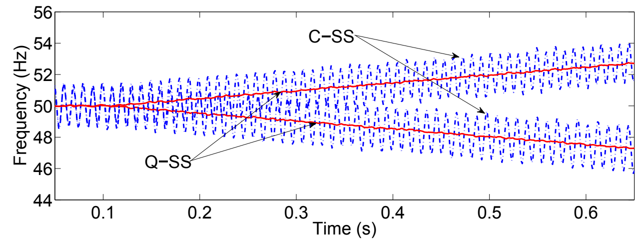

Fig. 3. Frequency estimation for a three-phase system experiencing a frequency rise or decay.

better than that of the complex-valued C-SS algorithm. The behavior of the Q-SS and C-SS algorithms during the fault are shown in Figure 5, where the Q-SS algorithm shows excellent ability to track the system frequency, while the C-SS algorithm became unreliable and lost track of the frequency of the system.

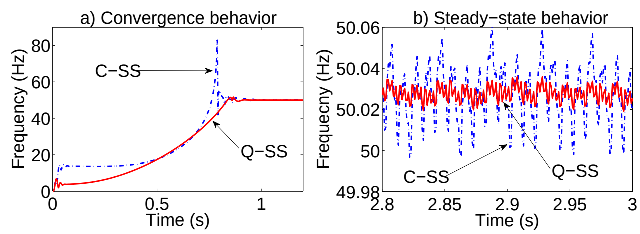

Fig. 4. Convergence and steady-state behavior of the proposed frequency estimation algorithm: a) convergence behavior, b) steady-state behavior.

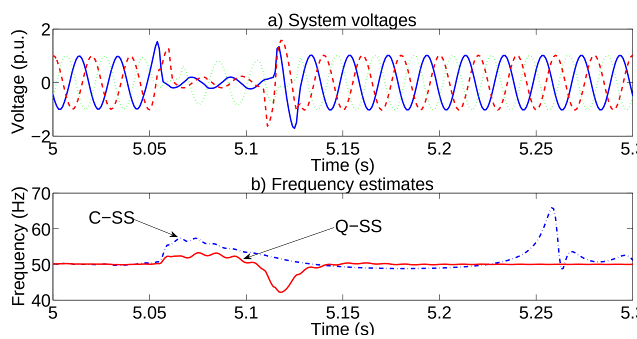

Fig. 5. Behavior of the proposed algorithm during a real-world voltage sag: a) system voltages, b) estimates of the system frequency.

5. CONCLUSION

A novel frequency estimation algorithm for three-phase power systems has been developed based on the extended quaternion Kalman filter (QEKF) and the $\mathbb{HR}$-calculus. The proposed algorithm has been shown to fully characterize both balanced and unbalanced three-phase power systems. The performance of the proposed algorithm has been assessed in a number of scenarios using both synthetic and real-world data, where it has been shown to outperform conventional complex linear estimators.

6. REFERENCES

[1] R.K. Varma, R.M. Mathur, G.J. Rogers, and P. Kundur, "Modeling effects of system frequency variation in long-term stability studies," *IEEE Trans. on Power Systems*, vol. 11, no. 2, pp. 827-832, May 1996.

[2] A. Baggini, "Handbook of Power Quality," *John Wiley & Sons*, New York, 2008.

[3] Y. Xia, S.C. Douglas, and D.P. Mandic, "Adaptive frequency estimation in smart grid applications: Exploiting noncircularity and widely linear adaptive estimators," *IEEE Signal Processing Magazine*, vol. 29, no. 5, pp. 44-54, Sept. 2012.

[4] Y. Xia and D.P. Mandic, "Widely linear adaptive frequency estimation of unbalanced three-phase power systems," *IEEE Trans. on Instrumentation and Measurement*, vol. 61, no. 1, pp. 74-83, Jan. 2012.

[5] D.H. Dini and D.P. Mandic, "Widely linear modeling for frequency estimation in unbalanced three-phase power systems," *IEEE Trans. on Instrumentation and Measurement*, vol. 62, no. 2, pp. 353-363, Feb. 2013.

[6] T. Lobos and J. Rezmer, "Real-time determination of power system frequency," *IEEE Trans. on Instrumentation and Measurement*, vol. 46, no. 4, pp. 877-881, Aug. 1997.

[7] V.V. Terzija, B.N. Djuric, and B.D. Kovacevic, "Voltage phasor and local system frequency estimation using Newton type algorithm," *IEEE Trans. on Power Delivery*, vol. 9, no. 3, pp.1368-1374, Jul. 1994.

[8] V. Eckhardt, P. Hippe, and G. Hosemann, "Dynamic measuring of frequency and frequency oscillations in multiphase power systems," *IEEE Trans. on Power Delivery*,vol. 4, no. 1, pp. 95-102, Jan. 1989.

[9] M.H.J. Bollen, "Voltage sags in three-phase systems," *IEEE Power Engineering Review*, vol. 21, no. 9, pp. 8-15, Sept. 2001.

[10] A.K. Pradhan, A. Routray, and A. Basak, "Power system frequency estimation using least mean square technique," *IEEE Trans. on Power Delivery*, vol. 20, no. 3, pp. 1812-1816, Jul. 2005.

[11] P.K. Dash, A.K. Pradhan, and G. Panda, "Frequency estimation of distorted power system signals using extended complex Kalman filter," *IEEE Trans. on Power Delivery*, vol. 14, no. 3, pp. 761-766, Jul. 1999.

[12] M. Akke, "Frequency estimation by demodulation of two complex signals," *IEEE Trans. on Power Delivery*, vol. 12, no. 1, pp. 157-163, Jan. 1997.

[13] P. Rodriguez, J. Pou, J. Bergas, J.I. Candela, R.P. Burgos, and D. Boroyevich, "Decoupled double synchronous reference frame PLL for power converter control," *IEEE Trans. on Power Electronics*, vol. 22, no. 2, pp. 584-592, Mar. 2007.

[14] J.B. Kuipers, "Quaternions and rotation sequences: A primer with applications to orbits, aerospace and virtual reality," *Princeton University Press*, Aug. 2002.

[15] T.A. Ell and S.J. Sangwine, "Hypercomplex Fourier transforms of color images," *IEEE Trans. on Image Processing*, vol. 16, no. 1, pp. 22-35, Jan. 2007.

[16] S. Said, N. Le Bihan, and S.J. Sangwine, "Fast complexified quaternion Fourier transform," *IEEE Trans. on Signal Processing*, vol. 56, no. 4, pp. 1522-1531, Apr. 2008.

[17] S. Miron, N. Le Bihan, and J.I. Mars, "Quaternion-MUSIC for vector-sensor array processing," *IEEE Trans. on Signal Processing*, vol. 54, no. 4, pp. 1218-1229, Apr. 2006.

[18] F.A. Tobar and D.P. Mandic, "Quaternion reproducing kernel Hilbert spaces: Existence and uniqueness conditions," *IEEE Trans. on Information Theory*, vol. 60, no. 9, pp. 5736-5749, Sept. 2014.

[19] J. Via, D.P. Palomar, L. Vielva, and I. Santamaria, "Quaternion ICA from second-order statistics," *IEEE Trans. on Signal Processing*, vol. 59, no. 4, pp. 1586-1600, Apr. 2011.

[20] D.P. Mandic, C. Jahanchahi, and C.C. Took, "A quaternion gradient operator and its applications," *IEEE Signal Processing Letters*, vol. 18, no. 1, pp. 47-50, Jan. 2011.

[21] C. Jahanchahi and D.P. Mandic, "A class of quaternion Kalman filters," *IEEE Trans. on Neural Networks and Learning Systems*, vol. 25, no. 3, pp. 533-544, Mar. 2014.

[22] E. Clarke, "Circuit analysis of A.C. power systems," *John Wiley & Sons*, New York, 1943.

[23] T.A. Ell, S.J. Sangwine, "Quaternion involutions and anti-involutions, " *Computers & Mathematics with Applications*, vol. 53, no. 1, pp. 137-143, Jan. 2007.

[24] C.C. Took and D.P. Mandic, "Augmented second-order statistics of quaternion random signals," *Signal Processing*, vol. 91, no. 2, pp. 214-224, Feb. 2011.

[25] J. Via, D. Ramirez, and I. Santamaria, "Properness and widely linear processing of quaternion random vectors," *IEEE Trans. on Information Theory*, vol. 56, no. 7, pp. 3502-3515, Jul. 2010.

[26] R.A. Silverman, "Modern calculus and analytical geometry," *Macmillan*, New York, 1969.