QUICKEST DETECTION OF SHORT-TERM VOLTAGE INSTABILITY WITH PMU MEASUREMENTS

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ABSTRACT

The quickest detection of short-term voltage instability in a smart grid is considered. The problem is formulated as a binary sequential composite hypothesis testing where the null hypothesis is a non-stationary process with an unknown exponentially decaying mean and the alternative is a non-stationary process with an unknown exponentially increasing mean. A sequential generalized likelihood ratio test (SGLRT) is proposed and analyzed. It is shown that the proposed SGLRT is asymptotically optimal.

***Index Terms*—** Sequential hypothesis testing, sequential generalized likelihood ratio test, voltage instability, Lyapunov exponents.

1. INTRODUCTION

1.1. Voltage Stability

Voltage stability in a power system refers to the ability of the system to maintain the load voltage within specified operating limits. The voltage stability problem is classified into short-term and long-term stability phenomena [1]. Short-term voltage instability phenomenon is mainly caused by heavy usage of reactive power by electronically controlled loads and induction motors.

Short-term instability can be characterized by the system Lyapunov exponents [2]. In particular, short-term voltage instability occurs if one of the Lyapunov exponents is positive. To detect voltage instability, it is therefore natural to use either the Lyapunov exponents or a proxy of Lyapunov exponents as the indicator for instability. Existing techniques estimate Lyapunov exponents (or related statistics) from phasor measurement unit (PMU) data or state estimates [2, 3]. These existing techniques are heuristic and do not provide any level of performance guarantee.

1.2. Main Results

In this paper, we study voltage instability detection based on classic sequential hypothesis testing theory. In the classic sequential hypothesis testing [4], the decision maker aims to infer the state of an underlying phenomenon from an i.i.d. sequence of observations $\{Y(t)\}_{t\geq 1}$ drawn from two different distributions depending on whether hypothesis H_0 or H_1 is true (assuming binary hypothesis testing). The objective is to minimize the detection delay subject to an error probability constraint. Voltage instability detection can be formulated as a sequential composite hypothesis testing problem with nonidentical distributions. Specifically, the Lyapunov exponents under both hypotheses are unknown, and the observations are non-stationary random processes with either exponentially decaying or exponentially increasing expectations.

We develop a sequential test for the detection of the presence of unknown positive Lyapunov exponents. Referred to as sequential generalized likelihood ratio test of exponents (SGLRT-exp), this test is shown to be both consistent and asymptotically optimal in the sense that the probability of error diminishes as the number of samples increases. Specifically, let the maximum Lyapunov exponent be denoted by λ_1, and two hypotheses be $H_0 : \lambda_1 < 0$ and $H_1 : \lambda_1 > 0$ corresponding to stable and instable states, respectively. Within a Bayes cost formulation of assigning cost one to either type of error and cost $c > 0$ to each observation, the performance of SGLRT-exp is analyzed. Moreover, asymptotic (as c goes to zero) lower bound on the performance of any sequential test within this formulation is established which shows the asymptotic optimality of the SGLRT-exp.

1.3. Related Work

The classic sequential hypothesis testing problem was pioneered by Wald [4]. Wald showed that the sequential probability ratio test (SPRT) is optimal in terms of minimizing the expected sample size subject to given error probability constraints. The composite hypotheses testing problem is fundamentally more difficult than simple hypothesis testing problem. The sequential generalized likelihood ratio test (SGLRT) was first studied by Schwartz for one-parameter exponential family with i.i.d. observations [5]. Adopting a similar Bayesian formulation as in this paper, Schwartz showed that SGLRT is asymptotically optimal when c approaches to zero. A refinement of [5] was studied by Lai [6, 7] which showed that for a multivariate exponential family, SGLRT asymptotically optimizes the Bayesian cost. A more general setting where a set of different experiments are available and the observations depend on the chosen experiment was studied by Chernoff [8]. Another well-studied test for sequential composite hypothesis testing is the sequential adaptive likelihood ratio test (SALRT) [9–11]. The advantage of SALRT is its computationally more efficient statistics. The disadvantage of SALRT is that poor early estimates can never be revised even if a large number of observations are available. All these classic results assume i.i.d. observations over time that is different from the specific non-stationary sequential test formulated in this paper.

The optimality of SPRT for sequential hypothesis testing with non-stationary observations was shown in [12]. The optimal SPRT in the non-stationary environment requires laborious calculation of a sequence of thresholds. The asymptotic optimality of SPRT with approximated thresholds, under certain assumptions on log-likelihood ratios, was shown in [13, 14]. Theses assumptions, however, do not apply to the voltage instability detection problem considered in this paper.

The Lyapunov exponents method in power system short-term voltage instability was proposed in [2]. An online short-term voltage stability monitoring algorithm was introduced in [3], where a model-free approach is developed based on a proxy of maximum Lyapunov exponent that can be computed from data. We adopt this particular idea in our formulation and develop sequential tests that are different from the algorithms considered in [3]. Our contribution lies in a formal approach to the detection of voltage instability in the presence of unknown system parameters and measurement noise. Our algorithm is also shown to be asymptotically optimal.

2. PROBLEM FORMULATION

2.1. Lyapunov Exponents

The Lyapunov exponents in a non-linear system are analogous to the eigenvalues of a linear system which provide information about the stability of the non-linear system. The Lyapunov exponents of a non-linear system are defined as follows [15].

Definition 1. *Consider a continuous time dynamical system $\dot{x} = f(x)$, with $x \in \mathcal{X} \in \mathbb{R}^n$. Let $\phi(t, x)$ be the solution of the differential equation. Define the following limiting matrix*

$$\Lambda(x) = \lim_{t\to\infty}[\frac{\partial\phi(t,x)^T}{\partial x}\frac{\partial\phi(t,x)}{\partial x}]^{\frac{1}{2t}}. \tag{1}$$

Let $\Lambda_i(x)$ be the eigenvalues of the limiting matrix $\Lambda(x)$. The Lyapunov exponents $\lambda_i(x)$ are defined as

$$\lambda_i(x) = \log \Lambda_i(x) \tag{2}$$

Order $\Lambda_i(x)$ such that $\lambda_1(x) \geq \lambda_2(x) \geq ... \geq \lambda_N(x)$. Then, $\lambda_1(x)$ is called the maximum Lyapunov exponent.

An algorithm for online computation of Lyapunov exponents with improved computational efficiency was proposed in [3]. Let $V_{m\Delta} \in \mathbb{R}^n$ be obtained data for time instances $m\Delta$, $m = 0, 1, ..., M$, for some $\Delta > 0$, which is the time interval between the measurements. Choose fixed small numbers, $0 < \epsilon_1 < \epsilon_2$, and an integer N, such that for $m = 1, 2, ..., N$, $\epsilon_1 < ||V_{m\Delta} - V_{(m-1)\Delta}|| < \epsilon_2$. The maximum Lyapunov exponent at time $k\Delta$, is obtained as follows [3]. For $k > N$

$$\exp(Nk\Delta\lambda_1) = \Pi_{m=1}^{N}\frac{||V_{(k+m)\Delta} - V_{(k+m-1)\Delta}||}{||V_{m\Delta} - V_{(m-1)\Delta}||}. \tag{3}$$

Let $N\Delta\lambda_1$ be denoted by θ and $k - N$ be denoted by t. Moreover, let $X(t)$ denote the statistic obtained from measurements (the right hand side of equation (3)), referred to as sample observation. Taking into account the effect of noise on the value of sample observation, the sample observations of the power system at time t are in the form of $X(t) = e^{\theta t} + \mathbf{n}(t)$, where $\mathbf{n}(t)$ is assumed to be the normally distributed noise. The parameter θ determines the stability of the system such that for $\theta < 0$ the system is stable and otherwise the system is instable.

2.2. Sequential Voltage Instability Testing

Consider a classic sequential hypothesis testing problem with observations $\{Y(t)\}_{t\geq 1}$ and two hypotheses H_0 and H_1. The goal is to design a sequential test $\pi = (\tau, \delta)$, where τ is the stopping time and δ is the terminal decision. After observation of τ samples, one of the two hypotheses is declared as the true one. Let $\delta = 0$ denote the declaration of hypothesis H_0 and $\delta = 1$ denote the declaration of hypothesis H_1 . Particularly, the objective is to minimize the expected sample number, $\mathbb{E}[\tau]$, subject to the following constraints on the probability of error

$$\mathbb{P}[\delta = 1|H_0] \leq \alpha, \tag{4}$$
$$\mathbb{P}[\delta = 0|H_1] \leq \beta, \tag{5}$$

for small positive α and β. The first type of error given in (4) is referred to as false alarm and the second type of error given in (5) is referred to as missed detection.

The sequential short term voltage instability detection problem considered in this work can be formulated as a sequential hypothesis testing problem with time-varying distribution of observations. In particular, under each hypothesis, observations are ruled by a non-stationary random process determined by a parameter θ. The null hypothesis corresponds to the stable system, $H_0 : \theta < 0$. The alternative hypothesis corresponds to the instable system, $H_1 : \theta > 0$. The objective, similar to the classic sequential hypothesis testing problem, is to minimize the expected sample number subject to the error constraints. To start with, we assume that, under each hypothesis, the parameter is known. See Sec. 3. In practical applications, however, the value of parameters may be unknown. In Sec. 4 , we study the sequential voltage instability detection problem with unknown parameters.

In our formulation, the distribution of the sample observation at time t is

$$f(X(t); \theta t) = \frac{1}{\sqrt{2\pi}}\exp(\frac{(X(t) - e^{\theta t})^2}{2}), \tag{6}$$

where θ is the true parameter. Note that the distribution of sample observations is time-varying within a specific model of exponential dependence to the parameter.

2.3. Preliminaries and Notations

Let us introduce some concepts and notations that are used throughout the paper. Let $X^{(t)} = X(1), X(2), ..., X(t)$, and $f(X^{(t)}; \theta)$ denote the joint distribution of $X^{(t)}$. The notation $l_t(\theta_1, \theta_0)$ denotes the log-likelihood ratio of two distributions with parameters θ_1 and θ_0 at time t,

$$l_t(\theta_1, \theta_0) = \log \frac{f(X(t); \theta_1 t)}{f(X(t); \theta_0 t)}. \tag{7}$$

The Kullback-Leibler (KL) divergence between the above two distributions, denoted by $I_t(\theta_1, \theta_0)$ is defined as the following expectation of the log-likelihood ratio.

$$I_t(\theta_1, \theta_0) = \mathbb{E}_{\theta_1} l_t(\theta_1, \theta_0), \tag{8}$$

where $\mathbb{E}_\theta[.]$ is the expectation operator when θ is the parameter determining the underlying distributions. Also, let

$$L_t(\theta_1, \theta_0) = \sum_{s=1}^{t} l_s(\theta_1, \theta_0), \tag{9}$$

and

$$g_{\theta_1,\theta_0}(t) = \sum_{s=1}^{t} I_s(\theta_1, \theta_0). \tag{10}$$

Let us define the inverse function $g^{-1}_{\theta_1,\theta_0}(z)$ as

$$g^{-1}_{\theta_1,\theta_0}(z) \triangleq \min\{t \in N : g_{\theta_1,\theta_0}(t) \geq z\}. \tag{11}$$

3. REDUCTION IN THE EXPECTED NUMBER OF OBSERVATIONS

To gain insight into the similarities and differences between the sequential voltage instability test and the classic sequential test, we first consider the simple hypothesis case where $H_0 : \theta = \theta_0 < 0$ and $H_1 : \theta = \theta_1 > 0$. The constraint on the first and second type of error is given by α and β, respectively, as in (4) and (5). The SPRT-exp, a modification of the classic SPRT, is as follows. Continue sampling as long as

$$\log B < L_t(\theta_1, \theta_0) < \log A, \tag{12}$$

stop sampling otherwise. The terminal decision is given by

$$\delta^{SPRT-exp} = \begin{cases} 0, & \text{if } L_t(\theta_1, \theta_0) \leq \log B, \\ 1, & \text{if } L_t(\theta_1, \theta_0) \geq \log A. \end{cases} \tag{13}$$

The thresholds A and B are designed such that the error probability constraints are met. Calculating the exact values of A and B is quite laborious. Instead of exact values of A and B, the so called Wald's approximation values can be used in practice. The Wald's approximations of the values are

$$A = \frac{1-\beta}{\alpha}, \quad B = \frac{\beta}{1-\alpha}.$$

An upper bound on the expected number of observations for the SPRT-exp is given in the following theorem.

Theorem 1. *The expected number of observations for the* $\pi^{SPRT-exp} = (\tau^{SPRT-exp}, \delta^{SPRT-exp})$ *satisfies*

$$\mathbb{E}_{\theta_0}[\tau^{SPRT-exp}] \leq g^{-1}_{\theta_1,\theta_0}(-(1-\alpha)\log B - \alpha \log A), \tag{14}$$
$$\mathbb{E}_{\theta_1}[\tau^{SPRT-exp}] \leq g^{-1}_{\theta_1,\theta_0}((1-\beta)\log A + \beta \log B). \tag{15}$$

Proof. Proof is omitted due to space limit. □

Recall that in the classic simple hypothesis testing considered by Wald it was shown that the average sample number equals to

$$\mathbb{E}_{\theta_0}[\tau] = \frac{-(1-\alpha)\log B - \alpha \log A}{D(\theta_1, \theta_0)}, \tag{16}$$
$$\mathbb{E}_{\theta_1}[\tau] = \frac{(1-\beta)\log A + \beta \log B}{D(\theta_1, \theta_0)}, \tag{17}$$

where $D(\theta_1, \theta_0)$ is the KL divergence between the distributions corresponding to two hypotheses. A comparison between Theorem 1 and the classic problem shows a logarithmic order of reduction in the average sample number for the voltage instability test.

4. COMPOSITE HYPOTHESIS TESTING WITH EXPONENTIALLY TIME-VARYING PARAMETER DEPENDENCE

In this section, we study the voltage instability detection problem where, as dictated by the practical application, the Lyapunov exponents are unknown. We analyze the performance of SGLRT-exp for the composite hypothesis testing problem. Conventionally the set of possible parameters Θ is partitioned to three disjoint sets. Under hypothesis H_0, $\theta \in \Theta_0$, under hypothesis H_1, $\theta \in \Theta_1$, where $\Theta_0 \cap \Theta_1 = \emptyset$, and $\mathcal{I} = \Theta / \{\Theta_0 \cup \Theta_1\} \neq \emptyset$ is an indifference set. In this problem the indifference set is assumed to be the $(-a, a)$ interval for some small $a > 0$. The sets Θ_0 and Θ_1 are the $(-d, -a)$ and (a, d) intervals, respectively. The Bayes cost assigns cost one for the declaration of hypothesis H_1 (or H_0) when hypothesis H_0 (or H_1) is the true one. Also, obtaining each sample observation incurs a cost of $c > 0$. The objective of a sequential test is to minimize the Bayes cost which is equivalent to

$$R_0^\pi = c\mathbb{E}_\theta[\tau] + \mathbb{P}_\theta[\delta = 1] \quad \textit{or} \tag{18}$$
$$R_1^\pi = c\mathbb{E}_\theta[\tau] + \mathbb{P}_\theta[\delta = 0], \tag{19}$$

when the true parameter θ is in Θ_0 or Θ_1, respectively. The

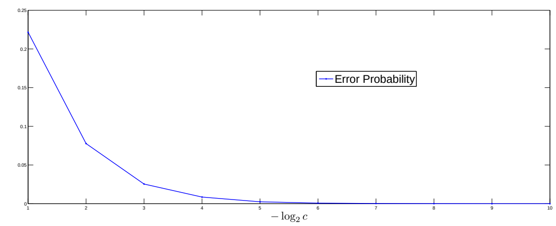

Fig. 1. The probability of error for SGLRT-exp.

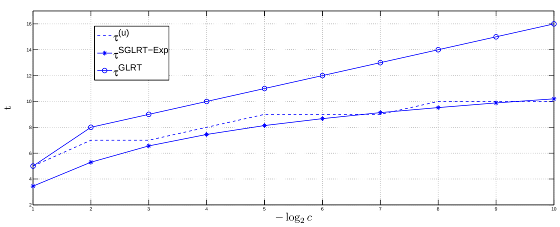

Fig. 2. The average sample number.

$\pi^{SGLRT-exp}$ is as follows. At time t calculate a maximum likelihood estimation of the parameter,

$$\widehat{\theta}_t = \arg \sup_{\theta \in \Theta/\mathcal{I}} f(X^{(t)}; \theta). \tag{20}$$

For any $\theta \in \Theta/\mathcal{I}$ define $\rho(\theta)$ the index of the alternative hypothesis of θ. In other words if $\theta \in \Theta_0$ let $\rho(\theta) = 1$, otherwise, if $\theta \in \Theta_1$ let $\rho(\theta) = 0$. Calculate the ϕ_t as

$$\phi_t = \arg \sup_{\theta \in \Theta_{\rho(\widehat{\theta}_t)}} f(X^{(t)}; \theta). \tag{21}$$

Continue observation of new samples as long as

$$L_t(\widehat{\theta}_t, \phi_t) < -\log c, \tag{22}$$

stop observation, otherwise. The terminal decision is given by

$$\delta^{SGLRT-exp} = 1 - \rho(\widehat{\theta}_\tau). \tag{23}$$

Next, we establish an upper bound on the performance of SGLRT-exp. Moreover, we provide a lower bound on the performance of any arbitrary sequential composite hypothesis test of exponents that shows the asymptotic optimality of SGLRT-exp. It is assumed the true parameter is $\theta_0 \in \Theta_0$. The similar results hold if the alternative hypothesis is the true one. For any $\theta \in \Theta/\mathcal{I}$, let

$$\psi(\theta) = \arg \inf_{\psi \in \Theta_{\rho(\theta)}} I_1(\theta, \psi). \tag{24}$$

Also, let $g_{\theta_0}(t) = g_{\theta_0, \psi(\theta_0)}(t)$, accordingly, $g_{\theta_0}^{-1}(z) = g_{\theta_0, \psi(\theta_0)}^{-1}(z)$.

Theorem 2. *The Bayes cost of the SGLRT-exp satisfies*

$$R_0^{SGLRT-exp} \leq (1+\epsilon) c \, g_{\theta_0}^{-1}(-\log c), \tag{25}$$

such that $\epsilon \to 0$ *as* $c \to 0$.

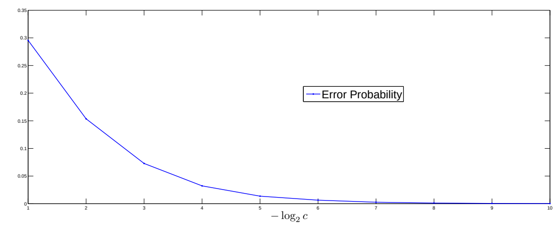

Fig. 3. The probability of error for SGLRT-exp.

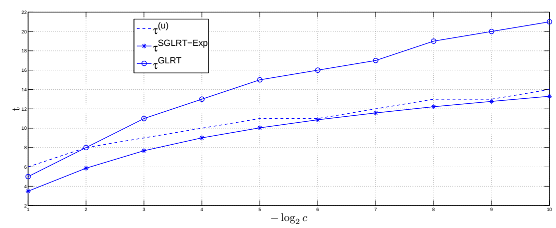

Fig. 4. The average sample number.

Theorem 3. *The Bayes cost of any sequential hypothesis test of exponents* π *satisfies*

$$R_0^{\pi} \geq (1-\epsilon) c \, g_{\theta_0}^{-1}(-(1-\epsilon)\log c), \tag{26}$$

such that $\epsilon \to 0$ *as* $c \to 0$.

Proofs are omitted due to space limit.

5. SIMULATIONS

In this section numerical analysis of the performance of the SGLRT-exp is provided. The numerical results as we shall see are close to asymptotic upper bounds provided in the paper. Furthermore, to show the efficiency of the sequential test, the average sample number of the SGLRT-exp is compared with a fixed size test with the same power. Fig. 1 shows the probability of error for the SGLRT-exp over different values of the cost c. For smaller c, the cost of obtaining observations is lower, thus a higher number of observations results in a smaller probability of error. Second figure, shows the average sample number for the SGLRT-exp algorithm ($\tau^{SGLRT-exp}$). From our analytical results in Theorem 2, the value of $\tau^{(u)} = g_{\theta_0}^{-1}(-\log c)$ is an approximation of the upper bound on the average sample number, which is illustrated in the figure. Also, denoted by τ^{GLRT}, the number of observations required in a fixed size GLRT to achieve the same probability of error is shown in Fig. 2 that confirms the efficiency of the sequential algorithm. In these simulations the values of d and a are assigned to 1 and 0.05, respectively. For the first two figures $\theta_0 = 0.1$. The second two figures show the same numerical analysis when $\theta_0 = -0.1$.

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