A TWO CHANNEL APPROACH FOR SYSTEM APPROXIMATION WITH GENERAL MEASUREMENT FUNCTIONALS

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ABSTRACT

The approximation of linear time-invariant (LTI) systems by sampling series is an important topic in signal processing. However, the convergence of the approximation series is not guaranteed: there exist stable LTI systems and bandlimited input signals such that the approximation series diverges, regardless of the oversampling factor and the sampling pattern. Recently, it has been shown that this divergence can be overcome by using measurement functionals instead of pointwise sampling. However, the bandwidth of the approximation series needs to be strictly larger than the signal bandwidth. In this paper we derive a two channel system approximation approach based on measurement functionals that converges for all stable LTI systems and all signals in the Paley–Wiener space $\mathcal{PW}_\pi^1$. Thanks to the two channel structure it is possible to achieve an approximation bandwidth that is equal to the signal bandwidth.

Index Terms— bandlimited signal, linear time-invariant system, approximation, measurement functional, two channel approach

1. INTRODUCTION

Sampling theory plays a fundamental role in modern signal and information processing, because it is the basis for today's digital world. In his seminal work [1] Shannon started this theory. The reconstruction of bandlimited signals from their samples is also widely used in other applications and theoretical concepts [2–4]. For an overview of existing sampling theorems see for example [2,5], and [6].

Although the sampling theorems are very important on their own, they do not reflect the actual purpose of signal processing. The core task of signal processing is to process data. This means that, usually, the interest is not in a reconstruction of the sampled signal itself, but in some processed version of it. This might be the derivative, the Hilbert transform or the output of any other stable linear system T. Thus, the goal is to approximate the desired transform Tf of a signal f by an approximation process which uses only finitely many, not necessarily equidistant, samples of the signal f.

A common approach to do this approximation is to use

$$\sum_{k=-\infty}^{\infty} f(t_k)(T\phi_k)(t), \quad t \in \mathbb{R}, \tag{1}$$

where $\{t_k\}_{k\in\mathbb{Z}}$ denotes the sequence of sampling points, and the ϕ_k are certain reconstruction functions. Exactly as in the case of signal reconstruction, the convergence and approximation behavior of (1) is important for practical applications [7].

In [8] it was shown that (1) is not always stable, i.e, that there exist stable LTI systems T and bandlimited signals f such that (1) diverges, regardless of the oversampling factor. However, as proved in [9], if suitable measurement functionals are used instead of pointwise sampling, then the divergence can be overcome and the obtained system approximation process is convergent for all systems and signals. It is important to note that in the approach of [9], the bandwidth of the reconstruction functions and thus the bandwidth of the approximation process needs to be strictly larger than the bandwidth of the input signal f.

In this paper we will develop a two channel system approximation process that is based on measurement functionals, and which has, in contrast to the approach in [9], exactly the same bandwidth as the input signal. Such a behavior is desirable for practical applications, where out-of-band noise due to an increase of the bandwidth by the system approximation process may not be permissible.

Before we describe the measurement functionals and the proposed method in more detail, we need to introduce some notation.

2. NOTATION

Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \le p < \infty$, is the space of all measurable, pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. For $\sigma > 0$ and $1 \le p \le \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of functions f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)\,\mathrm{e}^{iz\omega}\ \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma,\sigma]$. The norm for $\mathcal{PW}_\sigma^p$, $1 \le p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi)\int_{-\sigma}^{\sigma}|\hat{f}(\omega)|^p\ \mathrm{d}\omega)^{1/p}$.

We briefly review some definitions and facts about stable linear time-invariant (LTI) systems. A linear system $T : \mathcal{PW}_\pi^p \to \mathcal{PW}_\pi^p$, $1 \le p \le \infty$, is called stable if the operator T is bounded, i.e., if $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^p}\le 1}\|Tf\|_{\mathcal{PW}_\pi^p} < \infty$. Furthermore, it is called time-invariant if $(Tf(\,\cdot\, - a))(t) = (Tf)(t-a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$. For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi,\pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{h}_T(\omega)\,\mathrm{e}^{i\omega t}\ \mathrm{d}\omega, \quad t \in \mathbb{R}, \tag{2}$$

for all $f \in \mathcal{PW}_\pi^1$. Conversely, every function $\hat{h}_T \in L^\infty[-\pi,\pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}\|_{L^\infty[-\pi,\pi]}$. Furthermore, it can be shown that the representation (2) with $\hat{h}_T \in$

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$L^\infty[-\pi,\pi]$ is also valid for all stable LTI systems $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$. Therefore, every stable LTI system that maps $\mathcal{PW}_\pi^1$ in $\mathcal{PW}_\pi^1$ maps $\mathcal{PW}_\pi^2$ in $\mathcal{PW}_\pi^2$, and vice versa. Note that $\hat{h}_T \in L^\infty[-\pi,\pi] \subset L^2[-\pi,\pi]$, and consequently $h_T \in \mathcal{PW}_\pi^2$.

3. BASICS OF NON-EQUIDISTANT SAMPLING

In classical non-equidistant sampling, the goal is to reconstruct a bandlimited signal f from its non-equidistant samples $\{f(t_k)\}_{k\in\mathbb{Z}}$, where $\{t_k\}_{k\in\mathbb{Z}}$ is the sequence of sampling points. One possibility to do the reconstruction is to use the sampling series

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t), \quad t \in \mathbb{R}, \tag{3}$$

where the ϕ_k, $k \in \mathbb{Z}$, are certain reconstruction functions.

If the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ is a real complete interpolating sequence for $\mathcal{PW}_\pi^2$, which is ordered strictly increasingly, then the product

$$\phi(z) = z \lim_{N\to\infty} \prod_{\substack{|k|\leq N \\ k\neq 0}} \left(1 - \frac{z}{t_k}\right) \tag{4}$$

converges uniformly on $|z| \leq R$ for all $R < \infty$, and ϕ is an entire function of exponential type π [10]. Without loss of generality, we assumed in (4) that $t_0 = 0$. It can be seen from (4) that ϕ, which is often called generating function, has the zeros $\{t_k\}_{k\in\mathbb{Z}}$. Moreover, it follows that

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t-t_k)}$$

is the unique function in $\mathcal{PW}_\pi^2$ that solves the interpolation problem $\phi_k(t_l) = \delta_{kl}$, where $\delta_{kl} = 1$ if $k = l$, and $\delta_{kl} = 0$ otherwise.

Remark 1. Equidistant sampling with $t_k = k$, $k \in \mathbb{Z}$, is a special case of the more general non-equidistant setting. For equidistant sampling we have $\phi_k(t) = \operatorname{sinc}(t-k)$, $k \in \mathbb{Z}$, and (3) reduces to the ordinary Shannon sampling series.

4. GENERAL MEASUREMENT FUNCTIONALS

A key concept in signal processing is to process analog, i.e., continuous-time signals in the digital domain. The first step in this procedure is to convert the continuous-time signal into a discrete-time signal, i.e., into a sequence of numbers. As in (3), usually the point evaluation functionals $c_k : f \mapsto f(t_k)$ are employed to do this conversion. However, it is also possible to use more general measurement functionals [11–15]. For example, functionals that take the average of the signal over an interval, like in $c_k : f \mapsto \frac{1}{2\delta}\int_{t_k-\delta}^{t_k+\delta} f(t)\,\mathrm{d}t$, where δ is some small positive number.

The approximation of Tf by the system approximation process (1) can be seen as an approximation that uses the biorthogonal system $\{\mathrm{e}^{-i\,\cdot\,t_k}, \hat{\phi}_k\}_{k\in\mathbb{Z}}$. Here, the measurement functionals are given by $c_k(f) = f(t_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\,\mathrm{e}^{i\omega t_k}\,\mathrm{d}\omega$. Further, the functions $\phi_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\phi}_k(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$ serve as reconstruction functions in the approximation process (1).

For $f \in \mathcal{PW}_\pi^1$, even with oversampling, an approximation of Tf by using the process (1) is not possible in general, because there are signals $f \in \mathcal{PW}_\pi^1$ and stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ such that (1) diverges [8]. In [9] more general measurement functionals, which are based on a complete orthonormal system $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ in $L^2[-\pi,\pi]$, were considered.

Before we treat the $\mathcal{PW}_\pi^1$ case, we quickly review the situation for the space $\mathcal{PW}_\pi^2$. For $f \in \mathcal{PW}_\pi^2$ the situation is simple. Let $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ be a complete orthonormal system in $L^2[-\pi,\pi]$. Then, the measurement functionals $c_n : \mathcal{PW}_\pi^2 \to \mathbb{C}$, $n \in \mathbb{N}$, are given by

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\overline{\hat{\theta}_n(\omega)}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} f(t)\overline{\theta_n(t)}\,\mathrm{d}t$$

and the reconstruction functions by

$$\theta_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\theta}_n(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega.$$

Further, we have

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|f(t) - \sum_{n=1}^{N} c_n(f)\theta_n(t)\right|^2 \mathrm{d}t = 0$$

for all $f \in \mathcal{PW}_\pi^2$. Since $T : \mathcal{PW}_\pi^2 \to \mathcal{PW}_\pi^2$ is a stable LTI system, it follows that the system approximation process

$$\sum_{n=1}^{\infty} c_n(f)\,(T\theta_n)(t), \quad t \in \mathbb{R}, \tag{5}$$

converges in the L^2-norm, and consequently uniformly on the whole real axis.

For signals in $\mathcal{PW}_\pi^1$ the situation is more difficult. In order that

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\overline{\hat{\theta}_n(\omega)}\,\mathrm{d}\omega \tag{6}$$

is also a reasonable measurement procedure for $f \in \mathcal{PW}_\pi^1$, we need the functionals $c_n : \mathcal{PW}_\pi^1 \to \mathbb{C}$, defined by (6), to be continuous and uniformly bounded in n. Since $\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 1}|c_n(f)| = \|\hat{\theta}_n\|_{L^\infty[-\pi,\pi]}$, this means we additionally have to require that the functions of the complete orthonormal system $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ satisfy

$$\sup_{n\in\mathbb{N}}\|\hat{\theta}_n\|_{L^\infty[-\pi,\pi]} < \infty. \tag{7}$$

In [9] it was proved that for every $0 < \sigma < \pi$ there exist a complete orthonormal system $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ in $L^2[-\pi,\pi]$ satisfying (7) and an associated sequence of measurement functionals $\{c_n\}_{n\in\mathbb{N}}$ as defined by (6), such that, for all stable LTI systems $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and all signals $f \in \mathcal{PW}_\sigma^1$, the system approximation process (5) converges uniformly to Tf.

Thus, using oversampling and more general measurement functionals, it is possible to have a stable system approximation with the process (5). Note that the system approximation process in this result has the bandwidth π, because the functions θ_n are in $\mathcal{PW}_\pi^2$, but the input signal f has bandwidth $\sigma < \pi$, because $f \in \mathcal{PW}_\sigma^1$. Thus, the approximation process requires a larger bandwidth than the input signal. In the next section we will present a two-channel approach that does not require this increased bandwidth.

5. REDUCED BANDWIDTH APPROXIMATION

We first study the behavior of the two-channel system approximation process, which is depicted in Figure 1, for signals in $\mathcal{PW}_\pi^2$. The analysis for $\mathcal{PW}_\pi^2$ is simple, but will provide some useful insights for the $\mathcal{PW}_\pi^1$ case.

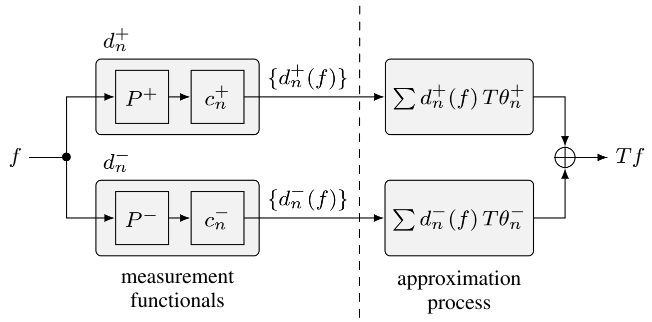

Fig. 1. Two-channel approach for system approximation with measurement functionals.

Let P^+ and P^- be the stable LTI systems defined by the transfer functions $\hat{h}_{P^+}(\omega) = \mathbf{1}_{[0,\pi]}(\omega)$ and $\hat{h}_{P^-}(\omega) = \mathbf{1}_{[-\pi,0]}(\omega)$. Clearly,

$$(P^+f)(t) = \int_{-\pi}^{\pi} \hat{f}(\omega)\mathbf{1}_{[0,\pi]}(\omega)\,\mathrm{e}^{i\omega t}\;\mathrm{d}\omega = \int_0^{\pi} \hat{f}(\omega)\,\mathrm{e}^{i\omega t}\;\mathrm{d}\omega$$

and

$$(P^-f)(t) = \int_{-\pi}^{\pi} \hat{f}(\omega)\mathbf{1}_{[-\pi,0]}(\omega)\,\mathrm{e}^{i\omega t}\;\mathrm{d}\omega = \int_{-\pi}^{0} \hat{f}(\omega)\,\mathrm{e}^{i\omega t}\;\mathrm{d}\omega$$

are the projections of f onto the positive and negative frequencies. Further, let $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ be two complete orthonormal systems in $L^2[-\pi,\pi]$. We will apply $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ to $\widehat{P^+f}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ to $\widehat{P^-f}$. The coefficient functionals associated with the bases $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ are given by

$$c_n^+(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\overline{\hat{\theta}_n^+(\omega)}\,\mathrm{d}\omega$$

and

$$c_n^-(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\overline{\hat{\theta}_n^-(\omega)}\,\mathrm{d}\omega.$$

Based on the projection operators and the coefficient functionals, we define the measurement functionals as

$$\begin{aligned} d_n^+(f) := c_n^+(P^+f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\widehat{P^+f})(\omega)\overline{\hat{\theta}_n^+(\omega)}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{0}^{\pi} \hat{f}(\omega)\overline{\hat{\theta}_n^+(\omega)}\,\mathrm{d}\omega \end{aligned} \tag{8}$$

and

$$\begin{aligned} d_n^-(f) := c_n^-(P^-f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\widehat{P^-f})(\omega)\overline{\hat{\theta}_n^-(\omega)}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} \hat{f}(\omega)\overline{\hat{\theta}_n^-(\omega)}\,\mathrm{d}\omega. \end{aligned} \tag{9}$$

For signals $f \in \mathcal{PW}_\pi^2$ it is easy to show, using the fact that $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ are bases for $L^2[-\pi,\pi]$, that the approximation process

$$\sum_{n=1}^{\infty} d_n^+(f)(T\theta_n^+)(t) + d_n^-(f)(T\theta_n^-)(t)$$

converges in the L^2-norm, and consequently uniformly on the real axis, to Tf.

Remark 2. Note that both $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ are bases for $L^2[-\pi,\pi]$, whereas $\widehat{P^+f}$ and $\widehat{P^-f}$ are supported on $[0,\pi]$ and $[-\pi,0]$, respectively. This fact will be important for the proof of Theorem 1 below. Although for the illustrating $\mathcal{PW}_\pi^2$ example here every complete orthonormal system will work, we need to use a special complete orthonormal system with specific properties later in the proof of the $\mathcal{PW}_\pi^1$ result.

Remark 3. Note that the measurement functionals d^+ and d^- cannot be implemented as a time-invariant system with subsequent sampling. This is a difference to classical multichannel sampling, where the signal is filtered by LTI systems and sampled subsequently.

The next theorem shows that the two-channel approach with sampling functionals, as depicted in Figure 1, provides a stable system approximation process with bandwidth equal to the input signal bandwidth that converges uniformly to Tf for all stable LTI systems T and all signals $f \in \mathcal{PW}_\pi^1$.

Theorem 1. *There exist two complete orthonormal systems $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ in $L^2[-\pi,\pi]$ satisfying* (7)*, two associated sequences of measurement functionals $\{d_n^+\}_{n\in\mathbb{N}}$ and $\{d_n^-\}_{n\in\mathbb{N}}$ as defined by* (8) *and* (9)*, and a constant C_1 such that for all stable LTI systems $T\colon \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ and all $f \in \mathcal{PW}_\pi^1$ we have*

$$\lim_{N\to\infty}\left\| Tf - \sum_{n=1}^{N} d_n^+(f)\,T\theta_n^+ + d_n^-(f)\,T\theta_n^- \right\|_\infty = 0.$$

Theorem 1 is not only an abstract existence result. The complete orthonormal systems $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$, which are used in Theorem 1, can be explicitly constructed by a procedure given in [16, 17].

Remark 4. It is necessary to use two different orthonormal systems. If we had only one single orthonormal system, we would have the situation that was analyzed in [9], for which divergence has been proved.

For the proof we need the following theorem from [16, 17].

Theorem 2 (Olevskii)**.** *Let $0 < \delta < 1$. There exists an orthonormal system $\{\psi_n\}_{n\in\mathbb{N}}$ of real-valued functions that is closed in $C[0,1]$ such that $\sup_{n\in\mathbb{N}}\|\psi_n\|_{L^\infty[0,1]} < \infty$ and such that there exists a constant C_2 such that for all $x \in [\delta,1]$ and all $N \in \mathbb{N}$ we have*

$$\int_0^1 \left|\sum_{n=1}^{N}\psi_n(x)\psi_n(\tau)\right|\;\mathrm{d}\tau \le C_2.$$

Remark 5. In the above theorem, we adopted the notion of "closed" from [18]. In [18] a system $\{\psi_n\}_{n\in\mathbb{N}}$ is called closed in $C[0,1]$ if every function in $C[0,1]$ can be uniformly approximated by finite linear combinations of the system $\{\psi_n\}_{n\in\mathbb{N}}$, that is if for every $\epsilon > 0$ and every $f \in C[0,1]$ there exists an $N \in \mathbb{N}$ and a sequence $\{\alpha_n\}_{n=1}^N \subset \mathbb{C}$ such that $\|f - \sum_{n=1}^N \alpha_n\psi_n\|_{L^\infty[0,1]} < \epsilon$.

Proof of Theorem 1. Using the functions ψ_n from Theorem 2 we define $\hat{\theta}_n^+(\omega) = \psi_n\left(\frac{\pi+\omega}{2\pi}\right)$ and $\hat{\theta}_n^-(\omega) = \psi_n\left(\frac{\pi-\omega}{2\pi}\right)$, $\omega \in [-\pi,\pi]$. Due to the properties of ψ_n, it follows that $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ are two complete orthonormal systems for $L^2[-\pi,\pi]$ and that $\sup_{n\in\mathbb{N}}\|\hat{\theta}_n^+\|_{L^\infty[-\pi,\pi]} = \sup_{n\in\mathbb{N}}\|\hat{\theta}_n^-\|_{L^\infty[-\pi,\pi]} < \infty$. Moreover, for $\omega \in [0,\pi]$, we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n=1}^{N}\hat{\theta}_n^+(\omega)\hat{\theta}_n^+(\omega_1)\right|\mathrm{d}\omega_1 &= \int_0^1\left|\sum_{n=1}^{N}\psi_n\Big(\frac{\pi+\omega}{2\pi}\Big)\,\psi_n(\tau)\right|\mathrm{d}\tau \\ &\le C_2, \end{aligned} \tag{10}$$

according to Theorem 2, because for $\omega \in [0,\pi]$ we have $(\pi+\omega)/(2\pi) \in [1/2,1]$, and for $\omega \in [-\pi,0]$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n=1}^{N}\hat{\theta}_n^-(\omega)\hat{\theta}_n^-(\omega_1)\right| \mathrm{d}\omega_1 \le C_2, \tag{11}$$

according to the same consideration, because for $\omega \in [-\pi,0]$ we have $(\pi-\omega)/(2\pi) \in [1/2,1]$. Next, for $f \in \mathcal{PW}_\pi^1$, we study the expression

$$\begin{aligned}(U_N\hat{f})(\omega) &:= \sum_{n=1}^{N} d_n^+(f)\hat{\theta}_n^+(\omega) + \sum_{n=1}^{N} d_n^-(f)\hat{\theta}_n^-(\omega)\\ &= \frac{1}{2\pi}\int_0^{\pi}\hat{f}(\omega_1)\sum_{n=1}^{N}\hat{\theta}_n^+(\omega)\hat{\theta}_n^+(\omega_1)\,\mathrm{d}\omega_1\\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{0}\hat{f}(\omega_1)\sum_{n=1}^{N}\hat{\theta}_n^-(\omega)\hat{\theta}_n^-(\omega_1)\,\mathrm{d}\omega_1.\end{aligned}$$

Applying Fubini's theorem, (10), and (11), it follows that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}|(U_N\hat{f})(\omega)|\,\mathrm{d}\omega\\ &\le \frac{1}{2\pi}\int_0^{\pi}|\hat{f}(\omega_1)|\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n=1}^{N}\hat{\theta}_n^+(\omega)\hat{\theta}_n^+(\omega_1)\right|\,\mathrm{d}\omega\right)\,\mathrm{d}\omega_1\\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{0}|\hat{f}(\omega_1)|\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n=1}^{N}\hat{\theta}_n^-(\omega)\hat{\theta}_n^-(\omega_1)\right|\,\mathrm{d}\omega\right)\,\mathrm{d}\omega_1\\ &\le C_2\|f\|_{\mathcal{PW}_\pi^1}\end{aligned} \tag{12}$$

Let $f \in \mathcal{PW}_\pi^1$ and $\epsilon > 0$ be arbitrary but fixed. There exists an $f_\epsilon \in \mathcal{PW}_\pi^2$ such that $\|f-f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$. We have

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)-(U_N\hat{f})(\omega)|\,\mathrm{d}\omega\\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)-\hat{f}_\epsilon(\omega)|\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon(\omega)-(U_N\hat{f}_\epsilon)(\omega)|\,\mathrm{d}\omega\\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}|(U_N(\hat{f}-\hat{f}_\epsilon))(\omega)|\,\mathrm{d}\omega\\ &\le \epsilon + C_2\epsilon + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon(\omega)-(U_N\hat{f}_\epsilon)(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}},\end{aligned} \tag{13}$$

where we used (12) and the Cauchy–Schwarz inequality in the last step. Let $\hat{f}_\epsilon^+ = \mathbf{1}_{[0,\pi]}\hat{f}_\epsilon$ and $\hat{f}_\epsilon^- = \mathbf{1}_{[-\pi,0]}\hat{f}_\epsilon$. Since $\hat{f}_\epsilon^+$ and $\hat{f}_\epsilon^-$ are in $L^2[-\pi,\pi]$, and since $\{\hat{\theta}_n^+\}_{n\in\mathbb{N}}$ and $\{\hat{\theta}_n^-\}_{n\in\mathbb{N}}$ are complete orthonormal systems in $L^2[-\pi,\pi]$, there exists a natural number $N_0 = N_0(\epsilon)$ such that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon^+(\omega)-(U_N\hat{f}_\epsilon^+)(\omega)|^2\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\hat{f}_\epsilon^+(\omega)-\sum_{n=1}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}_\epsilon^+(\omega_1)\hat{\theta}_n^+(\omega_1)\,\mathrm{d}\omega_1\right)\hat{\theta}_n^+(\omega)\right|^2\mathrm{d}\omega\\ &< \epsilon^2\end{aligned}$$

and, by the same considerations,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon^-(\omega)-(U_N\hat{f}_\epsilon^-)(\omega)|^2\,\mathrm{d}\omega < \epsilon^2$$

for all $N \ge N_0$. It follows that

$$\begin{aligned}&\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon(\omega)-(U_N\hat{f}_\epsilon)(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\\ &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon^+(\omega)-(U_N\hat{f}_\epsilon^+)(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\\ &\quad + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}_\epsilon^-(\omega)-(U_N\hat{f}_\epsilon^-)(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}} \le 2\epsilon.\end{aligned} \tag{14}$$

Hence, we see from (13), (14) that $\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)-(U_N\hat{f})(\omega)|\,\mathrm{d}\omega \le (3+C_2)\epsilon$ for all $N \ge N_0$. Since $\epsilon > 0$ was arbitrary, this shows that

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)-(U_N\hat{f})(\omega)|\,\mathrm{d}\omega = 0. \tag{15}$$

For arbitrary stable LTI systems $T\colon \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ we have

$$\begin{aligned}&(Tf)(t) - \sum_{n=1}^{N} d_n^+(f)(T\theta_n^+)(t) + d_n^-(f)(T\theta_n^-)(t)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigg(\hat{f}(\omega)\hat{h}_T(\omega)\,\mathrm{e}^{i\omega t}\\ &\quad - \sum_{n=1}^{N} d_n^+(f)\hat{h}_T(\omega)\hat{\theta}_n^+(\omega)\,\mathrm{e}^{i\omega t} + d_n^-(f)\hat{h}_T(\omega)\hat{\theta}_n^-(\omega)\,\mathrm{e}^{i\omega t}\Bigg)\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\hat{f}(\omega)-(U_N\hat{f})(\omega))\hat{h}_T(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega\end{aligned}$$

and consequently

$$\begin{aligned}&\left|(Tf)(t) - \sum_{n=1}^{N} d_n^+(f)(T\theta_n^+)(t) - d_n^-(f)(T\theta_n^-)(t)\right|\\ &\le \|\hat{h}_T\|_{L^\infty[-\pi,\pi]}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)-(U_N\hat{f})(\omega)|\,\mathrm{d}\omega\end{aligned} \tag{16}$$

for all $t \in \mathbb{R}$. Combining (16) and (15) yields the assertion. □

6. RELATION TO PRIOR WORK

The approximation of LTI systems by sampling series is a well-studied field [7,19–23]. The instability of the approximation process, which was observed in [8] for signals in $\mathcal{PW}_\pi^1$, can be overcome by using more general measurement functionals, as it was recently shown [9]. The result in [9] can be interpreted as a one channel approach, where we only have one sequence of measurement functionals, consisting of the identity operator $P = Id$ as pre-filter and a single orthonormal basis. Using this approach, oversampling is necessary to obtain a stable approximation, without oversampling we do not have convergence in general.

General measurement functionals have been analyzed before [11–15], but only for the signal reconstruction problem and not for the system approximation problem. In [24] a filter bank approach has been presented for sampling in atomic spaces, and in [25] sparsity has been considered for these spaces.

In this paper we show that, using a two channel approach and suitably chosen measurement functionals, it is possible to reduce the bandwidth of the approximation process to the bandwidth of the input signal. The proposed approach improves the result in [9] and extends the usual multichannel sampling setting [5,20,26–29] to incorporate the measurement functional idea. Interestingly, the transition from a one channel to a two channel approach is already sufficient for a stable implementation with minimum bandwidth.

7. REFERENCES

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