DESIGN OF SIGNAL-MATCHED CRITICALLY SAMPLED FIR RATIONAL FILTERBANK

Anupriya Gogna, Sri Harsha Gade, and Anubha Gupta

Department of Electronics and Communications Engineering
Indraprastha Institute of Information Technology Delhi, New Delhi
anupriyag, narayana1294, anubha@iiitd.ac.in

ABSTRACT

Wavelet transform is used for efficient signal analysis in various applications. The traditional wavelet system is implemented using integer decimation factors, although frequency tiling offered by rational decimation may better adapt to signal characteristics. In this paper, we propose a design methodology for signal-matched filterbank (FB) with rational decimation factors that achieves perfect reconstruction with FIR filters. We have applied the proposed design on some real world signals. With the proposed design, we obtain a more compressible transform domain representation than the dyadic standard wavelet transforms.

***Index Terms*—** Rational Wavelet Transform, Critically Sampled Multi-Channel Filterbank, Matched Wavelet

1. INTRODUCTION

Wavelet transform (WT) provides a way of performing multi-resolution analysis (MRA) on signals, giving simultaneous time-frequency information. A multilevel wavelet decomposition gives rise to filterbank with integer decimation ratios. Uniformly decimated critically sampled *M*-channel filterbanks decompose a signal into uniform frequency bands. Both these structures have been extensively researched and used in several applications[1][2][3][4]. However, in certain cases, such as speech signal analysis, it may be desirable to have a denser frequency domain representation. In such cases, wavelet transform with rational decimation ratios or rational filterbanks (RFBs) may prove to be advantageous compared to uniformly decimated FBs since they allow more flexible frequency tiling via nonuniform frequency partition [5].

Several works have been carried out to design perfect reconstruction filter banks (PRFB) with rational dilation factors. In [6], authors have proposed a method for the design of an overcomplete RFB in frequency domain. However, the filters obtained are not FIR. In [7], an iterative procedure with various linear constraints such as regularity is used for the design of orthonormal rational FB. However, because of the iterative nature of the design algorithm, the solution may not reach to the optimum point. Authors in [8] proposed an algorithm for fast rational orthogonal WT, but the filters obtained are IIR filters. In [5], authors have presented an algorithm to design an overcomplete orthogonal RFB with specified number of vanishing moments. Overcomplete structures, though less sensitive to aliasing, add redundancy, while critically sampled structures are not burdened by this drawback. In [9], authors designed a critically sampled bi-orthogonal RFB structure with perfect reconstruction and regularity properties. Their design involved solving a non-convex optimization problem with non-linear constraints requiring the use of a cumbersome iterative approach. So far, to the best of our knowledge, none of the methods have designed a RFB matched to a given input signal.

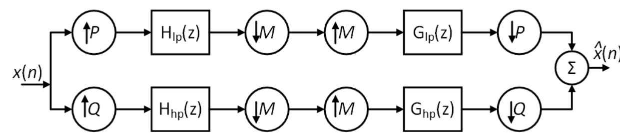

Fig. 1. Two Channel Rational Filter Bank Structure.

Moreover, most of the existing methods design filters in Fourier domain and hence, the resulting filters may not have compact support. In [10], authors have proposed a method to design filters matched to a given signal for a 2-channel dyadic discrete WT (DWT).

In this paper, we propose a design for signal-matched critically sampled RFB with different decimation factors. Both the analysis and synthesis end filters are designed to have compact support. Such a FB combines the advantages of a denser frequency tiling, simpler and more compact implementation of RFB (aided by the use of FIR filters) with a sparser representation of the input signal (due to the matched FB design). This paper is organized as follows. In section 2, our proposed approach is discussed with implementation details. Section 3 shows designed FBs for different input signals and results on sparsity are compared with standard wavelets. Conclusion is presented in section 4.

2. PROPOSED DESIGN

In this paper, we intend to design a two channel rational filter bank structure as shown in the Fig. 1. For RFB to be critically sampled, $P/M + Q/M = 1$. This RFB can be represented by an equivalent uniformly decimated *M*-band FB structure shown in Fig. 2, provided it satisfies the conditions (1), (2), (3) and (4) presented below [9]:

$$\mathrm{H}_{lp}(z) = \sum_{j=1}^{P} z^{M(j-1-P)} \mathrm{H}_{j+Q}(z^P) \tag{1}$$

$$\mathrm{G}_{lp}(z) = \sum_{j=1}^{P} z^{-M(j-1-P)} \mathrm{G}_{j+Q}(z^P) \tag{2}$$

$$\mathrm{H}_{hp}(z) = \sum_{j=1}^{Q} z^{M(Q-j)} \mathrm{H}_{j}(z^Q) \tag{3}$$

$$\mathrm{G}_{hp}(z) = \sum_{j=1}^{Q} z^{-M(Q-j)} \mathrm{G}_{j}(z^Q), \tag{4}$$

where *P* and *Q* are the upsampling factors in the lowpass and highpass branches of RFB, respectively, and $\mathrm{H}_i(z), i = 1, 2, ..., M$ and $\mathrm{G}_i(z), i = 1, 2, ..., M$ are the analysis and synthesis filters in the

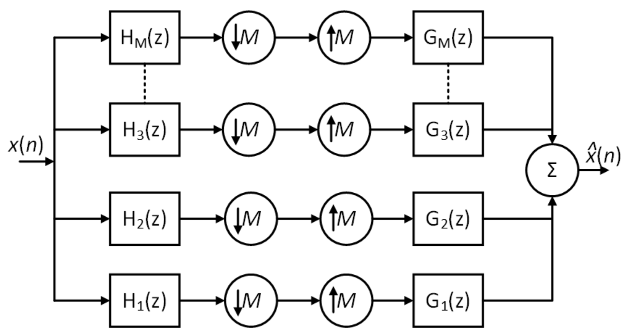

Fig. 2. *M*-Band Critically Sampled Filter Bank Structure.

equivalent uniformly decimated *M*-band FB as shown in Fig. 2. First, we design a signal-matched uniformly decimated filterbank and later, convert that to a rational FB.

2.1. M-Band Signal Matched Analysis Filterbank

In order to design analysis filters, first we use the least squares based approach of [10] for the design of highpass analysis filter. Next, we use the idea of signal spectrum clip removal of [11] together with the least squares approach of [10] to design bandpass filters as shown in Fig. 3. The spectral clip removal approach of [11] may lead to IIR filters at the synthesis end. Since we intend to design FIR filterbank, the lowpass filter is designed differently by imposing the condition that the determinant of the analysis polyphase matrix $\mathbf{E}(z)$ is a monomial [1].

2.1.1. Design of analysis highpass filter

This section briefly explains the method for the design of highpass filter using the approach mentioned in [10] for the sake of completeness of the paper. Consider Fig. 3 where $\mathbf{h}_1$ denotes the highpass filter, $x(n)$ is the input signal, and $d_{-1}(n)$ denotes the wavelet subspace coefficients.

$$d_{-1}(n) = \sum_k h_1(k)x(Mn-k) \tag{5}$$

Next, using least squares approach, we minimize the signal energy in this wavelet subspace to estimate highpass filter from a given input signal. On setting the center weight of $\mathbf{h}_1$ to unity (without loss of generality), the signal energy in wavelet subspace is given as (6)

$$\begin{aligned} E &= \sum_n (d_{-1}(n))^2 \\ E &= \sum_n (x(Mn-n_1))^2 + \sum_n \mathbf{W}^T\mathbf{X}(n)\mathbf{X}^T(n)\mathbf{W} \\ &\quad - 2\sum_n \mathbf{W}^T\mathbf{X}(n)x(Mn-n_1) \end{aligned} \tag{6}$$

where $\mathbf{W}$ is the highpass filter vector except its center weight, n_1 is the center weight index, and $\mathbf{X}(n)$ consists of downsampled (by *M*) signal vector. By computing derivative of *E* with respect to $\mathbf{W}$ and equating it to zero, we obtain

$$\sum_n \sum_{k=0}^{N} h_1(k)x(Mn-k)x(Mn-m) = 0 \quad \text{for } m = 0,1,...,n_1-1,n_1+1,...,N, \tag{7}$$

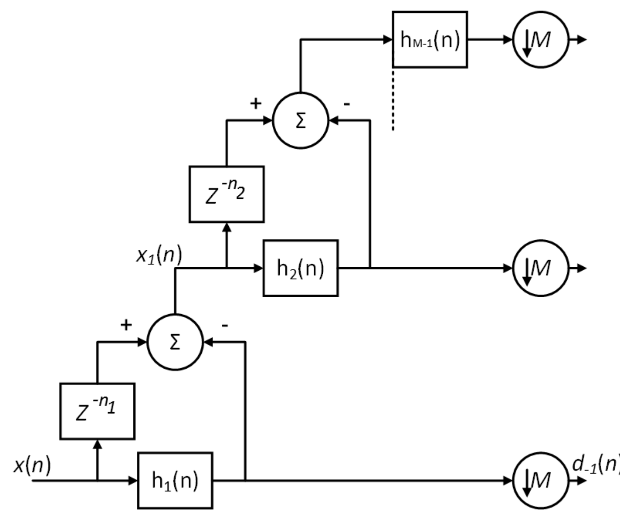

Fig. 3. *M*-Channel Matched Analysis Filterbank.

where *N* is length of the highpass filter $\mathbf{h}_1$.

This method leads to a closed form expression (7) that is the deterministic autocorrelation of the downsampled input signal. The *N*-1 linear equations can then be solved to obtain the analysis wavelet filter $\mathbf{h}_1$.

2.1.2. Design of analysis bandpass filters

Next, the highpass filtered spectrum of the input signal is subtracted from the original signal spectrum as shown in Fig. 3. This filtered signal $x_1(n)$ is used to design $\tilde{\mathbf{h}}_2$ using the same approach as outlined for highpass filter. This technique is followed for finding the next $M-2$ branches except the lowpass branch. The equivalence between filter structures in Fig. 2 and Fig. 3 can be stated mathematically as below:

$$h_2(n) = (\delta(n-n_1) - h_1(n)) \, * \, \tilde{h}_2(n) \tag{8}$$

where n_1 is the signal advancement inserted corresponding to the center (unity valued) weight position of filters $\mathbf{h}_1$. Similar equivalence holds true for other bandpass filters as well.

2.1.3. Design of analysis lowpass filter

The filter coefficients for the low pass filter, $\mathbf{h}_M$, that is the top most branch in Fig. 2 are found under the constraint that the determinant of the polyphase decomposition matrix $\mathbf{E}(z)$ is a monomial[1], i.e.,

$$det\ \mathbf{E}(z) = cz^{-d}, \tag{9}$$

$$\text{where } \mathbf{E}(z) = \begin{bmatrix} \mathrm{H}_{M1}(z) & \mathrm{H}_{M2}(z) & ..... & \mathrm{H}_{MM}(z) \\ . & . & ..... & . \\ . & . & ..... & . \\ . & . & ..... & . \\ \mathrm{H}_{21}(z) & \mathrm{H}_{22}(z) & ..... & \mathrm{H}_{2M}(z) \\ \mathrm{H}_{11}(z) & \mathrm{H}_{12}(z) & ..... & \mathrm{H}_{1M}(z) \end{bmatrix} \tag{10}$$

$$\mathrm{H}_i(z) = \mathrm{H}_{i1}(z^M) + z\mathrm{H}_{i2}(z^M) + ... + z^{M-1}\mathrm{H}_{iM}(z^M) \tag{11}$$

for $i = 1,2,...,M$. To ensure that $h_M(n)$ is a low pass filter, we impose the condition shown in (12),

$$\mathrm{H}_M(z) = (1+z)\ \mathrm{T}(z) \tag{12}$$

where $\mathrm{T}(z)$ is a polynomial to be determined. In (10), only the first row of $\mathbf{E}(z)$ is unknown. Readers may refer to [1] to know more on polyphase matrices in the context of multirate filterbanks.

On using (9), (10), (11), and (12), we get a system of linear equations (13) that can be solved to obtain the polynomial $\mathrm{T}(z)$ or the coefficients of the lowpass filter, $\mathbf{h}_M$

$$\mathbf{D}\ \mathbf{h}_M = [\mathbf{0}^T\ b\ \mathbf{0}^T]^T \tag{13}$$

where $\mathbf{D}$ is the coefficient matrix associated with the powers of z in the determinant of $\mathbf{E}(z)$, $\mathbf{h}_M$ is the vector of lowpass filter coefficients, and b is a positive constant. The position of constant b determines the net delay introduced in the reconstructed signal.

2.2. Design of FIR Synthesis Filterbank for Perfect Reconstruction

In order to attain PR, synthesis filters should satisfy the condition mentioned below [1]:

$$\mathbf{R}(z)\ \mathbf{E}(z) = \mathbf{I}_{M\times M} \tag{14}$$

where $\mathbf{R}(z)$ is the polyphase component matrix of the synthesis filters and $\mathbf{I}_{M\times M}$ is an identity matrix. From (14), the matrix $\mathbf{R}(z)$ is obtained as shown in (15) and (16). Each column of the $\mathbf{R}(z)$ matrix contains the polyphase components of the corresponding filter branch on the synthesis side. They are read in reverse order from the last row to the first as shown in (17).

$$\mathbf{R}(z) = \frac{\mathrm{Adj}\ \mathbf{E}(z)}{\det\ \mathbf{E}(z)} \tag{15}$$

$$\mathbf{R}(z) = \begin{bmatrix} \mathrm{G}_{MM}(z) & ..... & \mathrm{G}_{2M}(z) & \mathrm{G}_{1M}(z) \\ . & ..... & . & . \\ . & ..... & . & . \\ . & ..... & . & . \\ \mathrm{G}_{M2}(z) & ..... & \mathrm{G}_{22}(z) & \mathrm{G}_{12}(z) \\ \mathrm{G}_{M1}(z) & ..... & \mathrm{G}_{21}(z) & \mathrm{G}_{11}(z) \end{bmatrix} \tag{16}$$

$$\mathrm{G}_i(z) = \mathrm{G}_{i1}(z^M) + z\mathrm{G}_{i2}(z^M) + ... + z^{M-1}\mathrm{G}_{iM}(z^M) \tag{17}$$

for $i = 1, 2, ..., M$. From the above designed uniformly decimated PRFB, the rational FB is obtained using (1) and (2).

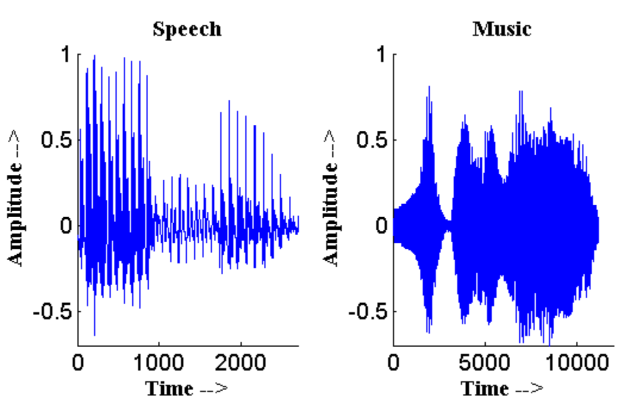

Fig. 4. Test Signals.

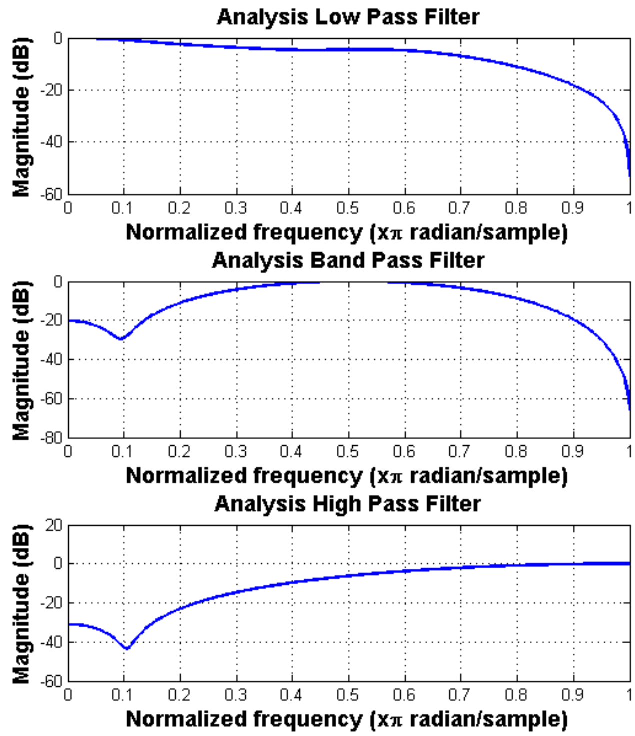

Fig. 5. Analysis End Filters for Music Clip Input Signal.

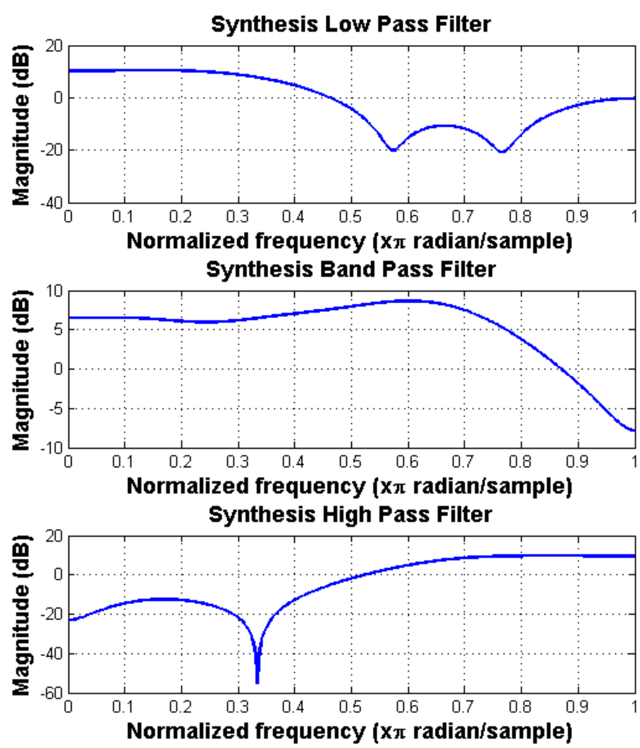

Fig. 6. Synthesis End Filters for Music Clip Input Signal.

Table 1. Filter Coefficients for (2/3, 1/3) RFB with Different Input Signals

S.No	Input Signal	Filter Coefficients
1	**Music Clip** Sampling frequency: $f_s = 11.025KHz$ Number of Samples = 11218	$\mathbf{h}_{lp}$: [0 0 0 0 0 0 0 0 -0.3638 -0.3023 -0.3638 -0.1077 0 0.3880 0.1634 -0.1066 0.1634 -0.3001 -0.0576 0 -0.0767 0 -0.0191] $\mathbf{h}_{hp}$: [0 0 0 -0.2742 0.4525 -0.2733] $\mathbf{g}_{lp}$: [0 0 0 0.7715 0 -0.7715 -1.2867 -2.0424 -1.4623 -0.3467 -1.1305 0.3467 -1.0097 0.9177 -0.2254 0.3644 0.6345 0.0814 0 -0.2290] $\mathbf{g}_{hp}$: [0 0 0 -0.8507 0.8507 -1.3951 0.0822 -0.0822 -0.2177 0.0034 0.0828 0.1333 0.0705 0.0157 -0.0443]
2	**Speech Signal** Sampling frequency: $f_s = 11.025KHz$ Number of Samples = 2713	$\mathbf{h}_{lp}$: [0 0 0 0 0 0 -0.4174 0 -0.4174 -0.2371 0 -0.0157 -0.0696 0.4756 -0.0696 -0.0325 0 -0.2867 -0.0076 0 -0.0130 0 -0.0054] $\mathbf{h}_{hp}$: [0;0;0;-0.2466;0.4860;-0.2674] $\mathbf{g}_{lp}$: [0 0 0 1.7974 0 0.6560 -0.7001 -0.6560 -1.0621 0.2997 -1.3338 0.1094 -0.5085 -0.1094 -0.0511 0.0406 0.4506 0.0041 0 -0.0360] $\mathbf{g}_{hp}$: [0 0 0 -0.5432 1.1667 -1.1667 0.5330 0.1945 -0.1945 0.0865 0.0288 -0.0370 0.0103 0.0010 -0.0092]

3. EXPERIMENTAL RESULTS

In this section, we demonstrate the performance of our approach for the design of a critically sampled RFB on two real signals shown in Fig. 4. As an example, we show the design approach for RFB with decimation ratios of 2/3 and 1/3 in the lowpass and highpass filter branch, respectively. The RFB can be represented as an equivalent *M*-band (*M*=3) uniformly decimated filter bank as discussed in the previous section.

Results are shown for the following filter lengths: 3 (highpass), 5 (bandpass), and 9 (lowpass). The length of lowpass filter is kept higher to provide enough degrees of freedom to satisfy the constraint (9) on the determinant of **E**(z). PR can be achieved with other filter lengths as well, as long as sufficient degrees of freedom are provided. Although this has been verified, results are not shown here for brevity.

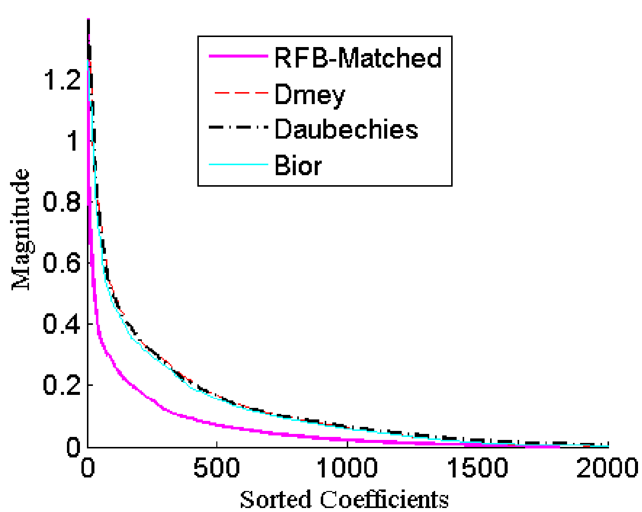

Fig. 7. Comparison of Magnitude of the Coefficients for Different Wavelet Structures.

Fig. 5 and 6 show the designed filters for a 3-channel PRFB corresponding to the 'speech' signal. The filter coefficients for two different input signals are shown in Table 1. With both the input signals, the proposed method achieves perfect reconstruction with RFB and the NMSE is of the order 10^{-15}.

The magnitude plot of the sorted subband coefficients of both the lowpass and highpass branch of a one-level decomposition of RFB is shown in Fig. 7. We compare the proposed method with Daubechies (Orthogonal 8-length filters), biorthogonal-9/11, and discrete Meyer wavelets for the 'speech' input signal. Our design of RFB with matched filters gives better results compared to all these wavelets in terms of compressibility of transform domain coefficients, due to much larger number of coefficients with very low magnitude.

4. CONCLUSION

In this paper, we have proposed a method for the design of a critically sampled rational wavelet transform with freedom to select decimation ratios. The filters in the RFB are derived from the given input signal (signal matched), and hence, yield a compact transform domain representation. While there is no signal matched approach for the design of rational filterbank, most of the existing techniques for the design of rational filter bank are also either computationally intensive or rely on IIR filters to achieve perfect reconstruction. Our method for design of rational filterbank is able to achieve 1) FIR perfect reconstruction filterbank, 2) follows simple procedure involving solution of a linear system of equations, and 3) designs signal-matched structure giving improved compressibility.

5. REFERENCES

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1993.

[2] Martin Vetterli and Jelena Kovačevic, *Wavelets and Subband Coding*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1995.

[3] V.X. Afonso, Willis J. Tompkins, T.Q. Nguyen, and Shen Luo, "Ecg beat detection using filter banks," *Biomedical Engineering, IEEE Transactions on*, vol. 46, no. 2, pp. 192–202, Feb 1999.

[4] T.D. Tran, R.L. De Queiroz, and T.Q. Nguyen, "Linear-phase perfect reconstruction filter bank: lattice structure, design, and application in image coding," *Signal Processing, IEEE Transactions on*, vol. 48, no. 1, pp. 133–147, Jan 2000.

[5] I Bayram and IW. Selesnick, "Overcomplete discrete wavelet transforms with rational dilation factors," *Signal Processing, IEEE Transactions on*, vol. 57, no. 1, pp. 131–145, Jan 2009.

[6] I Bayram and IW. Selesnick, "Frequency-domain design of overcomplete rational-dilation wavelet transforms," *Signal Processing, IEEE Transactions on*, vol. 57, no. 8, pp. 2957–2972, Aug 2009.

[7] T. Blu, "A new design algorithm for two-band orthonormal rational filter banks and orthonormal rational wavelets," *Signal Processing, IEEE Transactions on*, vol. 46, no. 6, pp. 1494–1504, Jun 1998.

[8] A. Baussard, Fred Nicolier, and Frdric Truchetet, "Rational multiresolution analysis and fast wavelet transform: application to wavelet shrinkage denoising.," 2004, pp. 1735–1747.

[9] Si Tran Nguyen Nguyen and Brian W. H. Ng, "Bi-orthogonal rational discrete wavelet transform with multiple regularity orders and application experiments," *Signal Process.*, vol. 93, no. 11, pp. 3014–3026, Nov. 2013.

[10] Anubha Gupta and ShivDutt Joshi, "On the concept of intrinsic wavelet functions," in *Signal Processing and Communications, International Conference on*, July 2014.

[11] Anubha Gupta, S.D. Joshi, and S. Prasad, "A new approach for estimation of statistically matched wavelet," *Signal Processing, IEEE Transactions on*, vol. 53, no. 5, pp. 1778–1793, May 2005.