RAMANUJAN FILTER BANKS FOR ESTIMATION AND TRACKING OF PERIODICITIES

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ABSTRACT

We propose a new filter-bank structure for the estimation and tracking of periodicities in time series data. These filter-banks are inspired from recent techniques on period estimation using high-dimensional dictionary representations for periodic signals. Apart from inheriting the numerous advantages of the dictionary based techniques over conventional period-estimation methods such as those using the DFT, the filter-banks proposed here expand the domain of problems that can be addressed to a much richer set. For instance, we can now characterize the behavior of signals whose periodic nature changes with time. This includes signals that are periodic only for a short duration and signals such as chirps. For such signals, we use a time *vs* period plane analogous to the traditional time *vs* frequency plane. We will show that such filter banks have a fundamental connection to Ramanujan Sums and the Ramanujan Periodicity Transform.

***Index Terms*—** Period Estimation, Time *vs* Period Plane, Periodicity Filter Banks, Ramanujan Sums, Periodicity Transforms.

1. INTRODUCTION

A discrete time signal $x(n)$ is said to be periodic with period P if P is the smallest positive integer such that

$$x(n+P) = x(n) \quad \forall \ n \in \mathbb{Z} \tag{1}$$

Given a finite length interval of such a signal, one often wishes to estimate its period P. Sometimes, a periodic signal might be generated as a sum of signals with much smaller periods. For example, a sum of signals with periods 11 and 15 can result in a signal with period $11 \times 15 = 165$. For such a signal, we want to be able to detect these 'hidden periods' 11 and 15. More generally, a signal might exhibit periodic behavior that changes with time. Examples include signals that are periodic only in a localized region, signals such as chirps, and so on.

The limitations of traditional spectrum estimation techniques such as those using DFT, when used for period estimation, have been noticed in the past [20], [18]. As an alternative, Sethares and Staley in [20] proposed several algorithms based on comparing the projection energies of a given periodic signal on a series of subspaces representing different periodicities. These subspaces have a concatenated structure due to which their algorithms involve an intricate sequence of projections. A different approach was suggested in [14], where the authors introduced the Farey dictionary as a high-dimensional representation for periodic signals. This dictionary was based on the union of columns of several DFT matrices, and was used to estimate the period by finding sparse representations for periodic signals. In a recent paper [18], we generalized the Farey dictionary to much simpler real valued dictionaries by introducing alternatives to the DFT matrix in the context of period estimation. In addition, we reformulated the period recovery problem into a convex program with a closed form linear solution that offers several orders of magnitude faster solutions than the sparsity based techniques.

A limitation of the above methods is that they were designed for signals whose periodic behavior doesn't change with time. To use them for characterizing more general periodic behavior such as chirps, we will have to break the signal into multiple blocks and apply these methods on each block. In such schemes, it would be beneficial to use smaller block lengths for detecting smaller periods and larger block lengths for detecting larger periods to obtain good localization.

In this paper, we propose an elegant way to achieve this. The linear solutions for the techniques presented in [18] enable us to design filter-bank implementations for the same. So in Sec. 2, we will briefly review the dictionary approaches of [18, 14]. In Sec. 3, we illustrate how to arrive at a filter bank structure starting from the dictionary methods. We also present examples of time *vs* period planes obtained using the proposed filter bank and discuss their advantages over the traditional Short-Time Fourier Transform (STFT) based time frequency plane. Our method yields a non-uniform tiling of the time *vs* period plane similar in spirit to the wavelet tiling of the time frequency plane [8, 15], unlike the above mentioned block-based schemes and [7], which give a uniform tiling like the STFT. Finally, in Sec. 4, we interpret the period identifying capability of the proposed filter bank in the frequency domain.

2. DICTIONARY METHODS FOR PERIOD ESTIMATION

The set of all signals that satisfy $x(n+P) = x(n)$ for all n forms a vector space, call it $\mathcal{V}_P$. It includes signals that have divisors of P as periods. Let $d_i, 1 \leq i \leq K$ denote the divisors of P in increasing order, so that $d_1 = 1$ and $d_K = P$. Let $\phi(\cdot)$ denote the Euler totient function. Now consider a matrix of the form

This work was supported in parts by the ONR grant N00014-11-1-0676, and the Information Science and Technology (IST) initiative of Caltech.

$$\mathbf{A} = [\mathbf{C}_{d_1} \quad \mathbf{C}_{d_2} \quad \ldots \quad \mathbf{C}_{d_K}] \tag{2}$$

with the following properties:

1. Each $\mathbf{C}_{d_i}$ is a $P \times \phi(d_i)$ matrix so that the total number of columns in $\mathbf{A}$ is (from [3]) $\sum_{d_i|P} d_i = P$. Thus $\mathbf{A}$ is a $P \times P$ matrix.
2. Each column of $\mathbf{C}_{d_i}$ is a length P segment of a sequence with period d_i.
3. $\mathbf{A}$ has full rank P.

Such a matrix will be referred to as a **periodicity matrix**. The columns of $\mathbf{A}$ form a basis for $\mathbb{C}^P$, so that by periodically extending them, we can obtain a basis for $\mathcal{V}_P$. They were introduced in [18] as generalizations of the Ramanujan Periodicity Transform (RPT) Matrices [12]. In [11], sequences known as Ramanujan sums [19] were identified as having several useful properties for studying periodicities. They were used to construct bases similar to (2) for $\mathcal{V}_P$, called the RPT matrices in [12]. Apart from the RPT, the DFT and the Walsh-Hadamard matrices are also examples of periodicity matrices. Taking $\phi(d_i)$ columns with period d_i for every $d_i|P$ ensures that such a basis for $\mathcal{V}_P$ also contains a basis for $\mathcal{V}_{d_i}$ for every divisor d_i of P [18].

Periodicity matrices have several important properties. For example, if $\mathbf{x}$ is a vector consisting of P successive samples of a signal $x(n)$ in $\mathcal{V}_P$ with period $d_i|P$, and $\mathbf{y}$ is such that $\mathbf{x} = \mathbf{A}\mathbf{y}$, then the lcm of the periods of all those columns of $\mathbf{A}$ that are multiplied by non-zero entries in $\mathbf{y}$ is exactly equal to the period of the signal $x(n)$, namely d_i.

The above technique gives us a way of estimating the period of a signal if it is known that it belongs to a particular $\mathcal{V}_P$. We will refer to this as *the lcm method*. But in general, such information might not be available a priori. So in [14] and [18], we propose methods based on dictionaries constructed by combining periodicity matrices of different sizes. For instance, let $\mathbf{x}$ be a vector consisting of N successive samples of a periodic signal $x(n)$ whose period we want to estimate. Consider a particular family of periodicity matrices - for example, the RPT matrices [12]. For every d in $1 \leq d \leq P_{max}$, where P_{max} is the largest expected period, construct a $d \times d$ periodicity matrix and take only its $\phi(d)$ columns that have period d. Extend these columns periodically to length N, truncating the last period if necessary. We form a dictionary $\mathbf{A}$ by collecting such columns for every d in $1 \leq d \leq P_{max}$. If $x(n)$ had period less than P_{max}, then it *has to be* a linear combination of the columns of $\mathbf{A}$. This is because, if it had period P, then the columns whose periods are divisors of P must be able to span it, since they come from a $P \times P$ periodicity matrix.

But the dictionary is likely to be fat since, P_{max} is usually of the order of the length of the data, while the sum of Euler totient function from 1 to P_{max} is $O(\frac{3{P_{max}}^2}{\pi^2})$ [3]. So it is likely that $\mathbf{x} = \mathbf{A}\mathbf{y}$ has multiple solutions for $\mathbf{y}$. We are interested in the one that involves subspaces corresponding to period P and its factors so that, as in the situation when $\mathcal{V}_P$ is known, we might be able to estimate the period of the signal by taking the lcm of the periods of the columns of $\mathbf{A}$ that are present in the solution. In [14], this was formulated as a sparse vector recovery problem [1, 2, 4, 9, 16]. But in [18], we showed that the recovery problem can reformulated into a simpler convex program with a closed form solution. The idea was to look at the period estimation problem as trying to fit the given signal with signals having as small periods as possible. Consider the following:

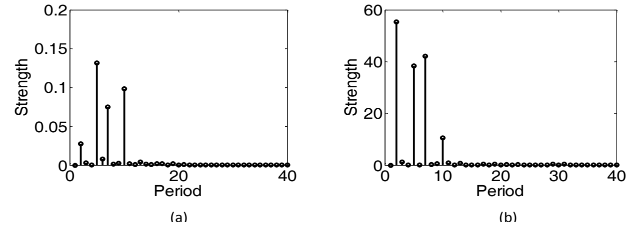

Fig. 1. Parts (a) and (b)- Strength vs period plots for a period 70 signal using (4) and (6) respectively (see text for details). Plots shown only till period 40 for clarity since all the further values are zeros.

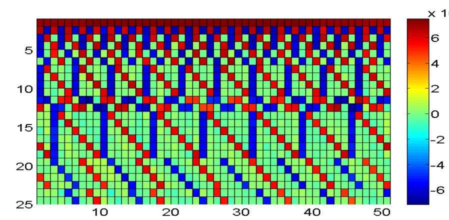

Fig. 2. The first 50 rows of the Pseudo-inverse matrix in (5) for $N = 200$ and $P_{max} = 200$.

$$\min \|\mathbf{D}\mathbf{y}\|_2 \quad s.t. \quad \mathbf{x} = \mathbf{A}\mathbf{y} \tag{3}$$

where $\mathbf{D}$ is a diagonal matrix whose i^{th} diagonal entry is $f(P_i)$, where P_i is the period of the i^{th} column of $\mathbf{A}$ and $f(\cdot)$ is some increasing function. By introducing $\mathbf{D}$ in (3), the columns in $\mathbf{A}$ that have larger periods contribute more towards the objective function than those with smaller periods for similar entries in $\mathbf{y}$. So in a way, columns with larger periods are being penalized more and the algorithm will try to use columns of $\mathbf{A}$ with as small periods as possible to fit $x(n)$. Problem (3) has a closed form solution:

$$\mathbf{y}_\star = \mathbf{D}^{-2}\mathbf{A}^T \left(\mathbf{A}\mathbf{D}^{-2}\mathbf{A}^T\right)^{-1} \mathbf{x} \tag{4}$$

Fig 1 (a) shows the results of solving (3) using a Ramanujan dictionary for a period 70 signal with two complete and a third incomplete period. It was generated as a sum of a period 7 and a period 10 signal. The penalty function was chosen to be $f(P) = P^2$ and P_{max} as 90. For each period, the plots show the sum of squares of those components of the optimal solution $y_\star$ of (3) that correspond to columns of the dictionary with that particular period. We do not show the period 1 component, since it is just a DC signal. We can see peaks at periods 2, 5, 10 and 7, and using the lcm method, we conclude that the original signal had period 70. In addition, we can also conclude that it was actually generated by adding a period 7 and a period 10 signal.

3. FROM DICTIONARIES TO FILTER BANKS

Among the different choices of periodicity matrices for constructing the dictionary in (3), we consistently observed that the Ramanujan design performs the best under perturbations such as noise. Looking more closely at the form of its left-inverse in (4),

$$\mathbf{P} = \mathbf{D}^{-2}\mathbf{A}^T \left(\mathbf{A}\mathbf{D}^{-2}\mathbf{A}^T\right)^{-1} \tag{5}$$

we noticed that its rows have an interesting pattern. Similar to (the transpose of) the dictionary itself, the rows of the left-inverse seem to be periodic, with exactly $\phi(P)$ rows with period P. For instance, Fig. 2 shows a section of the first 50 rows of a left-inverse matrix obtained from a Ramanujan dictionary with parameters $N = 200$ and $P_{max} = 200$. Clearly, the first row has period 1, the second has period 2, the third and fourth have period 3 and so on. Moreover, for many of the periods, the $\phi(P)$ rows corresponding to them are approximately shifted versions of each other. Such a strong pattern raises the question if we could directly design suitable 'left-inverses' with the same structure *without having to formulate it as an optimization problem like* (3). For instance, we experimentally observed that the following expression instead of (4) gives equally good results (see Fig. 1 (b)):

$$\mathbf{y}_\star = \mathbf{D}^{-1}\mathbf{A}^T\mathbf{x} \tag{6}$$

Using (6) has an advantage. We can implement it efficiently using a filter bank. To see this, consider the case when the input data is of infinite length (streaming data). One way to process the signal in that case is to apply (6) on successive input blocks of length N, each shifted by one sample. When we use the Ramanujan dictionary, for every period P we are taking the inner product of the input blocks with the following $\phi(P)$ vectors (assuming that N, the data length, is reasonably larger than the signal's period P):

$$\mathbf{C}_P^{(0)} = [c_P(0) \quad c_P(1) \quad c_P(2) \quad \dots \quad c_P(N-1)]^T$$
$$\mathbf{C}_P^{(1)} = [c_P(1) \quad c_P(2) \quad c_P(3) \quad \dots \quad c_P(N)]^T$$

and so on till $\mathbf{C}_P^{(\phi(P)-1)}$, where $c_P(n)$ represents the P^{th} Ramanujan sum [11]. After dividing these inner products by the penalty function $f(P)$, we sum their squares to define the *strength* of that period in Fig. 1 (b):

$$y_P = \sum_{i=0}^{\phi(P)-1} \left| \frac{\langle \mathbf{x}, \mathbf{C}_P^{(i)} \rangle}{f(P)} \right|^2 \tag{7}$$

Notice that the $\mathbf{C}_P^{(i)}$'s are periodic vectors with period P. Moreover, they are *nearly shifted* versions of the same vector $\mathbf{C}_P^{(0)}$. Based on this observation, we propose a slightly modified implementation for (7):

$$y_P(n) = \sum_{i=n-\phi(P)+1}^{n} \left| \frac{(x * h_P)(n)}{f(P)} \right|^2$$

where $(x * h_P)$ denotes convolution, and

$$h_P = \{c_P(0) \quad c_P(-1) \quad \dots \quad c_P(-LP+1)\} \tag{8}$$
$$= \{c_P(0) \quad c_P(1) \quad c_P(2) \quad \dots \quad c_P(LP-1)\} \tag{9}$$

for some integer L (Ramanujan sums are symmetric: $c_P(n) = c_P(-n)$). That is, it contains L complete periods of the Ramanujan sum $c_P(n)$. Choosing the filter length as LP instead of a fixed N as in (6) effectively enables us to detect smaller periods using smaller blocks and larger periods using larger blocks. A collection of such filters for all periods going from 1 to P_{max}, as shown in Fig. 3, is what we call as the Ramanujan Filter Bank (RFB). A plot of the outputs $y_P(n)$ for different P across different n will be referred to as **the time *vs* period plane**.

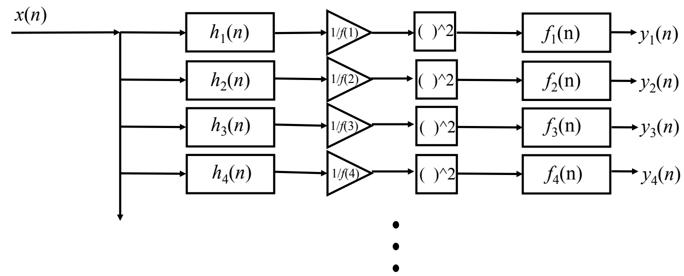

Fig. 3. Block diagram of the proposed Ramanujan filter bank.

We show two examples here. Fig. 4(a) shows the time *vs* period plane for a length 668 signal that has a randomly generated period 3 component between samples 201 and 218 and a sum of randomly generated period 15 and period 11 components from samples 319 to 469. The sum of the period 15 and period 11 signals is actually a signal with period $15 \times 11 = 165$. $L = 15$ and $f(P) = P^2$ were chosen for the RFB. In part (a), the localized period 3 component is detected initially. This is followed by periods 3, 5, 11 and 15 showing up, and using the lcm method, we can conclude that the signal exhibits a periodicity of 165 and is a sum of period 15 and period 11 components. The period 1 DC component is not shown. The outputs of all the filters beyond period 50 were 0 and hence not shown. Note that the q^{th} Ramanujan filter's output is delayed by $qL/2$ due to the causal implementation (9). This causes different divisors of 15 to be detected with different delays. To avoid this, part (b) was obtained from part (a) by advancing the output of each Ramanujan filter h_q by $\lfloor qL/2 \rfloor$ so that all the divisors of a particular period are expressed concurrently. Parts (c) and (d) show the time vs frequency plane using STFT (assuming 1 Hz sampling rate). In part (c), we had to use a rectangular window of size 128 to reasonably identify the period 11 and 15 components. The peaks in the spectrogram correspond to periods 15.06, 11.13, 7.53, 5.56, 5.02, 3.82, 3.66, 3.01, 2.75, 2.51, 2.21 and 2.13. These numbers roughly correspond to 11, 15 and their harmonics. But this window was too wide to detect the period 3 component present between samples 201 and 218. So in part (d), a window of size 32 was chosen. Although the localized period 3 component gets detected well, this window is not sufficient to identify the period 11 and 15 components. We do not have to worry about having different analysis for different periods in the RFB since the length of the each filter was chosen proportional to its period. Moreover, if the periodic signal is a superposition of a number of signals with smaller periods such as the 11 and 15 case, then using the lcm method of the RFB might be more convenient than searching for fundamental frequencies in spectrograms.

In the second example, we consider the inverse chirp signal $x(t) = \sin(1/at)$ in the interval $t \in [2, 10]$ seconds, with $a = 0.01/2\pi$, sampled every 0.01s (Fig. 5(a)). The instantaneous period of this signal is $2\pi a t^2$. This quadratic behavior is evident in the time vs period plane in part (b) (L=5). Part (d) shows the time frequency plane obtained from STFT using a length 32 rectangular window. It captures the small

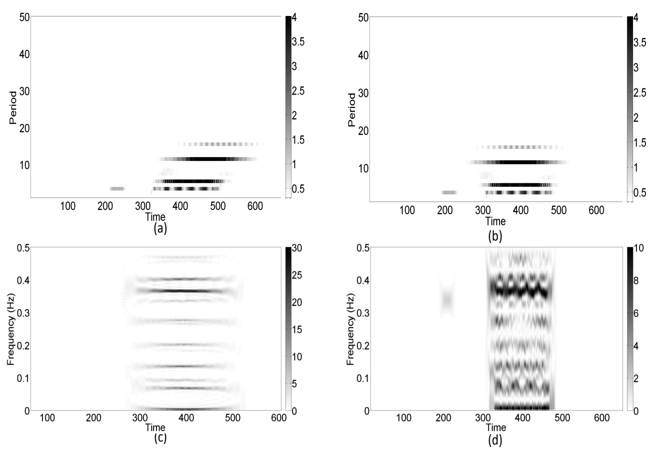

Fig. 4. Parts (a) and (b) - The time vs. period plane for a signal exhibiting localized periodicities using RFB and shifted RFB. Parts (c) and (d) show the time-frequency plane using STFT with window sizes 128 and 32 respectively. Refer text for details.

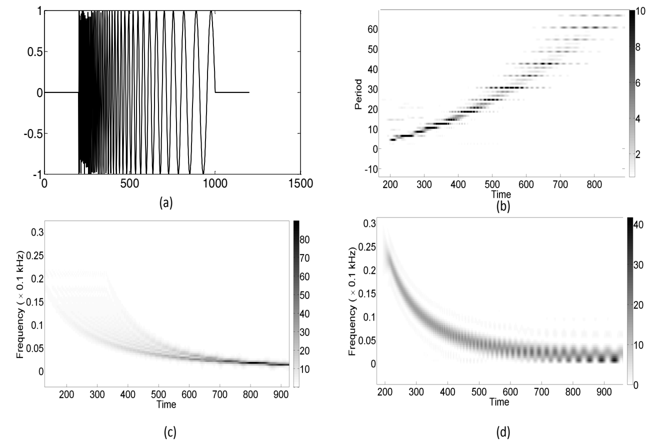

Fig. 5. Part (a) - Sampled inverse chirp signal. Part (b) - The time vs. period plane using shifted RFB. Parts (c) and (d) show the time-frequency plane using STFT with window sizes 256 and 32 respectively. Refer text for details.

Table 1. Period Estimation using STFT and the RFB

t	Instantaneous Period	STFT(32)	STFT(256)	RFB
2.1 s	44.1 ms	44.9 ms	71.1 ms	40 ms
7.5 s	562 ms	639 ms	512 ms	560 ms

periods well as shown in Table 1. But the larger periods are mis-estimated. When the frequency is very small, the finite frequency resolution of STFT limits the accuracy of the $P = 1/f$ estimate. If we increase the window size to 256 to better estimate the higher periods, as in Fig. 5 (c), the smaller periods are smeared out in the time frequency plane. The estimate for larger periods is still not very accurate (Table 1). The RFB on the other hand offers good estimates for both small and large periods. $f(P)$ was chosen as $(\phi(P))^2$ here to show that a wide choice is available for its selection.

4. RAMANUJAN FILTER BANKS - A MORE FUNDAMENTAL APPROACH

In Sec. 3, we developed the RFB as an approximate but convenient alternative to (4). We had experimentally observed that among the various choices for the dictionary in (6), the RPT based dictionary gave the best results. So we used it to construct the filter bank, and noticed that the lcm method could still be used to estimate the periods. The observation that Ramanujan filters can be used to estimate periods using the lcm method is not just a coincidence, as shown by the following theorem (essentially follows from Theorem 12 of [11]) :

Theorem 1. *Any periodic signal $x(n)$ can be expressed as a sum of exponentials in a unique way as*

$$x(n) = \sum_{i=1}^{K} \alpha_i e^{\frac{j2\pi k_i}{q_i} n} \qquad \alpha_i \neq 0 \tag{10}$$

where k_i and q_i are integers satisfying $\gcd(k_i, q_i) = 1 \ \forall \ i$. The period of $x(n)$ is exactly equal to $\mathrm{lcm}\{q_i\}$, rather than a proper divisor of it. ♢

Now consider a signal $x(n)$ that is periodic with period P. It can be expressed as a sum of exponentials in the form of (10) in a *unique* way through its Fourier series expansion:

$$x(n) = \sum_{k=1}^{P} \alpha_k e^{j\frac{2\pi k}{P} n} \quad , \quad \alpha_k = \frac{1}{P} \sum_{n=1}^{P} x(n) e^{-j\frac{2\pi k}{P} n} \tag{11}$$

by reducing each $\frac{k}{P}$ to its lowest form. From Theorem 1 and (11), to estimate the period of $x(n)$ we need to find among the set of all frequencies of the form $\{\frac{2\pi k_i}{q_i} \ : \ \gcd(k_i, q_i) = 1\}$, the ones at which the signal's spectrum has non-zero energy. We can then take the lcm of the periods q_i of the exponentials with those frequencies as an estimate for the signal's period.

This is exactly what is happening in the RFB as the length of the filters tends to infinity. The spectrum of the q_i^{th} Ramanujan filter with impulse response $c_{q_i}(n)$ is non-zero only at those frequencies $\frac{2\pi k_i}{q_i}$, where $\gcd(k_i, q_i) = 1$. So its output will be non-zero *if and only if* $x(n)$'s decomposition into the form (10) has a q_i periodic exponential. So taking the lcm of the indices of those Ramanujan filters that have non-zero output (the lcm property) is indeed a valid estimate for the period of the signal.

5. CONCLUSION AND FUTURE WORK

In this work, we introduced the idea of Ramanujan filter banks as a tool for identifying periodic patterns in data. The property that makes the Ramanujan filters useful is their support in the frequency domain. We cannot replace $c_q(n)$ with an arbitrary q periodic impulse response and expect the lcm property to still hold. Nevertheless, we can construct a family of filters with the same support by convolving the Ramanujan sums $c_q(n)$ with arbitrary sequences whose spectrum is not zero at the frequencies $\frac{2\pi k_i}{q}$, where $\gcd(k_i, q) = 1$.

Such filter banks might be of interest in various real-world settings, for instance in detecting seizures using EEG data or to identify repetitive structures in protein molecules [6]. These will be part of our future efforts.

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