# MULTIDIMENSIONAL RAMANUJAN-SUM EXPANSIONS ON NONSEPARABLE LATTICES

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## ABSTRACT

It is well-known that the Ramanujan-sum $c_q(n)$ has applications in the analysis of periodicity in sequences. Recently the author developed a new type of Ramanujan-sum representation especially suited for finite duration sequences $x(n)$. This is based on decomposing $x(n)$ into a sum of signals belonging to so-called Ramanujan subspaces $\mathcal{S}_{q_i}$. This offers an efficient way to identify periodic components using integer computations and projections, since $c_q(n)$ is integer valued. This paper revisits multidimensional signals with periodicity on possibly nonseparable integer lattices. Multidimensional Ramanujan-sum and Ramanujan-subspaces are developed for this case. A Ramanujan-sum based expansion for multidimensional signals is then proposed, which is useful to identify periodic components on nonseparable lattices.

***Index Terms*—** Ramanujan-sum on lattices, periodicity lattices, periodic subspaces, integer basis.

## 1. INTRODUCTION

The Ramanujan-sum [7], introduced by Srinivasa Ramanujan in 1918, is known to have applications in the analysis of periodicity in sequences [4], [5], [6], [8], [10]. The Ramanujan-sum is defined as

$$c_q(n) = \sum_{\substack{k=1 \\ k,q \text{ coprime}}}^{q} e^{j2\pi kn/q} \tag{1}$$

It is well known [7] that $c_q(n)$ is integer-valued. In recent work [17]–[18] we developed new Ramanujan-sum representations especially suited for finite duration (FIR) signals. This is based on decomposing $x(n)$ into a sum of signals belonging to so-called *Ramanujan subspaces* $\mathcal{S}_{q_i}$, which were introduced and studied in detail in [17]. This representation expresses $x(n)$ as a sum of orthogonal sequences $x_{q_i}(n)$:

$$x(n) = \sum_{q_i|N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n)}, \quad 1 \le n \le N \tag{2}$$

where each $x_{q_i}(n) \in \mathcal{S}_{q_i}$ and has period $q_i$. The notation $q_i|N$ means that $q_i$ is a *divisor* of $N$.

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In (2) $c_{q_i}(n)$ is the $q_i$th Ramanujan sum, and there are $\phi(q_i)$ circularly shifted versions $c_{q_i}(n-l)$ involved in the expansion, where $\phi(q_i)$ is the *Euler totient* (Sec. 1.2). Since $\sum_{q_i|N} \phi(q_i) = N$ (see [2]), Eq. (2) is equivalent to a linear combination of $N$ basis functions $c_{q_i}(n-l)$, where $N$ is also the signal duration. Since $c_{q_i}(n)$ are integers, $\{c_{q_i}(n-l)\}$ is an *integer basis* where the basis functions are periodic. The periodicity components $x_{q_i}(n)$, which are orthogonal projections of $x(n)$ onto the Ramanujan subspaces $\mathcal{S}_{q_i}$, can be computed using projection operators that are integer matrices (up to a scalar factor) [18]. A practical generalization of this representation, which removes some of the restrictions of (2) (such as the requirement that the hidden periods be divisors of $N$) is presented in [11] based on the so-called Ramanujan dictionaries, which itself is an improvement over Farey dictionaries introduced in [16].

### 1.1. Scope and outline

In this paper we generalize the periodic decomposition (2) to the case of multidimensional (MD) signals. This involves two things. First, the definition of Ramanujan sums should be extended to MD signals with integer *periodicity matrices* to describe periodicity over *lattices*. Second, the double-sum representation (2) should be generalized. We have shown that both of these are possible. Since the results are quite involved mathematically, the emphasis here will be on explaining the main ideas. Details will be presented in a more extensive publication in the future [19].

Section 2 reviews multidimensional periodic signals on lattices, and Sec. 3 develops the transition from the Fourier representation to the proposed multidimensional Ramanujan style representation. This development is completed in Sec. 4 by defining the Ramanujan sum for multidimensional signals and using it in the representation.

### 1.2. Notations and preliminaries

The notation $a|b$ means that $a$ is a divisor of $b$. $(k, q)$ denotes the greatest common divisor (gcd) of integers $k$ and $q$. The Euler totient function $\phi(q)$ is the number of integers in $1 \le k \le q$ coprime to $q$. We use $|\mathbf{Q}|$ to denote $|\det \mathbf{Q}|$. In multidimensions the definition of Ramanujan sums involves coprimality of an integer vector $\mathbf{k}$ and an integer matrix $\mathbf{Q}$. The following definitions, admittedly dense, are therefore crucial:

1. $\mathbb{Z}^D$ denotes the set of $D$-dimensional integer column vectors, and $\mathbb{Z}^{p\times r}$ denotes the set of $p \times r$ integer matrices. $\mathbf{U} \in \mathbb{Z}^{p\times p}$ is called unimodular if $\det \mathbf{U} = \pm 1$.

2. $\mathbf{L} \in \mathbb{Z}^{p\times p}$ is a *left divisor* of $\mathbf{P} \in \mathbb{Z}^{p\times r}$ if $\mathbf{P} = \mathbf{LR}$ for some $\mathbf{R} \in \mathbb{Z}^{p\times r}$.

3. $\mathbf{L} \in \mathbb{Z}^{p\times p}$ is a *left common divisor* (lcd) of $\mathbf{P} \in \mathbb{Z}^{p\times r}$ and $\mathbf{Q} \in \mathbb{Z}^{p\times l}$ if it is a left divisor of both $\mathbf{P}$ and $\mathbf{Q}$. And $\mathbf{G}$ is a *greatest left common divisor* or **glcd** of $\mathbf{P}$ and $\mathbf{Q}$ if it is a lcd, and any left common divisor $\mathbf{L}$ is such that $\mathbf{G} = \mathbf{LG}_1$ for some $\mathbf{G}_1 \in \mathbb{Z}^{p\times p}$. It can be shown that the glcd exists if and only if $[\mathbf{P}\ \mathbf{Q}]$ has rank $p$. We say $\mathbf{P}$ and $\mathbf{Q}$ are **left coprime** if every glcd is unimodular.

4. *Smith-form:* Any rank-$\rho$ matrix $\mathbf{P} \in \mathbb{Z}^{p\times r}$ can be written as $\mathbf{P} = \mathbf{U\Lambda V}$ where $\mathbf{U}$ and $\mathbf{V}$ are unimodular and $\mathbf{\Lambda} \in \mathbb{Z}^{p\times r}$ is a diagonal matrix (Smith-form of $\mathbf{P}$) with first $\rho$ diagonal elements $\neq 0$ and such that $\lambda_i | \lambda_{i+1}$.

## 2. MULTIDIMENSIONAL PERIODIC SIGNALS

A $D$-dimensional sequence $x(\mathbf{n})$ is said to be periodic [1] if there is $\mathbf{M} \in \mathbb{Z}^{D\times D}$ such that $x(\mathbf{n}) = x(\mathbf{n} + \mathbf{Mm})$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^D$. We call $\mathbf{M}$ a *repetition matrix*. The period itself should be defined more carefully. We say $\mathbf{P}$ is the *periodicity matrix* for the periodic signal $x(\mathbf{n})$ if $\det \mathbf{P} > 0$ and

$$x(\mathbf{n}) = x(\mathbf{n} + \mathbf{Pm}) \tag{3}$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^D$, and furthermore, $\mathbf{P}$ is a left divisor of all repetition matrices $\mathbf{M}$, i.e., $\mathbf{M} = \mathbf{PR}$ for some $\mathbf{R} \in \mathbb{Z}^{D\times D}$. Clearly $\det \mathbf{P} \leq |\det \mathbf{M}|$ as long as $\det \mathbf{M} \neq 0$. Thus the period has the smallest determinant among all nonsingular repetition matrices. Now consider the set of all vectors of the form $\mathbf{v} = \mathbf{Px}$ where $\mathbf{x} \in [0,1)^D$, that is, each component $x_i$ satisfies $0 \leq x_i < 1$. This region is called the *fundamental parallelepiped* of $\mathbf{P}$ [12], and is denoted as $FPD(\mathbf{P})$. The values of the periodic signal $x(\mathbf{n})$ for $\mathbf{n} \in FPD(\mathbf{P})$ constitute the fundamental period, and these values repeat. Figure 1(a) shows $FPD(\mathbf{P})$ for the periodicity matrix

$$\mathbf{P} = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \tag{4}$$

The integers $\mathbf{n} \in FPD(\mathbf{P})$ are shown by little circles. It can be shown that there are exactly $\det \mathbf{P}$ integers in $FPD(\mathbf{P})$ (ten in this case). The *lattice* generated by $\mathbf{P}$, denoted as $LAT(\mathbf{P})$, is the set of all integers of the form $\mathbf{n} = \mathbf{Pm}$ where $\mathbf{m}$ is an integer vector. Figure 1(a) shows some integers in $LAT(\mathbf{P})$ using black circles. So the periodicity matrix defines a lattice such that the values of $x(\mathbf{n})$ in $FPD(\mathbf{P})$ are repeated at the periodically shifted points defined by the lattice. Given any unimodular matrix $\mathbf{U}$, it is readily shown that $\mathbf{P}$ and $\mathbf{PU}$ generate the same lattice. So if $\mathbf{P}$ is the period, then so is $\mathbf{PU}$. But the *FPD* generated by $\mathbf{PU}$ is different.

## 3. FROM FOURIER TO RAMANUJAN

Now, let $x(\mathbf{n})$ be a finite duration signal (FIR signal) with support in $FPD(\mathbf{P})$. Or equivalently we can think of it as a periodic signal with repetition matrix $\mathbf{P}$. For such a signal we would like to develop a representation similar to the recent Ramanujan representation (2). Since the Ramanujan sums $c_{q_i}(n)$ are defined as in Eq. (1), the representation is closely related to the DFT representation as explained in Sec. V.E

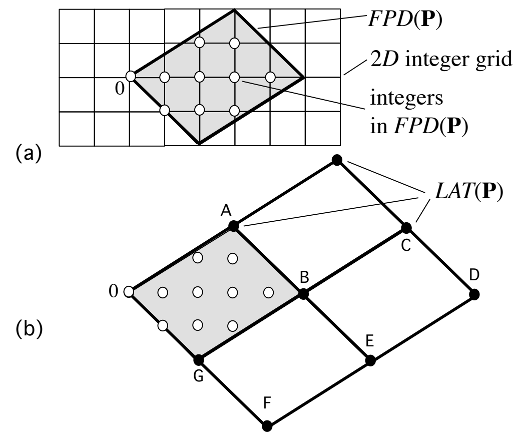


**Fig. 1**. (a) $FPD(\mathbf{P})$ and (b) $LAT(\mathbf{P})$ where $\mathbf{P}$ is as in Eq. (4). The origin is in the FPD, and is a lattice point as well. Other lattice points are shown by black circles.

of [18]. We shall therefore start from the multidimensional DFT representation [1] for a signal supported in $FPD(\mathbf{P})$, and define an appropriate set of left divisors $\mathbf{P}_i$ for $\mathbf{P}$ (analogous to $q_i|q$ in Eq. (2)). For each such divisor we will define a sum analogous to Eq. (1), and eventually arrive at the desired representation. Stated this way, it appears to be a straightforward extension of the 1D case, but there are a number of subtleties involved. For example, unlike (2) where all divisors $q_i$ are involved, only a subset of left divisors of $\mathbf{P}$ will be involved, called the *legitimate* divisors. Secondly, it turns out that the definition of multidimensional Ramanujan sum for $\mathbf{P}_i$ becomes degenerate unless $\mathbf{P}_i$ is a legitimate divisor of $\mathbf{P}$. The beauty of the development is that, in spite of these restrictions, the representation to be developed works out alright for any $\mathbf{P}$ we start with. The restrictions are only on the left divisors $\mathbf{P}_i$, and there are always enough legitimate divisors for any $\mathbf{P}$, to admit the proposed representation.

With a 1D signal $x(n)$ supported in $0 \leq n \leq P-1$ the DFT representation is $x(n) = \sum_{k=0}^{P-1} a(k) e^{j2\pi kn/P}$, where $a(k)$ are the DFT coefficients (more precisely $a(k) = X[k]/P$). In the same way it is well known [1], [12] that if $x(\mathbf{n})$ is an FIR signal supported in $FPD(\mathbf{P})$ where $\mathbf{P} \in \mathbb{Z}^{D\times D}$, we can represent it as

$$x(\mathbf{n}) = \sum_{\mathbf{k} \in FPD(\mathbf{P}^T)} a(\mathbf{k}) e^{j2\pi \mathbf{k}^T \mathbf{P}^{-1} \mathbf{n}}, \quad \mathbf{n} \in FPD(\mathbf{P}) \tag{5}$$

Viewed as a function of $\mathbf{n}$, the right hand side above is periodic with repetition matrix $\mathbf{P}$. Similarly, viewed as a function of $\mathbf{k}$, it is periodic with repetition matrix $\mathbf{P}^T$. Figure 2 demonstrates the difference between $FPD(\mathbf{P})$ and $FPD(\mathbf{P}^T)$ when $\mathbf{P}$ is as in Eq. (4).

There are $\det \mathbf{P}$ integer vectors $\mathbf{k}$ in $FPD(\mathbf{P}^T)$ as demonstrated in Fig. 2. Choose such a $\mathbf{k}$, and let $\mathbf{L}$ denote a glcd of $\mathbf{k}$ and $\mathbf{P}^T$, that is, $\mathbf{k} = \mathbf{Lk}_r$, $\mathbf{P}^T = \mathbf{LP}_r^T$, and $\mathbf{k}_r$ and $\mathbf{P}_r^T$ are left coprime. Thus $\mathbf{k}^T \mathbf{P}^{-1} = \mathbf{k}_r^T \mathbf{P}_r^{-1}$, and for each $\mathbf{k} \in FPD(\mathbf{P}^T)$ we can rewrite

$$e^{j2\pi \mathbf{k}^T \mathbf{P}^{-1} \mathbf{n}} = e^{j2\pi \mathbf{k}_r^T \mathbf{P}_r^{-1} \mathbf{n}} \tag{6}$$

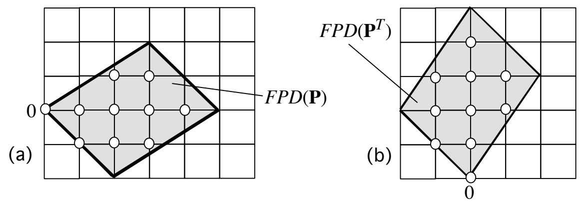


**Fig. 2**. (a) $FPD(\mathbf{P})$ and (b) $FPD(\mathbf{P}^T)$ where $\mathbf{P}$ is as in Eq. (4). In the DFT representation (5), $\mathbf{n} \in FPD(\mathbf{P})$ and $\mathbf{k} \in FPD(\mathbf{P}^T)$.

where $\mathbf{k}_r$ and $\mathbf{P}_r^T$ are left coprime, and $\mathbf{P}_r$ is a left divisor of $\mathbf{P}$ (since $\mathbf{P} = \mathbf{P}_r\mathbf{L}^T$). As we go through each $\mathbf{k} \in FPD(\mathbf{P}^T)$ to rewrite the exponential in reduced form (6), let us keep track of the left divisors generated in the process:

$$\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_K. \tag{7}$$

For each $\mathbf{P}_i$ we have a certain number of $\mathbf{k}_i$ such that $\mathbf{k}_i$ and $\mathbf{P}_i^T$ are left coprime. It can be shown [19] that the set of $\mathbf{k}_i$ thus associated with each $\mathbf{P}_i$ is a subset of the $\det \mathbf{P}_i$ *integer vectors* in $FPD(\mathbf{P}_i^T)$. Thus, the summation in (5) can be rearranged as a double summation:

$$x(\mathbf{n}) = \sum_{i=1}^{K} \underbrace{\sum_{\substack{\mathbf{k}_i \in FPD(\mathbf{P}_i^T) \\ (\mathbf{k}_i, \mathbf{P}_i^T)_l = \mathbf{I}}} a(\mathbf{k}_i) e^{j2\pi \mathbf{k}_i^T \mathbf{P}_i^{-1} \mathbf{n}}}_{\text{call this } x_i(\mathbf{n})} \tag{8}$$

Here $(\mathbf{k}_i, \mathbf{P}_i^T)_l = \mathbf{I}$ means that $\mathbf{k}_i$ and $\mathbf{P}_i^T$ are left coprime. A signal of the form $x_i(\mathbf{n})$ in Eq. (8) belongs to the space spanned by the basis functions

$$\{e^{j2\pi \mathbf{k}_i^T \mathbf{P}_i^{-1} \mathbf{n}}\}, \quad \mathbf{k}_i \in FPD(\mathbf{P}_i^T), \ (\mathbf{k}_i, \mathbf{P}_i^T)_l = \mathbf{I} \tag{9}$$

These constitute a subset of the $\det \mathbf{P}$ columns of the multidimensional IDFT matrix corresponding to $\mathbf{P}$, and are therefore orthogonal [12]. The space $\mathcal{S}_{\mathbf{P}_i} \subset \mathbb{Z}^{|\mathbf{P}|}$ spanned by (9) will be called the **Ramanujan subspace** corresponding to $\mathbf{P}_i$, by analogy with a similar development in the 1D case [17]. The spaces $\mathcal{S}_{\mathbf{P}_i}$ are orthogonal for different $i$. Notice that the exponentials in (9) can be regarded as column vectors by fixing an ordering convention for the vectors $\mathbf{n}_i \in FPD(\mathbf{P})$. The ambiguity of ordering can be avoided by using tensor notation: for example in 2D regard the signals as matrices (images), in 3D regard them as 3-dimensional tensors, and so on.

The signals $x_i(\mathbf{n}) \in \mathcal{S}_{\mathbf{P}_i}$ have repetition matrix $\mathbf{P}_i$, so Eq. (8) can be regarded as a *decomposition of* $x(\mathbf{n})$ *into* **periodic components**, the "smaller" periods $\mathbf{P}_i$ being some of the left divisors of $\mathbf{P}$. We next define multidimensional Ramanujan sums and explain how the decomposition (8) relates to them. We will also explain which left-divisors of $\mathbf{P}_i$ participate in the summation (8).

## 4. MULTIDIMENSIONAL RAMANUJAN-SUM

Given a rank-$D$ integer matrix $\mathbf{Q} \in \mathbb{Z}^{D\times D}$, define the sum

$$c_{\mathbf{Q}}(\mathbf{n}) = \sum_{\substack{\mathbf{k} \in FPD(\mathbf{Q}^T) \\ (\mathbf{k}, \mathbf{Q}^T)_l = \mathbf{I}}} e^{j2\pi \mathbf{k}^T \mathbf{Q}^{-1} \mathbf{n}} \tag{10}$$

where $\mathbf{n} \in \mathbb{Z}^D$. This will be called the $D$-dimensional Ramanujan-sum, in analogy with (1). The sum is performed over $\mathbf{k} \in FPD(\mathbf{Q}^T)$ such that $\mathbf{k}$ and $\mathbf{Q}^T$ are left coprime. It turns out [19] that there exist $\mathbf{k}$ in $FPD(\mathbf{Q}^T)$ coprime to $\mathbf{Q}^T$ if and only if the Smith decomposition $\mathbf{Q} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}$ is such that the Smith-form $\mathbf{\Lambda}$ takes the special form

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda \end{bmatrix} \tag{11}$$

where $\lambda = |\det \mathbf{Q}| > 0$. Furthermore when this is satisfied, the number of $\mathbf{k}$ in $FPD(\mathbf{Q}^T)$ coprime to $\mathbf{Q}^T$ is precisely

$$\phi(\lambda) = \phi(|\det \mathbf{Q}|) \tag{12}$$

where $\phi(.)$ denotes the Euler totient function. Thus the Ramanujan space $\mathcal{S}_{\mathbf{Q}}$ (defined in Sec. 3) has dimension $\phi(\lambda)$. There is a result [19] that connects the summation (10) to the 1D Ramanujan-sum: whenever $\mathbf{Q}$ has the Smith-form (11) it can be shown that

$$c_{\mathbf{Q}}(\mathbf{n}) = c_\lambda(m) \tag{13}$$

where $m = [\mathbf{U}^{-1}\mathbf{n}]_D$ (i.e., $m$ is the last element of $\mathbf{U}^{-1}\mathbf{n}$). So $c_{\mathbf{Q}}(\mathbf{n})$ is *integer valued* because the 1D Ramanujan sum $c_\lambda(m)$ is integer valued. Furthermore, as $\mathbf{n}$ takes on the $\det \mathbf{Q}$ values in $FPD(\mathbf{Q})$, the scalar integer $m$ takes all values in $0 \le m \le \lambda - 1$ in some order. Now, the beauty about the sum (10) is that, the space $\mathcal{S}_{\mathbf{Q}}$ can be spanned by an integer basis generated from $c_{\mathbf{Q}}(\mathbf{n})$ or equivalently $c_\lambda(m)$. To be more specific, consider the $\phi(\lambda)$ circularly shifted versions of $c_\lambda(m)$ as demonstrated below for $\lambda = 4$:

$$\begin{bmatrix} c_4(0) & c_4(3) \\ c_4(1) & c_4(0) \\ c_4(2) & c_4(1) \\ c_4(3) & c_4(2) \end{bmatrix} \tag{14}$$

Then for any $\mathbf{Q}$ with $\lambda = 4$, the Ramanujan space $\mathbf{S}_{\mathbf{Q}}$ is spanned by the above two columns (for appropriate ordering convention of the integer vectors $\mathbf{n}_i \in FPD(\mathbf{Q})$). Returning to the representation (8) it now follows that each inner sum has $\phi(\det \mathbf{P}_i)$ terms (by Eq. (12)). Since the total number of terms is $\det \mathbf{P}$ it follows that

$$\sum_{i=1}^{K} \phi(|\det \mathbf{P}_i|) = |\det \mathbf{P}| \tag{15}$$

It can be shown [19] that Eq. (8) can be rewritten in terms of $c_{\mathbf{P}_i}(\mathbf{n})$ as follows:

$$x(\mathbf{n}) = \sum_{i=1}^{K} \underbrace{\sum_{\mathbf{m}} \beta_{i,\mathbf{m}} c_{\mathbf{P}_i}(\mathbf{n} - \mathbf{m})}_{x_i(\mathbf{n})} \tag{16}$$

Here the summation over $\mathbf{m}$ has $\phi(\lambda_i)$ terms, where $\lambda_i = |\det \mathbf{P}_i|$. For example the precise set of $\mathbf{m}$ which participates in the $i$th inner sum can be taken to be those that satisfy $\mathbf{c}_{\mathbf{P}_i}(\mathbf{m}) = c_{\lambda_i}(m)$ for the first $\phi(\lambda_i)$ values of $m$, i.e.,

$0 \leq m \leq \phi(\lambda_i) - 1$. Once again the use of tensor notation will make this ordering easier to track but we shall not get into details here. We will refer to Eq. (16) as the multidimensional Ramanujan representation.

The matrices $\mathbf{P}_i$ in Eq. (8) and Eq. (16) are a subset of left divisors of $\mathbf{P}$. The question arises as to which divisors of $\mathbf{P}$ participate. The answer, as shown in [19], is that it is precisely the set of all divisors which have the restricted Smith form

$$\boldsymbol{\Lambda}_i = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda_i \end{bmatrix} \tag{17}$$

Such left divisors of $\mathbf{P}$ will be called the **legitimate** left divisors of $\mathbf{P}$. The relation (15) can be regarded as a generalization of the well-known 1D result $\sum_{q_i|q} \phi(q_i) = q$ proved in [2]. The following example will demonstrate some of these details. Thus consider the simple example

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{18}$$

First, what are the left divisors of $\mathbf{P}$? Some of them are

$$\mathbf{P}_1 = \mathbf{I},\ \mathbf{P}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ \mathbf{P}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{P} = 2\mathbf{I} \tag{19}$$

But there are more divisors. For example

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{20}$$

is a left divisor because $\mathbf{P} = \mathbf{P}_4\mathbf{P}_4^T$. Now consider the Smith-forms of these divisors. $\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{P}$ are already in Smith-form. Since $\det \mathbf{P}_3 = \det \mathbf{P}_4 = 2$ it follows that $\mathbf{P}_3$ and $\mathbf{P}_4$ also have Smith-form

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{21}$$

Only the Smith-form of $\mathbf{P}$ is not as in (17) which means that no $\mathbf{k}$ can be left-coprime to $\mathbf{P}^T$. Thus, the divisor $\mathbf{P}$ is *not included* in (7) and in the representations (8), (16). This is a sharp departure from the 1D case where all divisors $q_i$ of $q$, including $q$, are used in the representation (2). Each of the above $\mathbf{P}_i$ has precisely one $\mathbf{k}_i \in FPD(\mathbf{P}_i^T)$ such that $\mathbf{k}_i$ and $\mathbf{P}_i^T$ are coprime. These are listed below and shown in Fig. 3. To follow these details it is helpful to notice the following result proved in [19]: *Consider the integer vector and integer diagonal matrix below:*

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_D \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_D \end{bmatrix} \tag{22}$$

*These are left coprime if and only if* $(k_i, \lambda_i) = 1$ *and* $(\lambda_i, \lambda_j) = 1$ *for* $i \neq j$. Referring now to (19) and (20), we have the following:

1. The only vector in $FPD(\mathbf{P}_1^T)$ is $\mathbf{k}_{11} = [0 \quad 0]^T$, and it is coprime to $\mathbf{P}_1$ (in the degenerate sense that 0 is coprime to 1). The vectors in $FPD(\mathbf{P}_2^T)$ are $\mathbf{k}_{21} = [0 \quad 0]^T$ and $\mathbf{k}_{22} = [0 \quad 1]^T$. $\mathbf{k}_{22}$ is left coprime to $\mathbf{P}_2^T$. The vectors in $FPD(\mathbf{P}_3^T)$ are $\mathbf{k}_{31} = [0 \quad 0]^T$ and $\mathbf{k}_{32} = [1 \quad 0]^T$, and $\mathbf{k}_{32}$ is left coprime to $\mathbf{P}_3^T$.
2. Now consider $\mathbf{P}_4$. The vectors in $FPD(\mathbf{P}_4^T)$ are $\mathbf{k}_{41} = [0 \quad 0]^T$ and $\mathbf{k}_{42} = [0 \quad 1]^T$. With some effort we can prove that $\mathbf{k}_{42}$ is left coprime to $\mathbf{P}_4^T$.

Fig. 4 (top) shows an image $x(\mathbf{n})$ with periodicity matrix (18) and the four periodic components $x_i(\mathbf{n})$ corresponding to the periodicity matrices $\mathbf{P}_i$. These are the orthogonal projections of $x(\mathbf{n})$ onto the Ramanujan subspaces $\mathcal{S}_{\mathbf{P}_i}$. (We have added constants to make the images nonnegative for plotting.) Notice the 4th periodic component corresponding to $\mathbf{P}_4$ has a quincunx (nonseparable) periodic lattice [12], even though the original image has a separable periodic lattice.

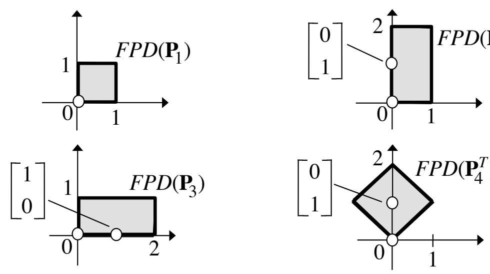


**Fig. 3**. The regions $FPD(\mathbf{P}_i)$ for the various divisors in the example are shown. Also shown are the integers inside these *FPD*s.

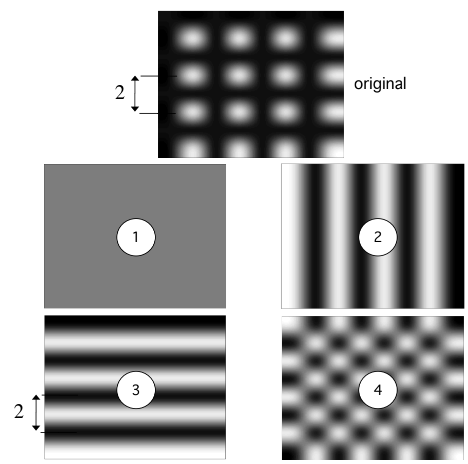


**Fig. 4**. Top: an arbitrary $2D$ image with periodicity matrix $\mathbf{P}$ as in Eq. (18). Middle and bottom: the four periodic components obtained by Ramanujan-space projections.

## 5. CONCLUDING REMARKS

We generalized the Ramanujan-sum for the multidimensional case from the viewpoint of periodicity on lattices. We also presented a way to decompose an arbitrary finite duration multidimensional signal in terms of this generalized Ramanujan sum. The techniques for calculating the periodic projections $x_i(n)$ are in principle similar to the 1D case, but the details will be developed in future work. Practical applications of this decomposition are yet to be developed.

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