SAMPLING SMOOTH SPATIO-TEMPORAL PHYSICAL FIELDS: WHEN WILL THE ALIASING ERROR INCREASE WITH TIME?

Karthik Sharma and Animesh Kumar

Department of Electrical Engineering
Indian Institute of Technology Bombay
Mumbai 400076, India
{skarthik,animesh}@ee.iitb.ac.in

ABSTRACT

Acquisition of physical fields, such as temperature along a path, using a distributed array of sensors (samples) is of interest. For smooth spatial fields, in a Nyquist style sampling {eq:aliasingerror} setup, the aliasing error is determined by the (spatial) spectral profile of a field. Physical fields and their spectral properties evolve with time. In this work, the spectral evolution of spatio-temporal fields is analyzed, where the field is given by physical law comprising of a constant coefficient linear partial differential equation. A procedure to examine whether field's spatial spectral profile will become worse with time, from aliasing point of view, is developed. The procedure is exemplified using a second-order PDE in this work. The analysis is extended to include simple point source terms. Techniques such as the Fourier transform, the unilateral Laplace transform, and root-locus plot from control theory are utilized in this work.

***Index Terms*—** signal sampling, signal reconstruction, geophysical signal processing, Fourier transforms

1. INTRODUCTION

Remote sensing (acquisition) of physical fields, such as temperature or pollution level, using a distributed array of precision-limited sensors is of interest [1]. Many physical phenomenons result in smooth fields due to underlying laws of physics (for example, temperature profile is "smooth" due to thermal exchange). Under smoothness conditions, a Nyquist style sampling approach—where a large bandwidth approximation of physical field is acquired—is natural. Physical fields cannot be filtered in the spatial dimension with a fixed array of sensors, where sampling fundamentally precedes filtering. In the Nyquist style sampling approach, it has been shown that the field approximation error depends on the (spatial domain) spectral decay of the physical field both in single and multidimensional setting [2, 3]. Specifically if $f(x)$ is a one-dimensional field with $\tilde{f}(\omega)$ as its Fourier spectrum and W is the reconstruction bandwidth, then

$$\text{Aliasing}(f, W) = \int_{|\omega|>W} |\tilde{f}(\omega)| \mathrm{d}\omega \tag{1}$$

is the aliasing error term due to nonbandlimitedness of the field [2, 3, 4]. The class of signals which can be sampled and reconstructed with a bounded aliasing error is fairly rich and includes all signals that have exponential and polynomial decay in their spectra. This aliasing error term while sampling a field is due to absence of an anti-aliasing prefilter. These results are valid for a *single* time-snapshot of the physical field.

On the other hand, a spatial field will evolve with time due to governing laws of physics. Acquisition of spatial fields while taking their laws of physics in account has been discussed in the literature [5, 6]. The evolution of many spatio-temporal fields can be modelled using partial differential equations. The Fourier spectrum of the spatial field and hence the spectral decay of spatio-temporal fields, in general, evolves with time.

In this work, we will investigate the following question at large: whether the aliasing error term in (1) increases with time when a physical field evolves according to a PDE based spatio-temporal evolution law? If not, then it means that a target reconstruction bandwidth (and associated Nyquist sampling rate or density) at time zero will be sufficient to sample the field at any time in the future.

Main contributions: For spatio-temporal fields evolving according to a constant-coefficient linear partial differential equation (PDE), we establish a procedure to examine whether their spectral decay will become worse with time for various (spatial) frequencies. We exemplify our procedure with a second-order PDE in this work. We extend our analysis to include simple point source terms as well. Our analysis technique uses the Fourier transform, the unilateral Laplace transform, and Agashe's algorithm for examination of roots of a complex-coefficient polynomial [7, 8].

Notation: The scalar variable x will be used for space and t for time. Correspondingly, ω will be used for spatial

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(angular) frequency. The variable s will represent a point in the complex plane. The spatio-temporal field will be denoted by $u(x,t)$, and its (spatial) Fourier transform will be denoted by $\tilde{u}(\omega,t)$. The unilateral Laplace transform of $\tilde{u}(\omega,t)$, with respect to time, will be denoted by $\tilde{U}(\omega,s)$. And, $\sqrt{-1}$ will be denoted by j.

Organization: Sec. 2 presents the assumptions made on the evolution of spatio-temporal field. Sec. 3 consists of the main result of this work. In Sec. 3.1, Fourier and Laplace transforms are used to convert a PDE evolution into a rational transfer function. In Sec. 3.2, the time and frequency based evolution of poles of the rational transfer function is studied. In Sec. 3.3, point source is added to the existing analysis. Finally, conclusions are presented in Sec. 4.

2. ASSUMPTIONS AND SYSTEM MODEL

Let $u(x,t)$ be a spatially one-dimensional and temporally evolving physical field of interest. Such field will model a time-dependent isotropic (radially symmetric) physical field or a time-dependent field in one dimension. It will be assumed that the evolution of $u(x,t)$ is governed by a partial differential equation (PDE), which models the laws of physics. For example, the heat equation will model the evolution of a spatial temperature field with time [9]. Throughout the paper, the spatial Fourier transform of the field $u(x,t)$

$$\tilde{u}(\omega,t) := \int_{x\in\mathbb{R}} u(x,t)\exp(-j\omega x)\mathrm{d}x$$

will be used. It is assumed that $u(x,t)$ for each fixed time has finite energy for the Fourier transform $\tilde{u}(\omega,t)$ to exist [10].

It will be assumed that $u(x,t)$ is sufficiently smooth to have a decaying (spatial) Fourier spectrum. This will enable Nyquist sampling and associated approximate reconstruction meaningful in the presence of aliasing [2]. Decaying Fourier spectrum condition requires that $\int_{\omega\in\mathbb{R}}|\omega||\tilde{u}(\omega,t)|\mathrm{d}\omega$ is finite for every $t\in\mathbb{R}$ [2]. This condition ensures that $u(x,t)$, for each $t\in\mathbb{R}$, can be acquired by Nyquist-style sampling with aliasing error that diminishes with (spatial) sampling rate [2].

The spatial field's (spatio-temporal) evolution will be modeled by a second-order constant coefficient PDE. Such PDEs include the heat-equation or the wave-equation [9] from physics. The general form of this equation is given by

$$a_2\frac{\partial^2 u(x,t)}{\partial t^2} + a_1\frac{\partial u(x,t)}{\partial t} + a_0 u(x,t) = b_1\frac{\partial u(x,t)}{\partial x} + b_2\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (2)$$

where a_2, a_1, a_0 and b_2, b_1 are real-valued constants. It will be assumed that $a_2 \neq 0$. For example, the wave equation has $a_2 = 1, a_1 = 0, a_0 = 0$ and $b_2 > 0, b_1 = 0$. With initial conditions $u(x,0) = u_0(x)$ and $\partial u(x,0)/\partial t = u_1(x)$, the solution $u(x,t)$ is unique for all $x\in\mathbb{R}$ and $t>0$ [9].

As mentioned in Sec. 1, the main result of this work will focus on the eventual spectral decay in $\tilde{u}(\omega,t)$ for each $t>0$. It is of interest to show that the spectral decay in $\tilde{u}(\omega,t)$ is (up to a proportionality constant) at least as fast as the spectral decay in $\tilde{u}_0(x)$ and $\tilde{u}_1(x)$. We will show that there are necessary and sufficient conditions on a_2, a_1, a_0 and b_2, b_1 for this result to hold.

3. FIELDS GIVEN BY SECOND ORDER PDE

The main contribution of this work is presented in this section. The constant coefficient second order PDE will be analyzed. Conditions will be derived on its coefficients and initial conditions on the field, which ensure that the temporally evolving field has decaying spatial spectrum. Consider the PDE in (2) with the initial conditions $u(x,0) = u_0(x)$ and $\partial u(x,0)/\partial t = u_1(x)$.

3.1. Analysis using the Fourier and Laplace transforms

The temporal evolution of the spatial spectrum of the field is captured in the Fourier transform domain. A spatial Fourier transform in (2) results in

$$a_2\frac{\partial^2 \tilde{u}(\omega,t)}{\partial t^2} + a_1\frac{\partial \tilde{u}(\omega,t)}{\partial t} + (a_0 - jb_1\omega + b_2\omega^2)\tilde{u}(\omega,t) = 0, \quad (3)$$

with transform domain initial conditions $\tilde{u}(\omega,0) = \tilde{u}_0(\omega)$ and $\partial\tilde{u}(\omega,0)/\partial t = \tilde{u}_1(\omega)$. The temporal evolution of $\tilde{u}(\omega,t)$ will be analyzed by using the unilateral Laplace transform in (3) [7]. This results in

$$a_2 s^2\tilde{U}(\omega,s) + a_1 s\tilde{U}(\omega,s) + (a_0 - jb_1\omega + b_2\omega^2)\times \tilde{U}(\omega,s) - (s\tilde{u}_0(\omega) + \tilde{u}_1(\omega))a_2 - a_1\tilde{u}_0(\omega) = 0$$

for each $\omega\in\mathbb{R}$. The above equation can be rearranged into the following more familiar form

$$\tilde{U}(\omega,s) = \frac{a_2\tilde{u}_0(\omega)s + a_2\tilde{u}_1(\omega) + a_1\tilde{u}_0(\omega)}{a_2 s^2 + a_1 s + a_0 + b_2\omega^2 - jb_1\omega}. \quad (4)$$

Without loss of generality, let $a_2 = 1$. If $a_2 \neq 1$, the numerator and denominator in the right-hand side of (4) can be divided by a_2. The solution $\tilde{u}(\omega,t)$ is given by the (causal) inverse Laplace transform of $\tilde{U}(\omega,s)$. The inverse Laplace transform and associated solution for $t\geq 0$ is obtained by the partial fraction expansion (in s) and is given by [7]

$$\tilde{u}(\omega,t) = c_1(\omega)e^{r_1(\omega)t} + c_2(\omega)e^{r_2(\omega)t}, \quad (5)$$

where $r_1(\omega)$ and $r_2(\omega)$ are the roots of $s^2 + a_1 s + a_0 + b_2\omega^2 - jb_1\omega = 0$ and $c_1(\omega)$ and $c_2(\omega)$ are given by

$$\begin{bmatrix} c_1(\omega) \\ c_2(\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1(\omega) & r_2(\omega) \end{bmatrix}^{-1} \begin{bmatrix} \tilde{u}_0(\omega) \\ \tilde{u}_1(\omega) \end{bmatrix}. \quad (6)$$

Observe that $\tilde{u}(\omega, t)$ in (5) can become large as t increases in two ways: (i) if $r_1(\omega)$ or $r_2(\omega)$ have positive real parts; or, (ii) if $r_1(\omega)$ and $r_2(\omega)$ are "close" but not equal. Both these issues are addressed in the next section. If $\tilde{u}(\omega, t)$ becomes unbounded, then the spectral aliasing term corresponding to (1) becomes unbounded, i.e., $\int_{|\omega|>W} |\tilde{u}(\omega, t)| \mathrm{d}\omega$ will become unbounded as t increases. In this case, Nyquist style sampling may not be useful to acquire the field. For Nyquist style sampling setup to be useful, the real part of the roots $r_1(\omega)$ and $r_2(\omega)$ should be negative for all $\omega \in \mathbb{R}$. At this point, it should be noted that the presence of $r_1(\omega)$ or $r_2(\omega)$ or both only depends on the coefficients a_2, a_1, a_0 and b_2, b_1 which govern the physical law or the PDE.

Definition 3.1 *The spectral evolution of the PDE in* (3) *will be called as stable if both $r_1(\omega)$ and $r_2(\omega)$ in* (5) *lie in the left-half plane, i.e., Re$[r_1(\omega)] < 0$ and Re$[r_2(\omega)] < 0$ for all $\omega \in \mathbb{R}$.*

Next we will establish the conditions on a_2, a_1, a_0 and b_2, b_1 such that the spectral evolution in (3) is stable. Had the denominator polynomial in s in (4) consisted only of real-valued coefficients, Routh-Hurwitz algorithm could be utilized to find the conditions on a_2, a_1, a_0 and b_2, b_1 such that spectral evolution in (3) is stable. For rational Laplace transfer functions with complex-valued coefficients, an extension of Routh-Hurwitz algorithm is available. It will be referred to as Agashe's algorithm in what follows [8].

Once it is ensured that $r_1(\omega)$ and $r_2(\omega)$ are in the left-half plane, it will be shown that $r_1(\omega) \neq r_2(\omega)$ for $\omega \neq 0$. This will ensure that the matrix in (6) is invertible with a finite condition number.

3.2. Conditions on PDE for stable spectral evolution

To ensure that $r_1(\omega)$ and $r_2(\omega)$ in (5) lie in the left-half plane, Agashe's algorithm will be used on the denominator polynomial $D(s) = s^2 + a_1 s + a_0 + b_2\omega^2 - jb_1\omega$. Based on Agashe's algorithm, the quasi-real and quasi-imaginary parts are

$$D_{\mathrm{r}}(s) = s^2 + a_0 + b_2\omega^2 \text{ and } D_{\mathrm{i}}(s) = a_1 s - jb_1\omega.$$

Observe that $D(s) = D_{\mathrm{r}}(s) + D_{\mathrm{i}}(s)$ and $D^*(s) = D_{\mathrm{i}}(s) + R_{\mathrm{r}}(s)$ where $R_{\mathrm{r}}(s)$ is the remainder obtained after dividing $D_{\mathrm{r}}(s)$ by $D_{\mathrm{i}}(s)$. For the denominator $D(s)$, the remainder is given by $R_{\mathrm{r}}(s) = a_0 + \left(b_2 - \frac{b_1^2}{a_1^2}\right)\omega^2$. For both the roots $r_1(\omega)$ and $r_2(\omega)$ to lie in the left-half plane, Agashe's algorithm requires that

$$\frac{1}{a_1} > 0 \text{ and } \frac{a_1}{a_0 + \left(b_2 - \frac{b_1^2}{a_1^2}\right)\omega^2} > 0. \tag{7}$$

For (7) to hold for all values of ω, it is required that

$$a_1 > 0, a_0 > 0, \text{ and } b_2 > \left(\frac{b_1^2}{a_1^2}\right). \tag{8}$$

Once a second order PDE is available, which explains the evolution of a spatio-temporal field, the conditions in (8) can be checked to ensure stability. The condition in (8) is the culmination of our first claim.

Next, to ensure that matrix inverse in the right-hand side of (6) is well defined (with a finite condition number), it has to be argued that the trajectories of $r_1(\omega)$ and $r_2(\omega)$ do not intersect as ω is varied. Let

$$r_1(\omega) = \alpha_1(\omega) + j\beta_1(\omega) \text{ and } r_2(\omega) = \alpha_2(\omega) + j\beta_2(\omega).$$

Since $(s - r_1(\omega))(s - r_2(\omega)) = D(s)$, therefore,

$$[\alpha_1(\omega) + \alpha_2(\omega)] + j[\beta_1(\omega) + \beta_2(\omega)] = -a_1$$

and

$$[\alpha_1(\omega)\alpha_2(\omega) - \beta_1(\omega)\beta_2(\omega)] + \\ j[\alpha_1(\omega)\beta_2(\omega) + \alpha_2(\omega)\beta_1(\omega)] = a_0 + b_2\omega^2 - jb_1\omega.$$

Further simplification of the above equations lead to

$$\alpha_1(\omega) + \alpha_2(\omega) = -a_1, \tag{9}$$

$$\beta_1(\omega) + \beta_2(\omega) = 0, \tag{10}$$

$$\alpha_1(\omega)\alpha_2(\omega) - \beta_1(\omega)\beta_2(\omega) = a_0 + b_2\omega^2, \tag{11}$$

$$\alpha_1(\omega)\beta_2(\omega) + \alpha_2(\omega)\beta_1(\omega) = -\omega b_1. \tag{12}$$

Since $r_1(\omega)$ and $r_2(\omega)$ are in the left-hand plane, so $\alpha_1(\omega) < 0$ and $\alpha_2(\omega) < 0$. Substitution from (9) and (10) in (11) and (12) results in

$$\beta_1(\omega)^2 - a_1\alpha_1(\omega) - \alpha_1(\omega)^2 = a_0 + b_2\omega^2, \tag{13}$$

$$\beta_1(\omega)(-a_1 - 2\alpha_1(\omega)) = -\omega b_1. \tag{14}$$

It will be shown next that $\beta_1(\omega)$ will move away from 0 and consequently away from $\beta_2(\omega)$ (see (10)) as ω increases. Since $a_0 > 0$ and $b_2 > 0$ (see (8)), the right hand side of (11) is always positive and increases with ω. Therefore at least one among $\beta_1(\omega)^2$ and $-a_1\alpha_1(\omega) - \alpha_1(\omega)^2$ has to increase with ω. If $\beta_1(\omega)^2$ increases it implies that $\beta_1(\omega)$ moves away from 0 and consequently away from $-\beta_1(\omega) = \beta_2(\omega)$. In the case that $\beta_1(\omega)$ does not increase, $-a_1\alpha_1(\omega) - \alpha_1(\omega)^2$ will have to increase which will happen only if $\alpha_1(\omega)$ moves towards $\frac{-a_1}{2}$. But this will cause the magnitude of $-a_1 - 2\alpha_1(\omega)$ to decrease in (14) which will force the magnitude of $\beta_1(\omega)$ to increase since the magnitude of the right hand side of (14) increases as ω increases. This again implies that $\beta_1(\omega)$ moves away from 0 and therefore also away from $\beta_2(\omega)$. Since the imaginary parts of the two roots, $r_1(\omega)$ and $r_2(\omega)$ move away from each other as ω increases it follows that the two roots cannot come arbitrarily close to each other, thus preventing the condition number of the matrix in (6) from becoming very large. There are 3 cases for the position of roots at $\omega = 0$:

Case 1: Roots coincide at $\omega = 0$. Therefore, $r_1(0) = r_2(0) = -a_1/2$.

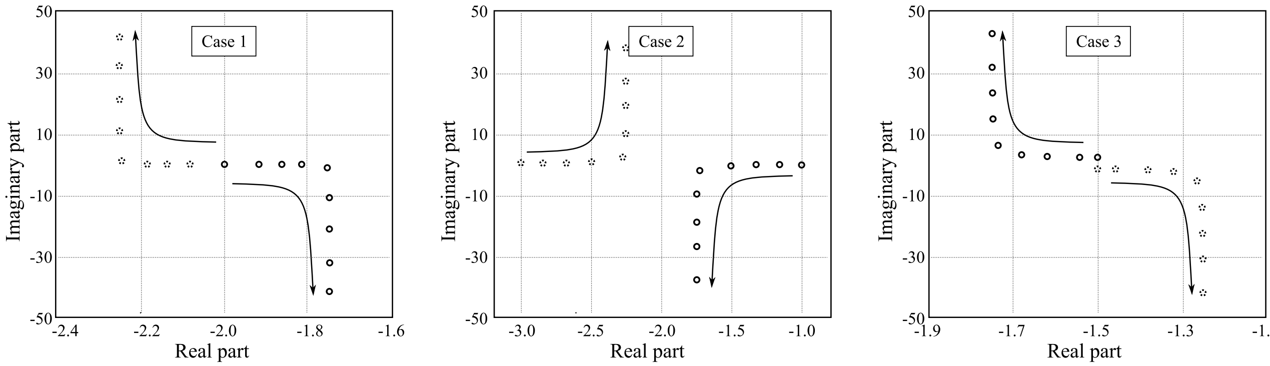

fig:rootlocus
Fig. 1. The evolution of $r_1(\omega)$ and $r_2(\omega)$, which characterize the spectral decay of $\tilde{u}(\omega, t)$ in (5), is plotted for various values of coefficients in second-order linear PDE. In all the plots, the direction of arrow indicates the movement of $r_1(\omega)$ and $r_2(\omega)$ as ω increases. Observe that the roots separate away as ω increases. So, the condition number of matrix in right-hand side of (6) is finite. **Case 1:** $a_1 = a_0 = 4, b_1 = -5, b_2 = 100$. **Case 2:** $a_1 = 4, a_0 = 3, b_1 = -5, b_2 = 100$. **Case 3:** $a_1 = 3, a_0 = 6$, $b_1 = -5, b_2 = 100$.

Case 2: Roots are real but distinct.

Case 3: Roots are complex conjugates of each other.

A sample trajectory for each of these cases is shown in Fig. 1.

Higher order PDEs: They can be treated similarly by applying Agashe's algorithm on the corresponding higher degree polynomial and imposing conditions on the coefficients such that all the roots of the polynomial are in the left-half plane. We will then have to also look at the trajectory of the roots as ω varies. The analysis gets increasingly complex as the degree of the PDE increases.

3.3. Addition of a point source term in the PDE

In what follows we will include a source term $s(x, t)$ in the right-hand side of the second order PDE in (2). In this case, the counterpart of (4) is

$$\tilde{U}_{\text{ps}}(\omega, s) = \frac{a_2\tilde{u}_0(\omega)s + a_2\tilde{u}_1(\omega) + a_1\tilde{u}_0(\omega) + \tilde{S}(\omega, s)}{a_2s^2 + a_1s + a_0 + b_2\omega^2 - jb_1\omega}, \tag{15}$$

where $\tilde{S}(\omega, s)$ is the unilateral Laplace transform of $\tilde{s}(\omega, t)$ which in turn is the (spatial) Fourier transform of $s(x, t)$, the source term. For further analysis, we will restrict the treatment to a simple point source where $s(x, t) = \delta(x)\delta(t - t_0)$ and $\tilde{S}(\omega, s) = e^{-t_0 s}$. Physically this point source represents an impulse of area one at origin (in space) and time t_0. For this point source, (15) reduces to

$$\tilde{U}_{\text{ps}}(\omega, s) = \frac{a_2\tilde{u}_0(\omega)s + a_2\tilde{u}_1(\omega) + a_1\tilde{u}_0(\omega) + e^{-t_0 s}}{a_2s^2 + a_1s + a_0 + b_2\omega^2 - jb_1\omega}.$$

Let $\tilde{U}_{\text{ps}}(\omega, s) = \tilde{U}_1(\omega, s) + \tilde{U}_2(\omega, s)$ where

$$\tilde{U}_1(\omega, s) = \frac{a_2\tilde{u}_0(\omega)s + a_2\tilde{u}_1(\omega) + a_1\tilde{u}_0(\omega)}{a_2s^2 + a_1s + a_0 + b_2\omega^2 - jb_1\omega},$$

$$\text{and } \tilde{U}_2(\omega, s) = \frac{e^{-t_0 s}}{a_2s^2 + a_1s + a_0 + b_2\omega^2 - jb_1\omega}.$$

Upon Laplace transform inversion $\tilde{u}_{\text{ps}}(\omega, t) = \tilde{u}_1(\omega, t) + \tilde{u}_2(\omega, t)$ is obtained where

$$\tilde{u}_1(\omega, t) = c_1(\omega)e^{r_1(\omega)t} + c_2(\omega)e^{r_2(\omega)t}, \text{ and}$$

$$\tilde{u}_2(\omega, t) = u(t - t_0)\left(k(\omega)e^{r_1(\omega)(t-t_0)} - k(\omega)e^{r_2(\omega)(t-t_0)}\right)$$

where $c_1(\omega)$ and $c_2(\omega)$ are as given before in (6) and $k(\omega) = \frac{1}{r_1(\omega) - r_2(\omega)}$.

If the constraints on the coefficients satisfy (7), then the spectral evolution of the field is stable. As a result, the spectral decay properties of the field in the presence of a point source only depends on the spectral decay properties of the initial conditions $\tilde{u}_0(\omega)$ and $\tilde{u}_1(\omega)$ (as in the case without a point source). This argument can be easily extended to multiple point sources.

4. CONCLUSIONS

The spectral evolution of spatio-temporal fields was analyzed, where the spectral evolution of the field was given by a constant coefficient linear PDE. It was shown that the spectral decay analysis of a field governed by a constant coefficient linear PDE naturally reduces into whether roots of a complex-coefficient polynomial lie in the left-half plane or not. This procedure was carried out for a second-order constant coefficient linear PDE. The analysis was extended to include point source terms.

5. REFERENCES

[1] J. M. Kahn, R. H. Katz, and K. S. J. Pister, "Next century challenges: Mobile networking for "smart dust"," in *ACM International Conference on Mobile Computing and Networking (MOBICOM)*, Aug 1999, pp. 271–278.

[2] Animesh Kumar, Prakash Ishwar, and Kannan Ramchandran, "Dithered A/D conversion of smooth non-bandlimited signals," *IEEE Trans. Signal Proc.*, vol. 58, no. 5, pp. 2654–2666, May 2010.

[3] Akshata Athawale and Animesh Kumar, "On dithered a/d conversion of multidimensional smooth non-bandlimited signals," in *Proc. of the International Conference on Signal Processing and Communications (SPCOM)*, 2012, pp. 1–5.

[4] John L. Brown Jr., "A Least Upper Bound for Aliasing Error," *IEEE Transactions on Automatic Control*, vol. 13, pp. 754–755, Dec. 1968.

[5] Baltasar Beferull-Lozano, Robert Konsbruck, and Martin Vetterli, "Rate-distortion problem for physics based distributed sensing," in *Proceedings of the Information Processing in Sensor Networks*, New York, NY, USA, Apr. 2004, pp. 330–339, IEEE.

[6] Juri Ranieri and Martin Vetterli, "Sampling and reconstructing diffusion fields in presence of aliasing," in *Proc. of the ICASSP 2013*, 2013.

[7] Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, *Signals and Systems*, Prentice Hall, USA, 2nd edition, 1996.

[8] S. Agashe, "A new general Routh-like algorithm to determine the number of RHP roots of a real or complex polynomial," *IEEE Trans. on Automatic Control*, vol. 30, no. 4, pp. 406–409, Apr. 1985.

[9] Ian N. Sneddon, *Elements of Partial Differential Equations*, Courier Dover Publications, 2006.

[10] Stéphane Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*, Academic Press, Burlington, MA, USA, 2009.