ANALYSIS OF H-SSSI PROCESSES USING THE CROSSING TREE: AN ALTERNATIVE TO WAVELETS

G. Decrouez[*]

National Research University
Higher School of Economics
Moscow, Russia

P.-O. Amblard

GIPSAlab/CNRS UMR 5283
Université de Grenoble
Grenoble, France

ABSTRACT

In this study we use the crossing tree of a signal for the purpose of analysis of H-sssi processes. The crossing tree performs an ad-hoc decomposition of a signal adapted to its dynamics, and represents a natural tool for the analysis of its local fluctuations. We present here a new multifractal formalism and a novel approach for estimating the spectrum of singularities of H-sssi processes using the crossing-tree. The performance of the crossing-tree based method is demonstrated in a numerical study. Its performance is also compared with state-of-the-art techniques based on wavelets, including wavelet-leaders.

Index Terms— H-sssi processes, crossing tree, multifractal formalism, adaptative decomposition, wavelets

1. INTRODUCTION

Scale invariance has been observed in time series in a wide range of applications, including hydrodynamic turbulence, high frequency finance, network traffic, signal and image processing. The dynamics of data presenting scale invariance can be understood from the relationship existing across a whole range of scales, which is summarized in the spectrum of singularities $D(h)$, where $D(h)$ is the Hausdorff dimension of the set of points with Hölder exponent h.

The spectrum of singularities conveys rich information about the scaling properties and regularity of a process, and deriving techniques for determining or estimating it (i.e. performing its multifractal analysis) has received a lot of attention. Due to the discrete nature of data, a direct computation of $D(h)$ cannot be performed in practice: estimation of the multifractal spectrum of a signal X typically happens in the context of the so-called *multifractal formalism*. Let $T_X(a, t)$ denote numerically computable quantities, summarizing the spatial displacement of X at time t and at a temporal scale a. It is usually obtained from a comparison of the original process with a reference pattern ψ dilated and located at different positions, $T_X(a, t) = a^{-1} \int X(u)\psi((u - t)/a)du$. The process X is said to possess scaling properties if the time averages of $T_X(a, t_k)$ follow a power law behaviour with respect to a,

$$n_a^{-1} \sum_{k=1}^{n_a} |T_X(a, t_k)|^q \sim C_q a^{\zeta(q)} \text{ as } a \to 0\,, \qquad (1)$$

where C_q is a positive constant depending on q, n_a is the number of $T_X(a, t_k)$ available at scale a, and where $\zeta(q)$ is referred to as the partition function. The multifractal formalism relates the spectrum of singularities $D(h)$ to the Legendre-Fenchel transform $\zeta^*(h) = \inf_q(1 + qh - \zeta(q))$ of ζ. When $D(h) = \zeta^*(h)$, the multifractal formalism is said to hold. In general, $\zeta^*(h)$ only provides an upper bound for $D(h)$.

The choice of ψ in (1) plays a central role in the estimation of the partition function. Multiresolution quantities based on a wavelet decomposition of the process are the most common tool to date. The contribution of the present study is to introduce a novel set of multiresolution quantities, defined in terms of the crossing tree.

Relation to earlier works. The crossing tree, defined in section 3, is a very general concept, and can easily be computed on real data. The crossing tree was used recently to construct a class of monofractal and multifractal processes, see [1, 2]. In [3] the crossing tree was used to estimate a time-change of a self-similar process, and in [4], it was used to characterise and test if a process is a continuous local martingale. In [5] it was applied to self-similar processes, to test for self-similarity and stationary increments, and to obtain an asymptotically consistent estimator of the Hölder exponent. Only information about the crossing tree structure was used there to estimate the Hölder exponent, and the present contribution differs from the kind of analysis found there, where information about the crossing durations is used for the purpose of analysis.

Compared to usual wavelet based multifractal formalisms (*e.g.* [6]), the crossing tree approach appears to be adaptive, and can be used for irregularly sampled signals. The contribution presented here extends earlier work presented in [7], see discussion at the end of the paper.

The paper is organised as follows. In Section 2 we define scale-invariant processes and in Section 3 we detail the construction of the crossing-tree. The crossing-tree formalism is presented in Section 4, and a numerical study follows in Section 5.

2. SCALE INVARIANT PROCESSES

A process $X(t)$ is said self-similar if there exists an $H \in (0, 1)$ such that the equality $X(ct) = c^H X(t)$ holds for all $c > 0$ in finite-dimensional distributions. If in addition the process $X(t)$ has stationary increments, then $X(t)$ is said to be H-sssi. The most-studied H-sssi processes are fractional Brownian motions (fBm), the only self-similar Gaussian processes with stationary increments. Their spectrum vanishes to a single point $D(h) = 1$ if $h = H$, and equal to $-\infty$ elsewhere. Let $\mathcal{B}(u)$ denote a Brownian motion. We also study Hermite processes, defined as

$$\mathcal{H}_H^k(t) = \int_{\mathbb{R}^k} \int_0^t \left(\prod_{j=1}^k (s - u_i)_+^{-(1/2+(1-H)/k)} \right) ds d\mathcal{B}(\mathbf{u})\,,$$

[*]Thanks to the Centre for Advanced Studies for funding.

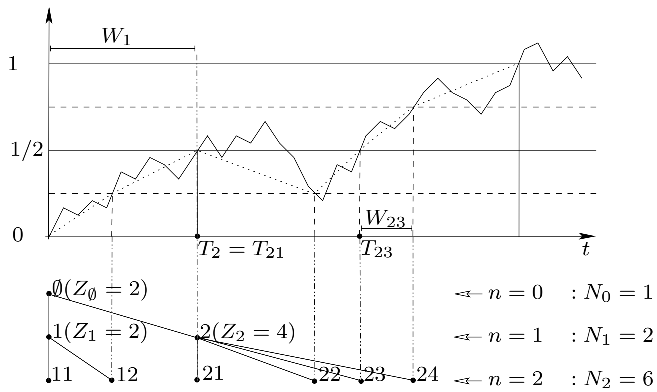

Fig. 1. Formation of the crossing tree from a sample path, and crossing tree notation. Variables are defined in the text.

where $d\mathcal{B}(\mathbf{u}) = d\mathcal{B}(u_1)\dots d\mathcal{B}(u_k)$, for $k \geq 1$, with $H \in (1/2,\, 1)$, and $x_+ = \max(0, x)$. The case $k = 1$ corresponds to the case of an fBm. For $k \geq 2$, Hermite processes are non-Gaussian.

The Weierstrass function is defined as

$$\mathcal{W}_H(t) = \sum_{k \in \mathbb{Z}} \lambda_0^{-kH} \Big(\cos(\varphi_k) - \cos(2\pi\lambda_0^k t + \varphi_k) \Big) ,$$

where H stands for the Hölder exponent and λ_0 is a fundamental harmonic. The Weierstrass function exhibits discrete scale invariance (DSI), with $\mathcal{W}_H(\lambda_0 t) = \lambda_0^H \mathcal{W}_H(t)$, in distribution. We consider here a stochastic version of this function, obtained by choosing the phases $\{\varphi_k\}_{k\in\mathbb{Z}}$ as a sequence of i.i.d. variables uniformly distributed over $[0,\ 2\pi]$. The spectrum of singularities of $\mathcal{W}_H$ is thus $D_{\mathcal{W}_H}(h) = 1$ if $h = H$, and equal to $-\infty$ elsewhere.

3. THE CROSSING TREE

Let $X : \mathbb{R}^+ \to \mathbb{R}$ be a process, with a.s. continuous sample paths and $X(0) = 0$. For $m \in \mathbb{Z}$ we define level-m crossing times T_k^m by putting $T_0^m = 0$ and

$$T_{k+1}^m = \inf\{t > T_k^m \mid X(t) \in 2^m\mathbb{Z},\ X(t) \neq X(T_k^m)\} ,$$

where $2^m\mathbb{Z} = \{x \mid x = 2^m a \text{ for } a \in \mathbb{Z}\}$. The k-th level-m (equivalently scale 2^m) crossing $C_k^m := \{(t, X(t)) \mid T_{k-1}^m \leqslant t < T_k^m\}$ is the bit of sample path from T_{k-1}^m to T_k^m. Denote by D_k^m the crossing duration of C_k^m. There is a natural tree structure to the crossings, as each crossing of size 2^m can be decomposed into a sequence of crossings of size 2^{m-1}. The nodes of the crossing tree are crossings and the offspring of any given crossing are the corresponding set of subcrossings at the level below. An example of a crossing tree is given in Figure 1. The crossing-tree can easily be computed for irregularly time-sampled signals, and as such the formalism developed later is adapted to this kind of data.

It is convenient to use the address space $I = \cup_{k=0}^{\infty} \mathbb{N}^k$, where $\mathbb{N}^k$ is the set of concatenations of k integers and $\mathbb{N}^0 = \emptyset$, to label the crossings of the process. For simplicity we will consider the first crossing from 0 to ± 1 and make this the root of our crossing tree. Label the root crossing $\emptyset$ and its subcrossings (each of size 1/2) 1 to $Z_\emptyset$. The subcrossings of a crossing $\mathbf{i} = i_1 i_2 \cdots i_n \in \mathbb{N}^n$ are then labelled $\mathbf{i}1, \dots, \mathbf{i}Z_\mathbf{i}$, where $Z_\mathbf{i}$ is the number of subcrossings of $\mathbf{i}$ and $\mathbf{i}j = i_1 i_2 \cdots i_n j$. Necessarily $Z_\mathbf{i}$ is an even integer larger or equal to 2. Denote by N_n the size of generation n. Each crossing $\mathbf{i}$ is one of two types, up or down, which we denote by $\sigma_\mathbf{i}$. Also let $W_\mathbf{i}$ be the duration of crossing $\mathbf{i}$, then the sample path is completely described by $\{(\sigma_\mathbf{i}, W_\mathbf{i}) \ : \ \mathbf{i} \in I\}$. Crossing-tree notation is summarized in Figure 1.

4. MULTIFRACTAL FORMALISMS

4.1. Wavelet-based formalisms

Estimation for H-sssi typically makes use of wavelet coefficients, see e.g. [8]. Consider the wavelet decomposition of a signal $X(t)$,

$$X(t) = \sum_{n,k\in\mathbb{Z}} c_{n,k}\psi(2^n t - k) ,$$

with $c_{n,k} = 2^n \int X(t)\psi(2^n t - k)$, for some mother wavelet ψ. If we denote by $\lambda_{n,k} = [k2^{-n}, (k+1)2^{-n})$ a dyadic cube at scale n, with $k \in \mathbb{Z}$, the wavelet-based structure function of X is $S_{wc}(q, n) = 2^{-n}\sum_k |c_{n,k}|^q$, where the sum is taken over all dyadic cubes $\lambda_{n,k}$ with non vanishing wavelet coefficients. Provided $S_{wc}(q, n)$ scales according to (1), we obtain the wavelet partition function,

$$\zeta_{wc}(q) = \liminf_{n\to+\infty} \left(\frac{\log S_{wc}(q,n)}{\log 2^{-n}} \right) , \quad q \in \mathbb{R} , \tag{2}$$

which leads to the multifractal formalism [9]

$$D_X(h) = \inf_{q\in\mathbb{R}} \{1 - \zeta_{wc}(q) + hq\} .$$

A major drawback of using wavelet coefficients is that the estimation of the structure function $S_{wc}(q, n)$ can be unstable for negative values of q, since wavelet coefficients can be arbitrarily small. The wavelet leaders formalism addresses this issue [6]. Put $3\lambda_{n,k} = \lambda_{n,k-1} \cup \lambda_{n,k} \cup \lambda_{n,k+1}$, which is the cube centered around $\lambda_{n,k}$, three times wider. The wavelet leaders $d_{n,k}$ of a bounded function $X(t)$ are defined as

$$d_{n,k} = \sup_{\{m,i \mid \lambda_{m,i} \subset 3\lambda_{n,k}\}} |c_{m,i}| .$$

The wavelet leader structure function is $S_{wl}(q, n) = 2^{-n}\sum_k |d_{n,k}|^q$, where the sum is taken over all non vanishing coefficients. The scaling function is

$$\zeta_{wl}(q) = \liminf_{n\to+\infty} \left(\frac{\log S_{wl}(q,n)}{\log 2^{-n}} \right) , \quad q \in \mathbb{R} , \tag{3}$$

which leads to the multifractal formalism

$$D_X(h) = \inf_{q\in\mathbb{R}} \{1 - \zeta_{wl}(q) + hq\} .$$

4.2. Crossing-tree based formalism

Let $\mathbf{i} \in \mathbb{N}^\infty$ be such that for each n, the size 2^{-n} crossing that contains t is $\mathbf{i}|n$, where $\mathbf{i}|n$ is $\mathbf{i}$ truncated to n places. Let $T_{\mathbf{i}|n}$ be the start time of crossing $\mathbf{i}|n$, then $T_{\mathbf{i}|n} \to t$ as $n \to \infty$, so we have

$$|X(t_0 + W_{\mathbf{i}|n}) - X(t_0)| \approx 2^{-n} = W_{\mathbf{i}|n}^{-n\log 2/\log W_{\mathbf{i}|n}} .$$

Thus (when everything works, for example for the Brownian motion, or more generally, for Canonical Embedded Branching Processes, see [2]) we get that

$$h(t_0) = \lim_{n\to\infty} -n\log 2/\log W_{\mathbf{i}|n} , \tag{4}$$

where

$$h(t_0) = \liminf_{\epsilon \to 0} \frac{1}{\log \epsilon} \log \sup_{|u-t_0|<\epsilon} |X(u) - X(t_0)|$$

is the Hölder exponent of the process at time t. Equation (4) gives the fundamental relationship between the multifractal spectrum and the crossing tree.

Given t, let $\mathbf{i} \in \mathbb{N}^\infty$ be such that for each n, the size 2^{-n} crossing that contains t is $\mathbf{i}|n$. Then, our analogue of the multiresolution quantity $T_X(2^{-n}, t)$ is $W_{\mathbf{i}|n}$. We say that the process $X(t)$ possesses scaling properties if time averages of the crossing durations follow a power law behaviour,

$$S_{ct}(n, q) = \frac{1}{N_n} \sum_{\mathbf{i}|n} |W_{\mathbf{i}|n}|^q \sim C'_q 2^{-n\zeta_{ct}(q)}, \qquad (5)$$

as $n \to \infty$, where the sum is taken over all crossings of size 2^{-n}. We call $S_{ct}(n, q)$ the structure function and ζ_{ct} the crossing tree partition function. The partition function can be obtained from the structure function as a limit,

$$\zeta_{ct}(q) = \liminf_{n \to \infty} \frac{\log S_{ct}(n, q)}{-n \log 2}. \qquad (6)$$

The difference between partition function (6) and wavelet-based partition functions (2) and (3), is that it relies on an adaptive decomposition of the signal: wavelet methods rely on a time decomposition of the signal whereas the proposed method relies on a space decomposition of the signal.

We proceed to some heuristic arguments. The definition of the crossing tree partition function at (6) implies that approximately $S_{ct}(n, q) \sim 2^{-n\zeta_{ct}(q)}$, and relation (4) roughly shows that $W_{\mathbf{i}|n} \sim 2^{-n/h(t_0)}$. For a given Hölder exponent h, when the $Z_{\mathbf{i}}$ are mutually independent and identically distributed with mean μ, there are about $\mu^{D(h)n}$ of the $W_{\mathbf{i}|n}$ that contribute to the sum in (5). Since $N_n \sim \mu^n$, it follows that the main contribution to the structure function at (5) comes from

$$2^{-n \log_2 \mu} \mu^{D(h)n} 2^{-nq/h} = 2^{-\{\log_2 \mu + q/h - D(h) \log_2 \mu\} n}.$$

Making use of the fact that for self-similar processes $H = \log 2 / \log \mu$, as we let $n \to \infty$, the dominant term comes from the smallest exponent, so that

$$\zeta_{ct}(q) = \inf_h \{(q+1)/h - D(h)/h\}. \qquad (7)$$

For a monofractal process with spectrum of singularities collapsing at a single point $h = H$, one gets $\zeta_{ct}(q) = q/H$. This result is proved formally for Brownian motion in [2] (Theorems 5.1 and 5.2), but remains an open conjecture for other self-similar monofractal processes. The numerical work supported in the next section supports representation (7). Wavelet based techniques also yield a linear partition function for monofractal signals, however with a slope equal to H for H-sssi processes, compared to $1/H$ using a crossing tree analysis. The crossing tree formalism is reversed compared to wavelet techniques: the decomposition of the signal is on a horizontal grid, instead on a vertical grid.

5. ESTIMATION PROCEDURE

5.1. Log-cumulants

Estimation of the crossing-tree partition function (7) is obtained from a weighted linear regression of $\log_2 S_{ct}(n, q)$ on n, $\hat{\zeta}_{ct}(q) = \sum_{n=n_1}^{n_2} w_n \log_2 S_{ct}(n, q)$, where the w_n satisfy $\sum n w_n = 1$, and $\sum w_n = 0$. They can be expressed as $w_n = b_n(Bn - B_n)/(BB_{nn} - B_n^2)$, where $B = \sum b_n$, $B_n = \sum n b_n$, and $B_{nn} = \sum n^2 b_n$. We take $b_n = N_n$, the number of crossing durations available at scale n. An alternative method is to first consider a polynomial expansion of the partition function

$$\zeta_{ct}(q) = \sum_{p \geq 1} \chi_p \frac{q^p}{p!}, \qquad (8)$$

and then to estimate the coefficients χ_p. Estimation of the coefficients χ_p is obtained from the estimation of the cumulants of the log of the crossing durations. Indeed, if we replace the time average with the statistical average, as first suggested in [10], then equation (5) reads $\mathbf{E}|W_{\mathbf{i}|n}|^q = \mathbf{E} e^{q \log W_{\mathbf{i}|n}} = C''_q 2^{-n\zeta_{ct}(q)}$. A cumulant expansion then yields

$$\log \mathbf{E} e^{q \log W_{\mathbf{i}|n}} = \sum_{p \geq 1} K_{p,n} \frac{q^p}{p!} = \log C''_q + \zeta_{ct}(q) \log 2^{-n}, \qquad (9)$$

where $K_{p,n}$ is the cumulant of order p of $\log W_{\mathbf{i}|n}$. Then from (9), the $K_{p,n}$ are of the form $K_{p,n} = k_p + \chi_p \log 2^{-n}$, so that

$$\sum_{p \geq 1} K_{p,n} \frac{q^p}{p!} = \sum_{p \geq 1} k_p \frac{q^p}{p!} + \left(\sum_{p \geq 1} \chi_p \frac{q^p}{p!} \right) \log 2^{-n}. \qquad (10)$$

Identifying the right hand side of (10) with (9) yields (8) as required. Estimation of the first three cumulants $K_{p,n}$ is done using $\hat{K}_{1,n} = s_1/N_n$, $\hat{K}_{2,n} = (N_n s_2 - s_1^2)/N_n^{[2]}$, and $\hat{K}_{3,n} = (N_n^2 s_3 - 3N_n s_1 s_2 + 2s_1^3)/N_n^{[3]}$, where $N_n^{[r]} = N_n(N_n - 1)\ldots(N_n - r + 1)$, and $s_j = \sum_{\mathbf{i}|n} (\log W_{\mathbf{i}|n})^j$, for $j = 1, 2, 3$. The κ_p are then estimated from a weighted linear regression of $\hat{K}_{p,n}$ on $\log 2^{-n}$, $\hat{\chi}_p = (\log_2 e) \sum_{n=n_1}^{n_2} w_n \hat{K}_{p,n}$, where the w_n are as before. In view of the discussion in the previous section, it is expected that $1/\chi_1 = H$ and $\chi_2 = 0$ for the class of H-sssi processes considered in section 2. We take $1/\hat{\chi}_1$ as an estimator of H. The derivation above is motivated from the techniques developed in [11], so that everything holds replacing the crossing tree partition function with $\zeta_{wl}(q)$. We denote by c_p^{wl} (resp. c_p^{wc}) the coefficients in a polynomial expansion of $\zeta_{wl}(q)$ (resp. $\zeta_{wc}(q)$). In particular, $\hat{c}_1^{\mathrm{wc}}/\hat{c}_1^{\mathrm{wl}}$ are estimators of H.

	Crossing tree	Wav. leaders	Wav. coef.
$\mathcal{H}_H^1(t)$	0.803 (0.768 ,0.841)	0.798 ± 0.064	0.800 ± 0.103
$\mathcal{H}_H^2(t)$	0.794 (0.732 ,0.867)	0.800 ± 0.121	0.799 ± 0.150
$\mathcal{H}_H^3(t)$	0.787 (0.744 ,0.836)	0.796 ± 0.107	0.800 ± 0.131
$\mathcal{H}_H^4(t)$	0.789 (0.753 ,0.827)	0.800 ± 0.103	0.803 ± 0.124
$\mathcal{W}_H(t)$	0.800 (0.778 ,0.824)	0.799 ± 0.049	0.801 ± 0.093

Table 1. Estimates $1/\hat{\chi}_1$, $\hat{c}_1^{\mathrm{wc}}$ and $\hat{c}_1^{\mathrm{wl}}$ with confidence limits $1/(\hat{\chi}_1 \pm 1.96\,\hat{\sigma}_{\chi_1})$ and $\hat{c}_1 \pm 1.96\,\hat{\sigma}_{c_1}$, where $\hat{\sigma}$. are estimates of the standard deviation from Monte-Carlo simulations, for Hermite processes, and the Weierstrass function for $H = 0.8$.

5.2. Numerical work

For each process presented in section 2, we compare the performance of the wavelet coefficients, wavelet leaders, and crossing tree based multifractal formalisms. The partition functions are estimated from an average of 1000 realizations of 2^{15} sample points each. The wavelet partitions functions are estimated using Daubechies'

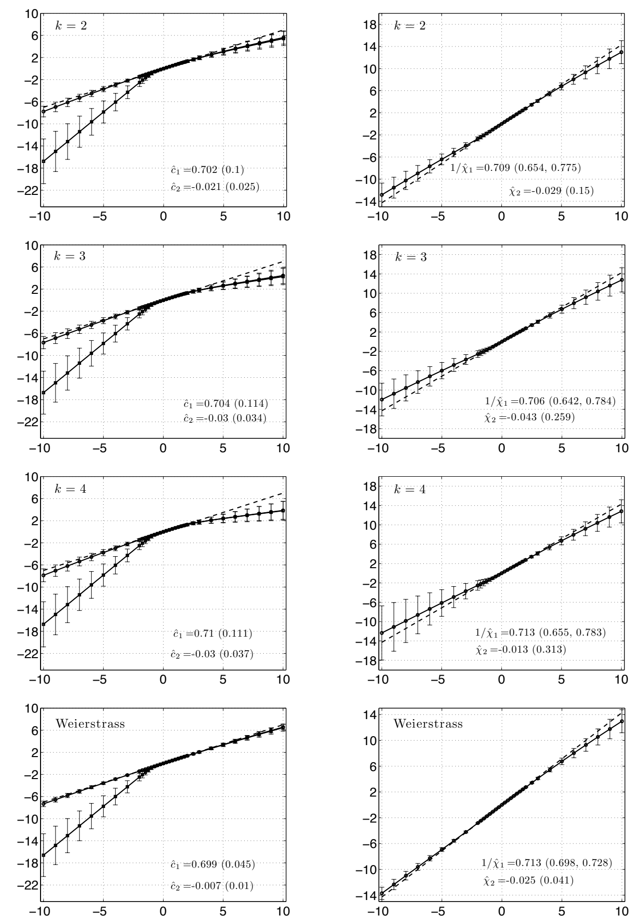

Fig. 2. Estimation of the partition function for q varying between -10 and 10 using wavelet coefficients (□) and wavelet leaders (○), left column, and using the crossing tree, right column. The dashed line is the theoretical line. From top to bottom, Hermite with $k = 2$, 3, and 4, and Weierstrass function, with $H = 0.7$.

wavelets with 3 vanishing moments, from scale 2^3 to 2^{12} using Matlab routines from [11, 12]. The crossing tree partition function is estimated from scale 2^2 to 2^5. Scales are chosen according to the temporal (wavelet) or spatial fluctuations (crossing-tree) of the signal, and thus differ in the two approaches. Table 1 presents estimate $1/\hat{\kappa}_1$ for the crossing tree, and $\hat{c}_1$ for the wavelet coefficients and wavelet leaders, which we take as estimates of the Hölder exponent H. The quality of the estimation using the crossing tree is comparable to estimates relying on wavelet leaders. Figure 2 presents the results obtained for Hermite processes with $H = 0.7$. In each case, the wavelet coefficient based formalism is not able to correctly estimate the partition function for negative values of q, compared to an estimation based on wavelet leaders or on the crossing tree. Other values of $H > 0.5$ give similar results but are not presented here. Estimation of the partition function for the fBm give similar results as for Hermite with $k = 2$, with lower variance of the estimate. The estimation for large positive q is better using the crossing tree than with wavelet leaders for the class of Hermite processes, at the expense of a higher variance for negative qs. Finally, estimate $\hat{\chi}_2$ is consistent with a linear partition function, for which $\chi_2 = 0$.

The present study extends earlier preliminary work presented in [7] in many ways. Firstly, the method is tested on a wider class of processes, indicating the validity of the approach for H-sssi processes. Secondly, it offers a comparison of the method with wavelet-based techniques, known to perform well on H-sssi processes, and shows similar performance on this class of processes. Next, the partition function is estimated from the cumulants of the log of the crossing durations, and an estimator of H is given. Finally, this study sets the grounds for further theoretical developments, the first step being formally proving the crossing-tree formalism (7).

Further preliminary numerical work not presented here indicates that the estimation of the partition function using the crossing-tree can be further improved if instead of using (5), we defined the structure function using the q-th moments of the sum of three consecutive crossing durations. Motivation for doing this comes from the study of the spectrum of singularities of non-negative multifractal measures, see for example [6], section 2.1. The bias observed on Figure 2 for negative values of q almost vanishes in that case. This work is under current investigation.

6. REFERENCES

[1] G. Decrouez and O.D. Jones, "A class of multifractal processes constructed using an embedded branching process," *Annals of Applied Probability*, vol. 22, pp. 2357–2387, 2012.

[2] G. Decrouez, B. Hambly, and O.D. Jones, "The hausdorff spectrum of a class of multifractal processes," *Stochastic Processes and their Applications. To appear*, 2015.

[3] W. Arroum and O.D. Jones, "A class of multifractal processes constructed using an embedded branching process," in *Complex Systems, A. Bender Ed. Article 60390F*. SPIE, 2006, vol. 6039.

[4] O.D. Jones and D.A. Rolls, "A characterisation of, and hypothesis test for, continuous local martingales," *Electronic Comm. Prob.*, vol. 16, pp. 638–651, 2011.

[5] O.D. Jones and Y. Shen, "Estimating the hurst index of a self-similar process via the crossing tree," *Signal Processing Letters*, vol. 11, pp. 416–419, 2004.

[6] S. Jaffard, B. Lashermes, and P. Abry, "Wavelet leaders in multifractal analysis," in *Applied and Numerical Harmonic Analysis*, pp. 219–264. Birkhauser Verlag Basel/Switzerland, 2006.

[7] G. Decrouez and P-O. Amblard O.D. Jones, "Estimation of the multifractal spectrum using the crossing tree," in *Proceedings of GRETSI, Brest, France*. Gretsi, 2013.

[8] D. Veitch and P. Abry, "A wavetlet-based joint estimator of the parameters of long-range dependence," *IEEE Trans. on Information Theory.*, vol. 45, pp. 878–897, 1999.

[9] G. Parisi and U. Frisch, "On the singularity structure of fully developed turbulence; appendix to," in *Fully developed turbulence and intermittency, by U. Frisch*. Proc. Int. Summer school Phys. Enrico Fermi, 1985, pp. 84–88.

[10] B. Castaing, Y. Gagne, and M. Marchand, "Log-similarity for turbulent flows," *Physica D*, vol. 68, pp. 387–400, 1993.

[11] H. Wendt, P. Abry, and S. Jaffard, "Bootstrap for empirical multifractal analysis," *IEEE Signal Proc. Mag.*, vol. 24, no. 4, pp. 38–48, 2007.

[12] H. Wendt and P. Abry, "Bootstrap tests for the time constancy of multifractal attributes," in *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Proc. (ICASSP)*, Las Vegas, USA, 2008.