MARGINAL WEISS-WEINSTEIN BOUNDS FOR DISCRETE-TIME FILTERING

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ABSTRACT

A marginal version of the Weiss-Weinstein bound (WWB) is proposed for discrete-time nonlinear filtering. The proposed bound is calculated analytically for linear Gaussian systems and approximately for nonlinear systems using a particle filtering scheme. Via simulation studies, it is shown that the marginal bounds are tighter than their joint counterparts.

Index Terms— Bayesian bounds, Weiss-Weinstein bound, nonlinear filtering.

1. INTRODUCTION

Consider the following discrete-time nonlinear system

$$x_{k+1} = f_k(x_k, v_k), \quad (1a)$$
$$z_k = h_k(x_k, w_k), \quad (1b)$$

where, $x_k \in \mathbb{R}^{n_x}$ is the state vector at discrete time k and $z_k \in \mathbb{R}^{n_z}$ is the measurement vector, and $f_k(\cdot)$ and $h_k(\cdot)$ are in general nonlinear mappings. The process and measurement noise vectors $v_k \in \mathbb{R}^{n_v}$ and $w_k \in \mathbb{R}^{n_w}$ are mutually independent white processes, assumed independent of the initial state x_0. In nonlinear filtering, one is interested in estimating the current state x_k from the sequence of measurements $Z_k = \{z_l\}_{l=1}^k$. In a Bayesian framework, this is equivalent to recursively computing the posterior density $p(x_k|Z_k)$, from which an optimal estimate with respect to any criterion can be extracted.

Unfortunately, the posterior pdf for the most general model (1b) is not available in closed form. Here, one has to resort to numerical approximations and the particle filter has become one of the most popular techniques due to its asymptotic properties in representing the posterior pdf [1–5]. For discrete-time linear systems with additive Gaussian noise

$$x_{k+1} = F \cdot x_k + v_k, \quad v_k \sim \mathcal{N}(0, Q_k), \quad (2a)$$
$$z_k = C \cdot x_k + w_k, \quad w_k \sim \mathcal{N}(0, R_k), \quad (2b)$$

a closed form solution for the posterior pdf is available, which is Gaussian $p(x_k|Z_k) = \mathcal{N}(x_k; \hat{x}_{k|k}(Z_k), \hat{P}_{k|k})$. The entities $\hat{x}_{k|k}(Z_k)$ and $\hat{P}_{k|k}$ are the state estimate and the covariance matrix provided by the celebrated Kalman filter.

While the area of developing estimators for the nonlinear filtering problem is relatively mature [4, 6, 7], the area of deriving corresponding fundamental performance limits is evolving rather slowly. The most preferred tool of assessing performance limits in a Bayesian context is the Bayesian Cramér-Rao lower bound (BCRB) [8, 9]. An important but much less explored alternative is to use the other Bayesian bounds in the literature, namely, Weiss-Weinstein, Bhattacharyya, and Bobrovsky-Zakai lower bounds [9]. All of the lower bounds mentioned above belong to a larger family of Bayesian bounds that is known today as the Weiss-Weinstein family of Bayesian bounds [10]. In the nonlinear filtering context, the recursive formulation of BCRB presented by Tichavský et al. was long considered as the state-of-the art [11], even though a couple of tighter alternatives exist [12]. The idea of [11] is to formulate the information matrix based on the joint density $p(X_k, Z_k)$ of the state and measurement sequence where $X_k = \{x_l\}_{l=1}^k$, from which the BCRB for estimating x_k can be extracted from the lower-right corner. This technique has been then adopted for the other Bayesian bounds in the Weiss-Weinstein family [13–15].

In this paper, a marginal version of the Weiss-Weinstein bound (WWB) is proposed using the marginal pdf $p(x_k, Z_k) = p(x_k|Z_k) \times p(Z_k)$. Similar to the BCRB case, the resulting bound turns out to be tighter after the marginalization [16]. For the linear Gaussian case, closed form solutions exist which exactly show this behavior. For nonlinear systems, a particle filter approximation is suggested, and based on a linear example with non-Gaussian noise it is shown that the same conclusions can be drawn.

2. GENERAL WEISS-WEINSTEIN BOUNDS

Weiss-Weinstein family of lower bounds is defined using the so-called *score functions* $\{\psi_i(x, z)\}_{i=1}^r$ which satisfy the property $\mathbb{E}_x[\psi_i(x, z)] = 0$ for $i = 1, \ldots, r$ and for all z. $\mathbb{E}_x[\cdot]$ denotes the expectation operator with respect to the variable x. The corresponding lower bounds in the family are then

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given as

$$\mathbb{E}_{x,z}\{[x-\hat{x}(z)][x-\hat{x}(z)]^T)\} \geq VG^{-1}V^T, \quad (3)$$

where the elements of the matrices $V \in \mathbb{R}^{n_x \times r}$ and $G \in \mathbb{R}^{r \times r}$ are defined as

$$[V]_{i,j} \triangleq \mathbb{E}_{x,z}[x_i \psi_j(x,z)], \qquad [G]_{i,j} \triangleq \mathbb{E}_{x,z}[\psi_i(x,z)\psi_j(x,z)],$$

with the notation $[\cdot]_{i,j}$ denoting the element of the argument matrix corresponding to the ith row and jth column and x_i being the ith element of x.

Weiss-Weinstein bound is obtained using the specific selection of the score functions given below

$$\psi_i(x,z) = L^{s_i}(z, x+h_i, x) - L^{1-s_i}(z, x-h_i, x), \quad (4)$$

for $i = 1, \dots, r$ where

$$L(z, x+h, x) \triangleq \frac{p(x+h,z)}{p(x,z)} = \frac{p(x+h|z)}{p(x|z)}. \quad (5)$$

The vectors $\{h_i \in \mathbb{R}^{n_x}\}_{i=1}^r$, which are column vectors called *test points*, and the scalars $\{s_i\}_{i=1}^r$ are generally free variables that have to be optimized. In the following, we fix $s_i = 1/2$, $i = 1, \dots, r$ according to the suggestion given in [10]. The mean square error matrix can be then lower bounded by

$$\mathbb{E}_{x,z}\{[x-\hat{x}(z)][x-\hat{x}(z)]^T)\} \geq HJ^{-1}H^T, \quad (6)$$

with matrix $H \in \mathbb{R}^{r \times r}$ given by $H = [\begin{array}{cccc} h_1 & h_2 & \cdots & h_r \end{array}]$. The elements of the matrix $J \in \mathbb{R}^{r \times r}$ can be written as follows

$$[J]_{i,j} = 2 \cdot \frac{e^{\mu(h_i - h_j)} - e^{\mu(h_i + h_j)}}{e^{\mu(h_i) + \mu(h_j)}}, \quad (7)$$

where $\mu(h)$ is known as the negative non-metric Bhattacharyya distance between the densities $p(x,z)$ and $p(x+h,z)$, which is defined as

$$\begin{aligned} \mu(h) &= \ln\left(\mathbb{E}_{x,z}\{\sqrt{L(z,x+h,x)}\}\right) \quad (8) \\ &= \ln\left(\iint \sqrt{p(x+h,z)p(x,z)}\,\mathrm{d}x\,\mathrm{d}z\right). \quad (9) \end{aligned}$$

In order to efficiently compute the WWB in closed form, analytical solutions for the expression $\mu(h)$ should be found.

3. JOINT WEISS-WEINSTEIN BOUND

The joint Weiss-Weinstein bound (J-WWB) is derived from the joint density $p(X_k, Z_k)$. It provides a lower bound on the MSE matrix for the sequence of states X_k rather than the current state x_k, and is given as follows

$$\mathbb{E}\{[X_k - \hat{X}_k(Z_k)][X_k - \hat{X}_k(Z_k)]^T)\} \geq H_{0:k} J_{0:k}^{-1} H_{0:k}^T, \quad (10)$$

where the expectation is taken with respect to $p(X_k, Z_k)$. In filtering applications, the MSE matrix of the current state x_k is of interest, which is located in the lower-right corner of the augmented MSE matrix. By choosing $H_{0:k}$ to be block diagonal and by making use of the specific structure of the matrix $J_{0:k}$, it has been shown in [13–15] that the inverse located in the lower-right corner of $[J_{0:k}]^{-1}$ can be computed recursively for all $k = 0, 1, \dots$, according to the following update formulae

$$\begin{aligned} A_k &= D_{k+1}^{11} - D_{k+1}^{10}[A_{k-1}]^{-1} D_{k+1}^{01}, \quad (11a) \\ \tilde{J}_{k+1} &= D_{k+1}^{22} - D_{k+1}^{21}[A_k]^{-1} D_{k+1}^{12}, \quad (11b) \end{aligned}$$

with the initial condition $[A_{-1}]^{-1} = 0$. The details on how the matrix elements $[D_{k+1}^{mn}]_{i,j}$ with $m, n \in \{0, 1, 2\}$ can be computed are not given here due to space constraints and can be found in [14]. As a result, a lower bound for the MSE matrix of the current state x_k is given as

$$\mathbb{E}\{[x_k - \hat{x}_k(Z_k)][x_k - \hat{x}_k(Z_k)]^T)\} \geq H_k \tilde{J}_k^{-1} H_k^T, \quad (12)$$

where the expectation is taken with respect to $p(x_k, Z_k)$. In the special case of linear state-space models with additive Gaussian noise structure, it has been shown in [15] that the matrix $\tilde{J}_{k+1}$ can be evaluated in closed form. In this case, $\tilde{J}_{k+1}$ for $k \geq 1$ can be evaluated recursively

$$\tilde{J}_{k+1} = D_{k+1}^{22} - D_{k+1}^{21}[D_{k+1}^{11} + \tilde{J}_k - B_k^{11}]^{-1} D_{k+1}^{12}, \quad (13)$$

with $B_0^{11} = \tilde{J}_0$ and $B_k^{11} = D_{k+1}^{22}$. For comparison purposes, the individual matrix elements $[D_{k+1}^{mn}]_{i,j}$ corresponding to this case are repeated here:

$$\begin{aligned} [D_{k+1}^{11}]_{i,j} &= 4\sinh\Big[\frac{1}{4} h_{k,i}^T \left(F^T Q_k^{-1} F \right. \\ &\qquad \left. + C^T R_k^{-1} C + Q_{k-1}^{-1}\right) h_{k,j}\Big], \quad (14a) \end{aligned}$$

$$[D_{k+1}^{12}]_{i,j} = -4\sinh\left[\frac{1}{4} h_{k,i}^T F^T Q_k^{-1} h_{k+1,j}\right] = [D_{k+1}^{21}]_{j,i}, \quad (14b)$$

$$[D_{k+1}^{22}]_{i,j} = 4\sinh\left[\frac{1}{4} h_{k+1,i}^T \left(C^T R_{k+1}^{-1} C + Q_k^{-1}\right) h_{k+1,j}\right]. \quad (14c)$$

The recursions are initiated at time $k = 0$ with

$$[D_1^{11}]_{i,j} = 4\sinh\left[\frac{1}{4} h_{0,i}^T \left(F^T Q_0^{-1} F + P_{0|0}^{-1}\right) h_{0,j}\right], \quad (14d)$$

and the initial matrix $\tilde{J}_0$ is given by

$$[\tilde{J}_0]_{i,j} = 4\sinh\left[\frac{1}{4} h_{0,i}^T P_{0|0}^{-1} h_{0,j}\right]. \quad (14e)$$

4. MARGINAL WEISS-WEINSTEIN BOUND

The marginal Weiss-Weinstein bound (M-WWB) is derived from the marginal density $p(x_k, Z_k)$ and hence from the posterior $p(x_k|Z_k)$. It provides a lower bound on the MSE matrix for the current state x_k directly, and is given as follows

$$\mathbb{E}_{x_k,Z_k}\{[x_k - \hat{x}_k(Z_k)][x_k - \hat{x}_k(Z_k)]^T)\} \geq H_k J_k^{-1} H_k^T. \quad (15)$$

4.1. Linear Systems

The M-WWB bound can be computed analytically thanks to the availability of a closed-form expression of the posterior density $p(x_k|Z_k) = \mathcal{N}(x_k; \hat{x}_{k|k}(Z_k), \hat{P}_{k|k})$. The computation of $\hat{P}_{k|k}$ can be expressed recursively using the following well-known formulas:

$$\hat{P}_{k|k-1} = F\hat{P}_{k-1|k-1}F^T + Q_k, \quad (16a)$$

$$\hat{P}_{k|k} = \hat{P}_{k|k-1} - K_k C\hat{P}_{k|k-1}, \quad (16b)$$

$$K_k = \hat{P}_{k|k-1}C^T(C\hat{P}_{k|k-1}C^T + R_k)^{-1}, \quad (16c)$$

where the recursions are initiated with $P_{0|0}$. For this case, the following Lemma applies.

Lemma 1. *For linear additive Gaussian systems, the negative non-metric Bhattacharyya distance is given by*

$$\mu(h) = -\frac{1}{8}h^T\hat{P}_{k|k}^{-1}h. \quad (17)$$

Proof. See Appendix □

Inserting (17) into (7) and performing some straightforward algebraic manipulations, the (i, j)-th element of the matrix J_k finally can be written as

$$[J_k]_{i,j} = 4 \cdot \sinh\left(\frac{1}{4}h_{k,i}^T\hat{P}_{k|k}^{-1}h_{k,j}\right). \quad (18)$$

By inspection, it is obvious that the matrix elements $\tilde{J}_k$ derived from the joint density approach are different to the matrix elements J_k obtained from the marginal density approach. The fact that $\tilde{J}_k$ is computed recursively introduces some temporal interdependency between the test points, while for the computation of J_k there is no such dependency. Hence, despite the lower computational complexity with which J_k can be evaluated compared to $\tilde{J}_k$, the optimization of $[J_k]_{i,j}$ via test points also seems to be much more easier.
In case we select $H_k = \epsilon \cdot I_{n_x}$, with $\epsilon \to 0$ and I_{n_x} denotes identity matrix, then this implies $h_{k,i}, h_{k,j} \to 0$ and we can simplify $\sinh(x) \approx x$, yielding $[J_k]_{i,j} = \epsilon^2[\hat{P}_{k|k}^{-1}]_{i,j}$. Inserting the result into (15) and considering the particular selection of H_k, the marginal WWB is given by $\hat{P}_{k|k}$ which is the BCRB for discrete-time linear filtering.

4.2. Nonlinear Systems

For nonlinear systems, closed-form expressions are available for neither M-WWB nor the posterior $p(x_k|Z_k)$. However, it is still possible to numerically approximate relevant quantities and compute an approximate marginal bound. One can use a particle filter (PF) to approximate the posterior [2, 4, 5]. In particle filters, the posterior density of the state $p(x_k|Z_k)$ is approximated by an empirical density of N particles and their weights as follows

$$\hat{p}(x_k|Z_k) = \sum_{p=1}^{N} w_k^{(p)}\delta(x_k - x_k^{(p)}), \quad (19)$$

where $x_k^{(p)}$ is a sample state at time k and $w_k^{(p)}$ is its corresponding weight. The approximate posterior $\hat{p}(x_k|Z_k)$ is propagated using the sequential importance sampling scheme. In this scheme, at any time k, first the samples, a.k.a. particles, are generated from a proposal distribution $q(x_k|x_{k-1}, z_k)$ and then the particle weights are updated according to

$$w_k^{(p)} \propto w_{k-1}^{(p)}\frac{p(z_k|x_k^{(p)})p(x_k^{(p)}|x_{k-1}^{(p)})}{q(x_k^{(p)}|x_{k-1}^{(p)}, z_k)}. \quad (20)$$

Particles are also resampled at required time instants to reduce the variance in the weights. By using the approximate density in (19), one can approximate $\mu(h)$ and thus J_k to compute the marginal bound. First, the cumbersome expectation appearing in (7) is approximated via Monte Carlo (MC) integration as follows

$$\mathbb{E}\{L^{1/2}(Z_k, x_k+h, x_k)\} \approx \frac{1}{N_{\text{mc}}}\sum_{l=1}^{N_{\text{mc}}}\sqrt{L(Z_k^{(l)}, x_k^{(l)} + h, x_k^{(l)})}, \quad (21)$$

where $x_k^{(l)}, Z_k^{(l)}$ with $l = 1, \ldots, N_{\text{mc}}$ are independent and identically distributed (i.i.d.) random vectors such that $(x_k^{(l)}, Z_k^{(l)}) \sim p(x_k, Z_k)$ holds. One can rewrite the pdf ratio as

$$L(Z_k, x_k + h, x_k) = \frac{p(z_k|x_k + h)\hat{p}(x_k + h|Z_{k-1})}{p(z_k|x_k)\hat{p}(x_k|Z_{k-1})}. \quad (22)$$

An approximation of the prediction density $\hat{p}(x_k + h|Z_{k-1})$ can be computed by using the particle approximation (19) at time $k - 1$ as follows

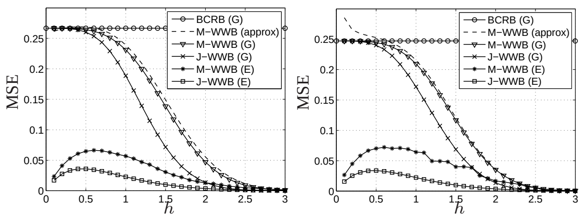
$$\hat{p}(x_k + h|Z_{k-1}) = \int p(x_k + h|x_{k-1})\hat{p}(x_{k-1}|Z_{k-1})\mathrm{d}x_{k-1}$$

$$\approx \sum_{p=1}^{N_{\text{pf}}} w_{k-1}^{(p)}p(x_k + h|x_{k-1}^{(p)}). \quad (23)$$

By plugging in the above particle approximation into (22) and averaging over the MC runs as in (21), we can compute $\hat{\mu}(h)$ for nonlinear systems.

5. SIMULATIONS

We consider the following simple one-dimensional linear state space model where it is possible to compute the analytical expressions for different bounds [15] and illustrate their



(a) $k = 1$ (b) $k = 19$

Fig. 1. MSE vs. test point h for different Bayesian bounds

differences with the M-WWB

$$x_{k+1} = x_k + w_k, \tag{24a}$$

$$y_k = x_k + v_k. \tag{24b}$$

Two examples are considered. In the first example, we assume that the prior and the noise terms are zero-mean Gaussian with variances $\sigma_0^2 = 0.4$, $\sigma_v^2 = 0.4$, $\sigma_w^2 = 0.4$. In the second example, the noise and prior are assumed to be exponential with $\mu_0 = \sigma_0$, $\mu_v = \sigma_v$, $\mu_w = \sigma_w$, where the exponential pdf with scale parameter μ is defined as: $p(n) = 1/\mu \cdot \exp -n/\mu$, for $n \geq 0$, and else $p(n) = 0$. Notice that, in the second case, it is not possible to compute M-WWB analytically and hence a PF approximation is required. The PF-based M-WWB is obtained by averaging over $N_{\text{mc}} = 5000$ MC runs where the bootstrap PF [1] uses $N_{\text{pf}} = 500$ particles except for the computation of the bound at $k = 19$ for the exponential distribution case where $N_{\text{pf}} = 10000$ particles are used and $N_{\text{mc}} = 1000$ MC runs are simulated. For the ease of exposition, we assume that the test points h are held fixed at each time step. In Fig. 1, the impact of the test point h on the different bounds at two different time steps is shown $k = 1$ and $k = 19$ [15]. It can be observed that the BCRB (G) assuming Gaussian prior and noise provide the tightest bound, which is also the optimal bound. The analytical J-WWB (G) and M-WWB (G) are generally looser, but approach the BCRB (G) when $h \to 0$. It can be also observed that the M-WWB (G) is tighter than the J-WWB (G) for this example and the chosen test points. For verification purposes, we also included the results of the PF implementation of the M-WWB for the linear Gaussian model, termed MM-WWB (approx). The approximation shows good agreement with the analytical results, but could be improved by increasing the number of MC runs and/or particles.

For the case of linear systems with exponential noise and prior, the BCRB does not exist because the exponential distribution violates the regularity conditions [9]. The M-WWB (E) obtained from the PF implementation is compared to the analytical J-WWB (E) presented in [15]. It can be observed that the M-WWB (E) is again tighter than the J-WWB (E). Increasing the number of MC runs and/or particles will yield smoother results. It is nevertheless noted, that using the PF with densities having finite support is often critical, since a significant number of particles is assigned a zero weight in the PF measurement update step. The strengths of the PF-based M-WWB implementation will pay off in situations where densities have infinite support, where the variance of the process and measurement noise are not too small, and where the state-space dimension is small.

6. CONCLUSION

In this paper, a marginal Weiss-Weinstein bound (WWB) is proposed for discrete-time nonlinear filtering. It is shown that for linear Gaussian systems the marginal WWB can be evaluated analytically and for nonlinear systems approximated numerically using a particle filtering approach. Results of two examples show that the marginal WWB is tighter than the joint WBB. Potential drawbacks of the PF-based approach are highlighted, as well as conditions are given, where the PF-implementation is expected to give satisfactory results with reasonable computational complexity.

7. APPENDIX

Application of Bayes' rule to the joint density $p(x_k, Z_k) = p(x_k|Z_k)p(Z_k)$ and inserting the result into (9), yields

$$\mu(h) = \ln\left(\iint \sqrt{p(x_k + h|Z_k)p(x_k|Z_k)}\,p(Z_k)\,\mathrm{d}x_k\,\mathrm{d}Z_k\right). \tag{25}$$

The expression under the square-root can be simplified by making using of the following identity

$$\mathcal{N}(x; x_1, P_1)\mathcal{N}(x; x_2, P_2) = \mathcal{N}(x_2; x_1, P_1 + P_2)\mathcal{N}(x; \bar{x}, \bar{P}),$$

where $\bar{x} = x_1 + P_1(P_1 + P_2)^{-1}(x_2 - x_1)$ and $\bar{P} = P_1 - P_1(P_1 + P_2)^{-1}P_1$. As a result (25), can be rewritten as

$$\begin{aligned}
\mu(h) &= \ln\Bigg(\sqrt{\mathcal{N}(h; 0, 2\hat{P}_{k|k})} \\
&\quad \times \iint \sqrt{\mathcal{N}(x_k; \hat{x}_{k|k}(Z_k) - h/2, \hat{P}_{k|k}/2)}\,p(Z_k)\,\mathrm{d}x_k\,\mathrm{d}Z_k\Bigg) \\
&= \ln\Bigg(\sqrt{\mathcal{N}(h; 0, 2\hat{P}_{k|k})}\,\frac{\sqrt{(2\pi)^n|\hat{P}_{k|k}|}}{\sqrt{\sqrt{(2\pi)^n|\hat{P}_{k|k}/2|}}} \\
&\quad \times \int \underbrace{\left[\int \mathcal{N}(x_k; \hat{x}_{k|k}(Z_k) - h/2, \hat{P}_{k|k})\,\mathrm{d}x_k\right]}_{=1} p(Z_k)\,\mathrm{d}Z_k\Bigg) \\
&= -\frac{1}{8}h^T\hat{P}_{k|k}^{-1}h. \qquad \square
\end{aligned}$$

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