A CONSTRAINED HYBRID CRAMÉR-RAO BOUND FOR PARAMETER ESTIMATION

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ABSTRACT

In statistical signal processing, hybrid parameter estimation refers to the case where the parameters vector to estimate contains both non-random and random parameters. Numerous works have shown the versatility of deterministic constrained Cramér-Rao bound for estimation performance analysis and design of a system of measurement. However in many systems both random and non-random parameters may occur simultaneously. In this communication, we propose a constrained hybrid lower bound which take into account of equality constraint on deterministic parameters. The usefulness of the proposed bound is illustrated with an application to radar Doppler estimation

Index Terms— Parameter estimation, hybrid Cramér-Rao bounds, equality constraints

1. INTRODUCTION

While Bayesian or non-Bayesian estimation algorithms are widely used in statistical signal processing, the technique called hybrid estimation has been developed more recently and suffers from a relative lack of results. Hybrid parameters mean the parameters vector to estimate contains both non-random and random parameters with a known prior probability density functions (p.d.f.). However, the hybrid estimation framework is not just a simple concatenation of the Bayesian and non-Bayesian techniques. Indeed, new estimator has to be derived as one can no longer use the Maximum Likelihood Estimator (MLE) for the non-Bayesian part and the Maximum *A Posteriori* estimator (MAP) for the Bayesian part since the parameters are generally statistically linked. Similarly, performance analysis methods of such estimators have to be modified accordingly, which is the aim of hybrid lower bounds.

Signal processing community generally use the Hybrid Cramér-Rao Bound (HCRB) [1] for which some asymptotic achievability results [2] are known. The HCRB, as well as the classical CRB, is known to be simple to obtain for various problems (see Part III of [3]) but suffers from some drawbacks. Indeed, these bounds are asymptotically tight only, in terms of number of samples or Signal to-Noise Ratio (SNR), and cannot predict the so-called threshold (i.e. large errors) on estimator mean square error (MSE) in non-linear estimation problems. This limitation can be overcome by resorting to other hybrid lower bounds, e.g. the Hybrid Barankin Bound (HBB) [4], the Hybrid Barankin/Weiss-Weinstein bound (HBWWB) [5] or the Hybrid Barankin/Ziv-Zakaï bound (HBZZB) [6]. Unfortunately, the computational cost of these hybrid "large-error" bounds is prohibitive in most applications when the number of unknown parameters increases. Therefore, provided that one keeps in mind the HCRB limitations, the HCRB is still a lower bound of great interest for system analysis and design in the asymptotic region.

As mentioned in the seminal paper [7] for deterministic parameter estimation, the standard form of the CRB is derived under the implicit assumption that the parameter space is an open subset of $\mathbb{R}^n$. However, in many applications, the vector of unknown parameters is constrained to lie in a proper non-open subset of the original parameter space. Since then, numerous works [8] have been devoted to extend the results introduced in [7]: 1) by providing useful technical results such as a general reparameterization inequality and the equivalence between parameterization change and equality constraints; 2) by studying the CRB modified by constraints either required by the model or required to solve identifiability issues; 3) by investigating the use of parameters constraints from a different perspective: the value of side (*a priori*) information on estimation performance. All these works have shown the versatility of deterministic constrained Cramér-Rao bound (CCRB) for estimation performance analysis and design of a system of measurement.

However not all system of measurement can be adequately modelled by resorting to deterministic parameters only, since both random and non-random parameters may occur simultaneously. One can cite, for example, the Gaussian generalized linear model [9], array shape calibration [1], time-delay estimation in radar signal [4], phase estimation in binary phase-shift keying transmission in a non-data-aided context [10], phase estimation of QAM modulated signals [11], cisoid frequency estimation [12], joint estimation of the pair dynamic carrier phase/Doppler shift and the time-delay in a digital receiver [13], parameters estimation in long-code DS/CDMA systems [14], bearing estimation for deformed towed arrays in the fluid mechanics context [15]. It is therefore the aim of this paper to provide an extension of the deterministic CCRB [16] to the hybrid parameter context yielding the Constrained HCRB (CHCRB). In this paper, we propose the CHCRB in the multivariate case for the esti-

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mation of random and non-random parameters with a set of equality constraints. The usefulness of the CHCRB is illustrated with an application to radar Doppler estimation.

2. RELATION TO PRIOR WORK

In deterministic parameter estimation, the CCRB has proven its usefulness for estimation performance analysis and design of a system of measurement by exploiting constraints between parameters to estimate. However, some systems of measurement cannot be adequately modelled by resorting to deterministic parameters only, since both random and non-random parameters may occur simultaneously. Therefore the purpose of the present paper is to extend the taking into account of equality constraint on deterministic parameters to the hybrid parameters context via the HCRB.

3. THE CONSTRAINED HYBRID CRAMÉR-RAO BOUND

3.1. Problem statement and notations

Let us first remind the estimation context in which the proposed bound can be useful. Consider Ω an observation space of points $\mathbf{x}$ and let $\boldsymbol{\theta} = \left[\boldsymbol{\theta}_d^T \ \boldsymbol{\theta}_r^T\right]^T$ denotes a $(D+R)-$dimensional hybrid real parameters vector to estimate, where $\boldsymbol{\theta}_d$ is a vector of unknown deterministic parameters belonging to $\Pi_d \subseteq \mathbb{R}^D$ and $\boldsymbol{\theta}_r$ is a vector of unknown random parameters belonging to $\Pi_r \subseteq \mathbb{R}^R$ with a known prior p.d.f. $f(\boldsymbol{\theta}_r; \boldsymbol{\theta}_d)$. Let $f(\mathbf{x}, \boldsymbol{\theta}) = f(\mathbf{x}, \boldsymbol{\theta}_r; \boldsymbol{\theta}_d)$ denotes the joint p.d.f. of $\mathbf{x}$ and $\boldsymbol{\theta}_r$ parameterized by $\boldsymbol{\theta}_d$. Additionally, the deterministic parameters $\boldsymbol{\theta}_d$ are assumed to be constrained in a non empty subset $\mathcal{C}$ of Π_d defined by $K < D$ non redundant equality constraints:

$$\mathcal{C} = \{\boldsymbol{\theta}_d \in \Pi_d \mid \mathbf{c}(\boldsymbol{\theta}_d) = \mathbf{0}\}, \qquad (1)$$

where $\mathbf{c}(\boldsymbol{\theta}_d)$ is a $K-$dimensional vector of derivable functions defined on Π_d. Let $\mathbf{C}(\boldsymbol{\theta}_d)$ denote the $K \times (D+R)$ matrix defined by

$$\mathbf{C}(\boldsymbol{\theta}_d) = \frac{d\mathbf{c}(\boldsymbol{\theta}_d)}{d\boldsymbol{\theta}^T} = \left[\frac{d\mathbf{c}(\boldsymbol{\theta}_d)}{d\boldsymbol{\theta}_d^T} \ \frac{d\mathbf{c}(\boldsymbol{\theta}_d)}{d\boldsymbol{\theta}_r^T}\right] = \left[\mathbf{C}_d(\boldsymbol{\theta}_d) \ \mathbf{0}\right], \qquad (2)$$

where $\mathbf{C}_d(\boldsymbol{\theta}_d)$ is a $K \times D$ matrix. Since the constraints are assumed to be non redundant, the rank of $\mathbf{C}_d(\boldsymbol{\theta}_d)$ is K for any $\boldsymbol{\theta}_d$ satisfying (1). Then there exists a $D \times (D-K)$ matrix $\mathbf{U}_d(\boldsymbol{\theta}_d)$ such that:

$$\mathbf{C}_d(\boldsymbol{\theta}_d)\mathbf{U}_d(\boldsymbol{\theta}_d) = \mathbf{0} \text{ and } \mathbf{U}_d^T(\boldsymbol{\theta}_d)\mathbf{U}_d(\boldsymbol{\theta}_d) = \mathbf{I}_{D-K}, \qquad (3)$$

where $\mathbf{I}_{D-K}$ denotes the identity matrix of size $D-K$. Moreover, if (3) holds, then the matrix $\mathbf{U}(\boldsymbol{\theta}_d) = \begin{bmatrix} \mathbf{U}_d(\boldsymbol{\theta}_d) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_R \end{bmatrix}$ satisfies

$$\mathbf{C}(\boldsymbol{\theta}_d)\mathbf{U}(\boldsymbol{\theta}_d) = \mathbf{0} \text{ and } \mathbf{U}^T(\boldsymbol{\theta}_d)\mathbf{U}(\boldsymbol{\theta}_d) = \mathbf{I}_{D+R-K}. \qquad (4)$$

Note that the column vectors of $\mathbf{U}_d(\boldsymbol{\theta}_d)$ is a basis of the kernel of $\mathbf{C}_d(\boldsymbol{\theta}_d)$ and the column vector of $\mathbf{U}(\boldsymbol{\theta}_d)$ is a basis of the kernel of $\mathbf{C}(\boldsymbol{\theta}_d)$. If the constraints are also applied over random parameters $\boldsymbol{\theta}_r$, then the matrix $\mathbf{U}$ will depend on $\boldsymbol{\theta}_r$, leading to a lower bound depending on the estimate of $\boldsymbol{\theta}_r$ (see section (3.3)).

3.2. Estimator class requirement and preliminary results

Let $\widehat{\boldsymbol{\theta}}(\mathbf{x})$ be an estimator of $\boldsymbol{\theta}$. The proposed bound is applicable for a class of estimator $\widehat{\boldsymbol{\theta}}$ which are unbiased, as for the classical HCRB [1][17], *i.e.*:

$$\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right] = \mathbf{0}. \qquad (5)$$

Any unbiased estimators satisfies the following relationship: for any integer $i \in [|1; D+R|]$, one has:

$$\begin{aligned}&\int_{\mathbb{R}^R}\int_{\mathbb{C}^N}\left(\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)\tfrac{\partial f(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d)}{\partial \theta_i}d\mathbf{x}d\boldsymbol{\theta}_r \\ &= \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right] + \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\tfrac{\partial}{\partial \theta_i}\left(\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)\right] \\ &= \mathbf{0} + \mathbf{e}_i,\end{aligned}$$

where $\mathbf{e}_i$ is a vector such that $\{\mathbf{e}_i\}_i = 1$ and $\{\mathbf{e}_i\}_{j \neq i} = 0$ where $\{\mathbf{e}_i\}_i$ denotes the i^{th} element of the vector $\mathbf{e}_i$. Thus, one has:

$$\int_{\mathbb{R}^R}\int_{\mathbb{C}^N}\left(\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)\frac{\partial f(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d)}{\partial \boldsymbol{\theta}^T}d\mathbf{x}d\boldsymbol{\theta}_r = \mathbf{I}_{D+R.} \qquad (6)$$

Additionally, let us set $\mathbf{v} = \frac{\partial \ln f(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d)}{\partial \boldsymbol{\theta}^T}$ then:

$$\begin{aligned}&\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\left(\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)\mathbf{v}^T\right] = \\ &\qquad \int_{\mathbb{R}^R}\int_{\mathbb{C}^N}\left(\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)\tfrac{\partial f(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d)}{\partial \boldsymbol{\theta}^T}d\mathbf{x}d\boldsymbol{\theta}_r\end{aligned} \qquad (7)$$

Finally, by mixing (6) and (7), one obtains:

$$\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\left(\widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}\right)\mathbf{v}^T\right] = \mathbf{I}_{D+R.} \qquad (8)$$

3.3. The proposed bound

In the following, for sake of legibility, let us set $\widetilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}$ and $\mathbf{U} = \mathbf{U}(\boldsymbol{\theta}_d)$. For any square matrix $\mathbf{M}$:

$$\begin{aligned}&\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\left(\widetilde{\boldsymbol{\theta}} - \mathbf{MUU}^T\mathbf{v}\right)\left(\widetilde{\boldsymbol{\theta}} - \mathbf{MUU}^T\mathbf{v}\right)^T\right] = \\ &\quad \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widetilde{\boldsymbol{\theta}}\widetilde{\boldsymbol{\theta}}^T\right] + \mathbf{MUU}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{vv}^T\right]\mathbf{UU}^T\mathbf{M}^T \\ &\quad -\mathbf{MUU}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}\widetilde{\boldsymbol{\theta}}^T\right] - \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widetilde{\boldsymbol{\theta}}\mathbf{v}^T\right]\mathbf{UU}^T\mathbf{M}^T.\end{aligned}$$

Since $\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\left(\widetilde{\boldsymbol{\theta}} - \mathbf{MUU}^T\mathbf{v}\right)\left(\widetilde{\boldsymbol{\theta}} - \mathbf{MUU}^T\mathbf{v}\right)^T\right]$ is positive semidefinite and, from (8), $\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widetilde{\boldsymbol{\theta}}\mathbf{v}^T\right] = \mathbf{I}_{D+R}$, one has:

$$\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widetilde{\boldsymbol{\theta}}\widetilde{\boldsymbol{\theta}}^T\right] \succeq \left(\begin{array}{c}\mathbf{MUU}^T + \mathbf{UU}^T\mathbf{M}^T - \\ \mathbf{MUU}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{vv}^T\right]\mathbf{UU}^T\mathbf{M}^T\end{array}\right). \qquad (9)$$

Since this inequality holds for any matrix $\mathbf{M}$, the tightest lower bound denoted CHCRB is obtained by maximizing the right hand side of (9) over $\mathbf{M}$:

$$\mathbf{CHCRB} = \max_{\mathbf{M}}\left(\begin{array}{c}\mathbf{MUU}^T + \mathbf{UU}^T\mathbf{M}^T - \\ \mathbf{MUU}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{vv}^T\right]\mathbf{UU}^T\mathbf{M}^T\end{array}\right). \qquad (10)$$

As $\mathbf{U}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{vv}^T\right]\mathbf{U}$ is symmetric positive definite, there exists an invertible diagonal matrix $\mathbf{D}$ and an unitary matrix $\mathbf{Q}$ such that $\mathbf{U}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{vv}^T\right]\mathbf{U} = \mathbf{QDQ}^T$. Consequently, (10) can be rewritten as:

$$\begin{aligned}&\mathbf{CHCRB} = \\ &\max_{\mathbf{M}}\left(\begin{array}{c}\mathbf{UQD}^{-1}\mathbf{Q}^T\mathbf{U} - \\ \left(\mathbf{UQD}^{-1} - \mathbf{MUQ}\right)\mathbf{D}\left(\mathbf{UQD}^{-1} - \mathbf{MUQ}\right)^T\end{array}\right)\end{aligned} \qquad (11)$$

Since $\mathbf{UQD}^{-1}\mathbf{Q}^T\mathbf{U}$ is independent of $\mathbf{M}$ and since the CHCRB is formulated as the difference of two positive semidefinite matrix, the maximum is achieved if and only if $\mathbf{MUQ} = \mathbf{UQD}^{-1}$, i.e.:

$$\mathbf{MU} = \mathbf{UQD}^{-1}\mathbf{Q}^T = \mathbf{U}\left(\mathbf{U}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{vv}^T\right]\mathbf{U}\right)^{-1}. \qquad (12)$$

Finally by substituting (12) in (10), one obtains:

$$\mathbf{CHCRB} = \mathbf{U}\left(\mathbf{U}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}\mathbf{v}^T\right]\mathbf{U}\right)^{-1}\mathbf{U}^T. \tag{13}$$

Remarks:

•Another possible derivation of the CHCRB can be obtained by using the covariance inequality [18, p.124][4]:

$$\mathbb{E}\left[\widetilde{\boldsymbol{\theta}}\widetilde{\boldsymbol{\theta}}^T\right] \succeq \mathbb{E}\left[\widetilde{\boldsymbol{\theta}}\boldsymbol{\psi}^T\right]\mathbb{E}^{-1}\left[\boldsymbol{\psi}\boldsymbol{\psi}^T\right]\mathbb{E}\left[\boldsymbol{\psi}\widetilde{\boldsymbol{\theta}}^T\right] \tag{14}$$

with $\boldsymbol{\psi} = \mathbf{U}^T\mathbf{v}$.

•In general, the proposed bound does not need the invertibility of the Fisher matrix $\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}\mathbf{v}^T\right]$ but of $\mathbf{U}^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}\mathbf{v}^T\right]\mathbf{U}$ only. This condition is also required for the CCRB in the deterministic estimation context [16].

•If the matrix $\mathbf{U}$ depends on $\boldsymbol{\theta}_r$ then $\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\widetilde{\boldsymbol{\theta}}\mathbf{v}^T\mathbf{U}\right] \neq \mathbf{U}$ and the lower bound will depend on $\widehat{\boldsymbol{\theta}}$, what is pointless.

3.4. Comparison with existing Cramér-Rao Bounds

3.4.1. The CHCRB versus the HCRB

The unconstrained HCRB is given by [1][17]:

$$\mathbf{HCRB} = \mathbb{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}\mathbf{v}^T\right], \tag{15}$$

where $\mathbf{F} \widehat{=} \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}\mathbf{v}^T\right]$ is the so-called hybrid Fisher information matrix. The HCRB can be obtained from the CHCRB when $K = 0$ leading to $\mathbf{U} = \mathbf{I}_{D+R}$. In other cases, the HCRB and the CHCRB are different. However, a comparison between the CHCRB and the HCRB is possible when $\mathbf{F}$ is non singular (otherwise the HCRB does not exist). Since $\mathbf{F}$ is symmetric positive definite, there exists a symmetric invertible matrix $\mathbf{F}^{\frac{1}{2}}$ such that $\mathbf{F} = \mathbf{F}^{\frac{1}{2}}\mathbf{F}^{\frac{1}{2}}$. Thus the CHCRB can be rewritten as:

$$\begin{aligned}\mathbf{CHCRB} &= \mathbf{F}^{-\frac{1}{2}}\mathbf{F}^{\frac{1}{2}}\mathbf{U}\left(\mathbf{U}^T\mathbf{F}^{\frac{T}{2}}\mathbf{F}^{\frac{1}{2}}\mathbf{U}\right)^{-1}\mathbf{U}^T\mathbf{F}^{\frac{T}{2}}\mathbf{F}^{-\frac{T}{2}}\\ &= \mathbf{F}^{-\frac{1}{2}}\mathbf{P}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}}\mathbf{F}^{-\frac{T}{2}}\end{aligned}$$

where $\mathbf{P}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}} = \mathbf{F}^{\frac{1}{2}}\mathbf{U}\left(\left(\mathbf{F}^{\frac{1}{2}}\mathbf{U}\right)^T\mathbf{F}^{\frac{1}{2}}\mathbf{U}\right)^{-1}\left(\mathbf{F}^{\frac{1}{2}}\mathbf{U}\right)^T$ is the projection matrix onto the column space of $\mathbf{F}^{\frac{1}{2}}\mathbf{U}$. Let $\mathbf{P}^{\perp}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}}$ denotes the projection matrix onto the vector space orthogonal to the previous one, then one has $\mathbf{P}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}} + \mathbf{P}^{\perp}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}} = \mathbf{I}$ and:

$$\begin{aligned}\mathbf{CHCRB} &= \mathbf{F}^{-\frac{1}{2}}\left(\mathbf{I} - \mathbf{P}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}}\right)\mathbf{F}^{-\frac{T}{2}}\\ &= \mathbf{F}^{-1} - \mathbf{F}^{-\frac{1}{2}}\mathbf{P}^{\perp}_{\mathbf{F}^{\frac{1}{2}}\mathbf{U}}\mathbf{F}^{-\frac{T}{2}} \preceq \mathbf{F}^{-1},\end{aligned}$$

therefore:

$$\mathbf{CHCRB} \preceq \mathbf{HCRB}. \tag{16}$$

This result is expected since the constraints can be interpreted as additional informations in order to estimate more accurately the parameters of interest. It has been shown in [19] that estimation algorithms which include parameters constraints could be lower than the unconstrained lower bounds. This is why the CHCRB, even lower than HCRB, is helpful in the hybrid estimation context with parameter constraints.

3.4.2. The CHCRB versus the marginal CCRB

Another question that we can ask is what is the difference between the CHCRB and the marginal CCRB for the deterministic parameters with constraints where in the first case, we estimate simultaneously non random parameters $\boldsymbol{\theta}_d$ and random parameters $\boldsymbol{\theta}_r$, whereas in second case, we estimate non-random parameters $\boldsymbol{\theta}_d$ only, $\boldsymbol{\theta}_r$ being regarded as nuisance parameters? To answer this question, note that, first, the CHCRB can be split into four blocks:

$$\mathbf{CHCRB} = \begin{bmatrix}\mathbf{CHCRB}_d & \mathbf{CHCRB}_{dr}^T\\ \mathbf{CHCRB}_{dr} & \mathbf{CHCRB}_r\end{bmatrix} \tag{17}$$

where the diagonal blocks $\mathbf{CHCRB}_d$ and $\mathbf{CHCRB}_r$ are respectively the lower bounds on the MSE of non-random parameters $\boldsymbol{\theta}_d$ and random parameters $\boldsymbol{\theta}_r$ *i.e.:*

$$\begin{aligned}\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\left(\widehat{\boldsymbol{\theta}}_d(\mathbf{x}) - \boldsymbol{\theta}_d\right)\left(\widehat{\boldsymbol{\theta}}_d(\mathbf{x}) - \boldsymbol{\theta}_d\right)^T\right] &\succeq \mathbf{CHCRB}_d\\ \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\left(\widehat{\boldsymbol{\theta}}_r(\mathbf{x}) - \boldsymbol{\theta}_r\right)\left(\widehat{\boldsymbol{\theta}}_r(\mathbf{x}) - \boldsymbol{\theta}_r\right)^T\right] &\succeq \mathbf{CHCRB}_r.\end{aligned}$$

Second, let $\mathbf{v}_d = \frac{\partial \ln f(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d)}{\partial \boldsymbol{\theta}_d}$ and $\mathbf{v}_r = \frac{\partial \ln f(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d)}{\partial \boldsymbol{\theta}_r}$. Then the Fisher information matrix can be decomposed as:

$$\mathbf{F} = \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\begin{bmatrix}\mathbf{v}_d\mathbf{v}_d^T & \mathbf{v}_d\mathbf{v}_r^T\\ \mathbf{v}_r\mathbf{v}_d^T & \mathbf{v}_r\mathbf{v}_r^T\end{bmatrix}.$$

Similarly:

$$\mathbf{CHCRB} = \mathbf{U}\mathbb{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\begin{bmatrix}\mathbf{U}_d^T\mathbf{v}_d\mathbf{v}_d^T\mathbf{U}_d & \mathbf{U}_d^T\mathbf{v}_d\mathbf{v}_r^T\\ \mathbf{v}_r\mathbf{v}_d^T\mathbf{U}_d & \mathbf{v}_r\mathbf{v}_r^T\end{bmatrix}\mathbf{U}^T \tag{18}$$

Let $\mathbf{S} = \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{U}_d^T\mathbf{v}_d\mathbf{v}_d^T\mathbf{U}_d\right] - \mathbf{R}$, where $\mathbf{R} = \mathbf{U}_d^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_d\mathbf{v}_r^T\right]\mathbb{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_r\mathbf{v}_r^T\right]\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_r\mathbf{v}_d^T\right]\mathbf{U}_d$, then an inversion by block of (18) leads to the following expression of the CHCRB:

$$\begin{aligned}&\mathbf{CHCRB} =\\ &\begin{bmatrix}\mathbf{U}_d & \mathbf{0}\\ -\mathbb{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_r\mathbf{v}_r^T\right]\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_r\mathbf{v}_d^T\right]\mathbf{U}_d & \mathbf{I}\end{bmatrix}\\ &\times\begin{bmatrix}\mathbf{S}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbb{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_r\mathbf{v}_r^T\right]\end{bmatrix}\\ &\times\begin{bmatrix}\mathbf{U}_d^T & -\mathbf{U}_d^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_d\mathbf{v}_r^T\right]\mathbb{E}^{-1}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_r\mathbf{v}_r^T\right]\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\end{aligned} \tag{19}$$

Then, by identification between (17) and (19), one has:

$$\mathbf{CHCRB}_d = \mathbf{U}_d\mathbf{S}^{-1}\mathbf{U}_d^T.$$

Since $\mathbf{R}$ is a positive semidefinite matrix, $\mathbf{S} \succeq \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{U}_d^T\mathbf{v}_d\mathbf{v}_d^T\mathbf{U}_d\right]$, which implies:

$$\mathbf{CHCRB}_d \preceq \mathbf{U}_d\left(\mathbf{U}_d^T\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}_d}\left[\mathbf{v}_d\mathbf{v}_d^T\right]\mathbf{U}_d\right)^{-1}\mathbf{U}_d^T. \tag{20}$$

The right hand side of (20) is the so-called marginal CCRB when $\boldsymbol{\theta}_r$ is considered as nuisance parameters. Consequently, the CHCRB is lower than the marginal CCRB. This is an extension of the order relation existing between the unconstrained hybrid lower bound and the unconstrained marginal lower bound [4].

4. APPLICATION TO DOPPLER ESTIMATION

We consider a radar system consisting of a 1-element antenna array receiving scaled, time-delayed, and Doppler-shifted echoes of a known complex bandpass signal $e_T(t)\, e^{j2\pi f_c t}$, where f_c is the carrier frequency and $e_T(t)$ is the envelope of the emitted signal. The antenna receives a pulse train (burst) of L pulses of duration T_0 and bandwidth B, with a pulse repetition interval (PRI) T, backscattered by a "slow" moving target in comparison with $e_T(t)$, i.e. [20]: $|2v(L-1)T| << \frac{c}{B}$ (no range migration) and $\frac{2v}{c}T_0 f_c << 1$ (Doppler effect on $e_T(t)$ is negligible), where c is the speed of light and v is the radial velocity of the target. Under the standard hypothesis of temporally white nuisance signal (thermal noise) of power σ_n^2 and a non fluctuating target during the burst duration, a simplified observation model for the l^{th}, $1 \le l \le L$, pulse is given by [20]:

$$x_l(t) = e_T(t-\tau)\,\alpha_l + n_l(t), \quad \alpha_l = \alpha e^{j2\pi f(l-1)}, \tag{21}$$

where $f = -2f_c \frac{v}{c} T$, $\frac{-1}{2} \le f \le \frac{1}{2}$, is the normalized Doppler frequency and α represents the complex amplitude of the target (including power budget equation). For the sake of simplicity, we assume that the target range is known. Therefore at the output of the delay/range matched filter at time $t = \tau$, the observation model is:

$$y_l = s e^{j2\pi f(l-1)} + n_l, \; s = \sqrt{BT_0}\alpha = r + jq, \tag{22}$$

and the vector of unknown parameters to estimate is $\boldsymbol{\theta} = (r, q, f)^T$ where (r, q) are assumed to be deterministic, f is assumed to be random with a known Gaussian prior distribution $\mathcal{N}\left(f_0, \sigma_f^2\right)$ and independent from the noise n_l assumed to be circular complex Gaussian distributed $n_l \sim \mathcal{CN}\left(0, \sigma_n^2\right)$. This scenario corresponds to a multi-function radar entering a tracking mode after a target detection in a surveillance mode. The radar budget, i.e. $|s|^2$, and f_0 associated to the target have been previously assessed by the detection step of the surveillance mode. However, during the inherent delay associated to the mode switch, the radial velocity of the target may vary, what we model by a prior distribution. An interesting question is whether it is worth taking into account this radar budget knowledge for the estimation of the f. Indeed, this amounts to introduce the following equality constraint: $r^2 + q^2 = |s|^2 = c$.
Therefore, the answer can be provided by a comparison between the CHCRB and the HCRB. Using (15), the classical HCRB is:

$$\begin{pmatrix} \frac{2L}{\sigma_n^2} & 0 & \frac{2\pi q L(1-L)}{\sigma_n^2} \\ 0 & \frac{2L}{\sigma_n^2} & \frac{2\pi r L(L-1)}{\sigma_n^2} \\ \frac{2\pi q L(1-L)}{\sigma_n^2} & \frac{2\pi r L(L-1)}{\sigma_n^2} & \frac{4\pi^2\left(r^2+q^2\right)L(L-1)(2L-1)}{3\sigma_n^2} + \frac{1}{\sigma_f^2 T^2} \end{pmatrix}^{-1} \tag{23}$$

The CHCRB is obtained using the following matrix $\mathbf{U}$ (13):

$$\mathbf{U} = \begin{pmatrix} \frac{q}{|s|} & \frac{-r}{|s|} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T. \tag{24}$$

In order to validate the proposed approach, we compute the MSE of the classical Maximum-*A Posteriori* MLE (MAPMLE) defined as:

$$\left(\widehat{r}, \widehat{q}, \widehat{f}\right) = \underset{(r,q)\in\mathbb{R}^2,\, f\in]-0.5;0.5]}{\arg\max} f_{\mathbf{y},F_D;r,q}\left(\mathbf{y}, f; r, q\right), \tag{25}$$

and the MSE of the Constrained MAPMLE (CMAPMLE) which restricts the (r, q) domain from $\mathbb{R}^2$ to $\mathcal{S} = \left\{(r,q)\,|r^2+q^2 = |s|^2\right\}$. The simulation settings are: $r = \frac{1}{\sqrt{2}}$, $|s|^2 = 0.8$, $f = 0.25$, $\sigma_f = 0.05$ and $L = 32$. The empirical MSE are assessed with 5000 Monte-Carlo trials and a frequency step $\delta f = 2^{-18}$. In figure (1), the total empirical MSE of the MAPMLE and the CMAPMLE are compared with the trace of HCRB and CHCRB. One can note that the CMAPMLE total MSE is lower than the classical HCRB whereas the CHCRB adequately predicts the asymptotic behavior of the CMAPMLE total MSE. In figure (2), the empirical MSE of $\widehat{f}$ is compared with the HCRB and the CHCRB. Since the HCRB and the CHCRB are identical, therefore the estimation of $\widehat{f}$ is independent of the knowledge of the radar budget at least in the asymptotic region. This theoretical result is confirmed by the same asymptotic performance of the MAPMLE and CMAPMLE. It is an extension of a well known property of the deterministic single tone estimation problem [21] to the random parameter case.

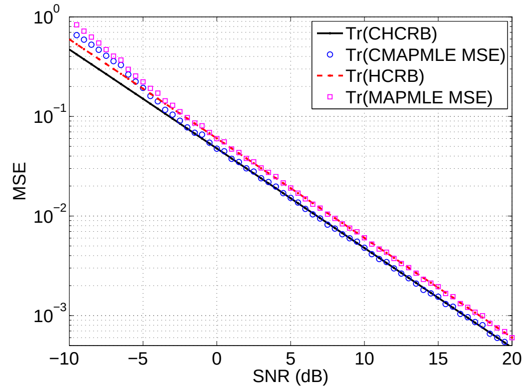

Fig. 1. Comparison of MAPMLE total MSE and HCRB versus SNR

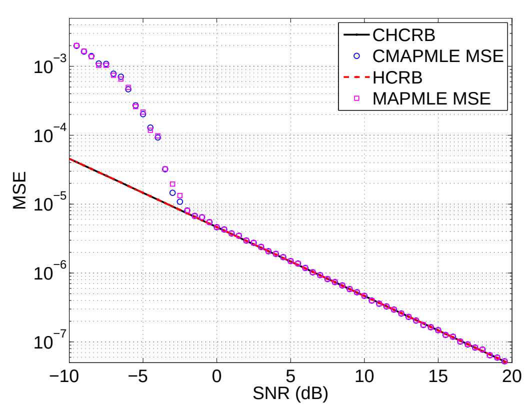

Fig. 2. Comparison of MAPMLE MSE of f and HCRB versus SNR

5. CONCLUSION

In this paper, a constrained hybrid lower bound, called the CHCRB, has been developed in order to take into account equality constraints between deterministic parameters. The CHCRB is not only the relevant bound to predict the asymptotic behavior of constrained estimators but also a versatile tool for estimation performance analysis and design of a system of measurement involving hybrid parameters.

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