AN ALGORITHM FOR THE PARAMETER ESTIMATION OF MULTIPLE SUPERIMPOSED EXPONENTIALS IN NOISE

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ABSTRACT

The parameter estimation of multiple superimposed complex exponentials in noise has been a popular research problem for decades due to its various practical applications. In this paper, we propose a simple yet accurate estimator for estimating the complex amplitudes and frequencies of the superimposed exponentials. Combining an efficient frequency estimator with a leakage subtraction scheme, the novel method iterates to consecutively estimate each component by gradually reducing the estimation error and increasing the estimation accuracy. Simulation results are presented to verify that the proposed algorithm is capable of obtaining estimation performance that is very close to the Cramer-Rao lower bound.

Index Terms— Frequency estimation, complex exponentials, interpolation algorithm, Fourier coefficients, leakage subtraction.

1. INTRODUCTION

The sum of multiple complex exponentials is adopted as the signal model in many applications in engineering and chemistry [1]. Estimating the parameters of the signal is always a fundamental and important research problem. In this paper, we focus on the parameter estimation problem of the signal model given by:

$$x(n) = \sum_{i=1}^{I} A_i e^{j2\pi f_i n} + w(n). \tag{1}$$

In (1), $n = 0 \ldots N-1$ and N is the total number of signal samples. I is the number of components and is assumed to be known as *a-priori* information. A_i and f_i are respectively the complex amplitude and the normalised frequency ($f_i \in [-0.5, 0.5]$) of the i^{th} component, which we aim to estimate. The noise terms $w(n)$ are assumed to be additive Gaussian noise with zero mean and variance σ^2.

In the past few decades, various algorithms have been proposed to deal with the estimation problem [2], and the time-domain high resolution parametric estimators are among the most popular ones [3, 4, 5, 6]. They utilise the singular value decomposition (SVD) to separate the noisy signal into pure signal and noise subspaces. These methods can achieve both high resolution, such that they can resolve peaks that are closely spaced, and considerably accurate estimation. But they also suffer from high computational cost due to the SVD operation, which has a complexity of $O(N^3)$. The frequency-domain estimators such as those proposed in [7, 8, 9], on the other hand, are computationally more efficient than the time-domain high resolution estimators. However, since they are usually developed for single-tone signals, they are lack of estimation accuracy due to estimation bias when applied to signals with multiple components. In this paper, we put forward a novel parametric estimation algorithm that operates in the frequency domain. The algorithm is computationally efficient as it does not rely on SVD operation and at the same time can achieve accurate estimation performance of the parameters that outperforms time-domain high resolution estimators.

The rest of the paper is organised as follows. In Section 2, we present the novel parameter estimation algorithm. In Section 3, we demonstrate the simulation results of the proposed algorithm by comparing with state-of-art parametric estimators and the Cramer-Rao lower bound (CRLB). Finally, conclusion is drawn in Section 4.

2. THE PROPOSED METHOD

The proposed estimation algorithm for multiple complex exponentials relies on the exact version of the A&M [7] estimator for the estimation of frequency of each component. However, the A&M estimator is not directly implemented but in combination with an iterative leakage subtraction scheme which eliminates the error of the interpolated Fourier coefficients introduced by other components in the signal.

From now on, we denote $\hat{\lambda}$ as the estimate of the parameter λ. We start the derivation of the novel method by reviewing the original A&M estimator. Assuming $x(n)$ to be the single tone signal ($I = 1$), the A&M estimator starts by finding the maximum bin of the periodogram of the signal as the coarse estimation of the frequency

$$\hat{m} = \arg\max_k |X(k)|^2 , \tag{2}$$

where $X(k) = \text{DFT}[x(n)]$. The true frequency is then given by

$$f = \frac{\hat{m} + \delta}{N}, \tag{3}$$

where $\delta \in [-0.5, 0.5]$ is the frequency residual. The coarse estimation is refined by finding δ using an estimator based on iterative interpolation on Fourier coefficients. The noise-free interpolated coefficients in each iteration are given by

$$\begin{aligned} X_{\pm 0.5} &= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta}\pm 0.5)n} \\ &= A \frac{1 + e^{j2\pi(\delta-\hat{\delta})}}{1 - e^{j\frac{2\pi}{N}(\delta-\hat{\delta}\mp 0.5)}}, \end{aligned} \tag{4}$$

where $\hat{\delta}$ is the estimated residual from the previous iteration and is initialised as zero. Putting $z = e^{j2\pi\frac{\delta-\hat{\delta}}{N}}$, an estimate of z^{-1} is constructed as

$$\hat{z}^{-1} = \cos\left(\frac{\pi}{N}\right) - j\frac{X_{0.5} + X_{-0.5}}{X_{0.5} - X_{-0.5}} \sin\left(\frac{\pi}{N}\right). \tag{5}$$

From $\hat{z}^{-1}$, the estimate of δ in each iteration can be given by

$$\hat{\delta} = -\frac{N}{2\pi} \Im\left\{\ln \hat{z}^{-1}\right\} + \hat{\delta}, \tag{6}$$

where $\Im\{\bullet\}$ denotes the imaginary part of $\bullet$. It has been shown in [7] that two iterations are sufficient for the estimator to obtain asymptotically unbiased frequency estimate with the variance only 1.0147 times the CRLB.

As the A&M estimator operates in the frequency domain and is developed for single-tone signals, estimation error will be introduced when it is applied to signals with multiple components due to the fact that leakage of other components is introduced which can lead to a deviation of the interpolated coefficients from their expected values for a single exponential. Assuming $x(n) = \sum_{i=1}^{I} x_i(n)$ now has I components, and denoting λ_i as the parameter λ corresponding to the i^{th} component, the noise-free interpolated Fourier coefficients to the i^{th} component in each iteration become

$$\tilde{X}_{i,\pm 0.5} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(\hat{m}_i+\hat{\delta}_i\pm 0.5)n} \tag{7}$$

$$= X_{i,\pm 0.5} + \sum_{l=1, l\neq i}^{I} \check{X}_{l,\pm 0.5}, \tag{8}$$

where $X_{i,\pm 0.5}$ are the expected coefficients for a single exponential as shown in (4). $\check{X}_{l,\pm 0.5}$ $(l = 1 \ldots I, l \neq i)$ are the leakage terms introduced by the other $I - 1$ existing components, which can be calculated by

$$\begin{aligned} \hat{\check{X}}_{l,\pm 0.5} &= \sum_{n=0}^{N-1} x_l(n) e^{-j\frac{2\pi}{N}(\hat{m}_i+\hat{\delta}_i\pm 0.5)n} \\ &= A_l \frac{1 + e^{j2\pi\hat{\Delta}_l}}{1 - e^{j\frac{2\pi}{N}(\hat{M}_l+\hat{\Delta}_l\mp 0.5)}}, \end{aligned} \tag{9}$$

where

$$\hat{M}_l = \hat{m}_l - \hat{m}_i, \quad \text{and} \quad \hat{\Delta}_l = \delta_l - \hat{\delta}_i. \tag{10}$$

Therefore, the reduction of the estimation error can be performed by subtracting the sum of leakage from the interpolated coefficients to obtain the estimates of the expected coefficients of a single exponential. Substituting (9) into (8) yields

$$\hat{X}_{i,\pm 0.5} = \tilde{X}_{i,\pm 0.5} - \sum_{l=1, l\neq i}^{I} A_l \frac{1 + e^{j2\pi\hat{\Delta}_l}}{1 - e^{j\frac{2\pi}{N}(\hat{M}_l+\hat{\Delta}_l\mp 0.5)}}. \tag{11}$$

The true values of δ_l and A_l are unknown and need to be estimated in the estimation process as well, though $\hat{m}_l$ can be obtained by maximum bin search (2). Therefore, we iterate the estimation process by consecutively estimate both parameters of each component with leakage subtraction incorporated. During the iteration process, to estimate the parameters of the i^{th} component in the each iteration, the previous estimation of the residuals and amplitudes of all the other $I - 1$ components are utilised to calculate the sum of the leakage terms, while the previous residual estimate of the i^{th} component itself is used to obtain the refined estimation of δ_i and A_i. It is straightforward that the estimation of δ_i can be obtained using (6) by substituting (11) into (5), while the estimation of the complex amplitude can be obtained by the maximum likelihood estimator

$$\hat{A}_i = \frac{1}{N}\left(\sum_{n=0}^{N-1} x(n) e^{-j2\pi\hat{f}_i n} - \sum_{l=1, l\neq i}^{I} \hat{\check{X}}_{l,\hat{f}_l}\right), \tag{12}$$

where

$$\hat{\check{X}}_{l,\hat{f}_l} = \hat{A}_l \frac{1 - e^{j2\pi N(\hat{f}_l-\hat{f}_i)}}{1 - e^{j2\pi(\hat{f}_l-\hat{f}_i)}}, \tag{13}$$

is the leakage term at frequency $\hat{f}_i$ introduced by the l^{th} component. As a result of the iterative procedure, the leakage terms (9) can be gradually better estimated as the number of iteration increases and the error between $\hat{X}_{i,\pm 0.5}$ and $X_{i,\pm 0.5}$ is gradually reduced. Thus the performance of the algorithm approaches the single component case for sufficient number of iterations, and the parameters can be accurately estimated.

The estimation procedure of the proposed algorithm combining the exact A&M estimator and the leakage subtraction scheme is finally summarised as follows.

Initialising $\hat{f}_{1,\ldots,I} = \hat{\delta}_{1,\ldots,I} = \hat{A}_{1,\ldots,I} = 0$, loop the following steps for Q iterations:

For $i = 1$ to I, do:

1. If $Q = 1$, calculate

$$X(k) = \text{DFT}\left(x(n) - \sum_{c=1}^{i-1} \hat{A}_c e^{j2\pi\hat{f}_c n}\right), \tag{14}$$

   and use (2) to find the maximum bin of $|X(k)|^2$;
2. Calculate $\hat{\check{X}}_{l,\pm 0.5}$, $(l = 1 \ldots I, l \neq i)$ using (9) and calculate $\hat{X}_{i,\pm 0.5}$ using (7) and (11);

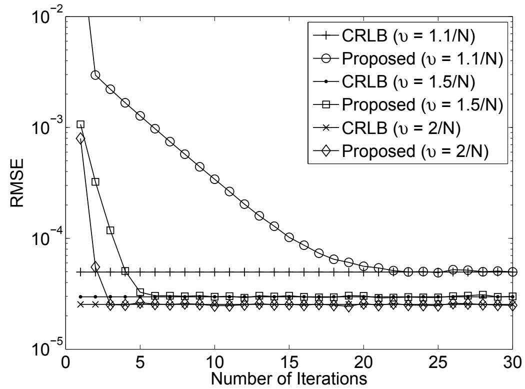

Fig. 1. RMSE of $\hat{f}_2$ obtained by the proposed method versus Q under various υ when SNR = 30dB and $\alpha = 1$. 1,000 Monte Carlo runs were used.

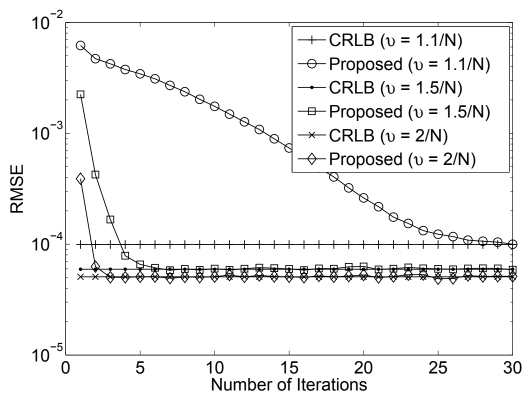

Fig. 2. RMSE of $\hat{f}_2$ obtained by the proposed method versus Q under various υ when SNR = 30dB and $\alpha = 0.5$. 1,000 Monte Carlo runs were used.

3. Find $\hat{z}(\hat{X}_{i,\pm 0.5})$ using (5) and renew $\hat{\delta}_i$ by (6);
4. Estimate $\hat{f}_i$ and $\hat{A}_i$ by (3) and (12) respectively.

It is worth pointing out that Step 1 of the algorithm (finding the maximum bins) can be efficiently implemented as calculating FFT for the first component and calculating the close form expressions for the rest of the components. As a result, the overall computational complexity of the proposed algorithm has the same order as that of the FFT operation, $O(N \log_2 N)$, which is more efficient than the SVD based high resolution methods that cost $O(N^3)$ for computation.

3. SIMULATION RESULTS

We test the proposed algorithm on the following signal

$$x(n) = e^{j2\pi f_1 n} + \alpha e^{j\phi} e^{j2\pi (f_1+\upsilon)+n} + w(n). \tag{15}$$

The signal length is set to $N = 64$. υ is the interval between the two frequencies. f_1 and ϕ are randomly selected in each

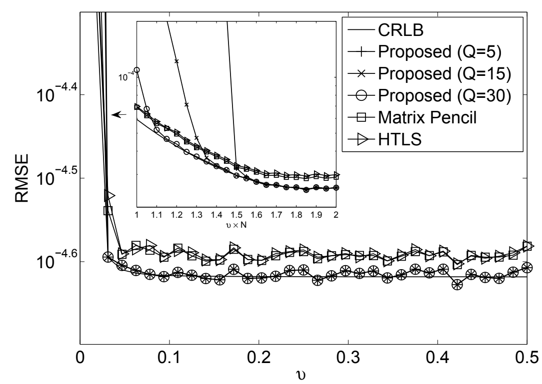

Fig. 3. RMSE of $\hat{f}_1$ versus υ obtained by various algorithm when SNR = 30dB and $\alpha = 0.5$. 5,000 Monte Carlo runs were used.

run such that $f_1 \in [-0.5, 0.5 - \upsilon]$ and $\phi \in [-\pi, \pi]$. $\alpha \leq 1$ is the ratio of the two magnitudes. We define the signal to noise ratio (SNR) of the signal as $\rho = 1/\sigma^2$.

We firstly examine the performance of the algorithm versus different the numbers of iterations Q. In this simulation we set SNR = 30dB. In Fig. 1 we plot the root mean square error (RMSE) of $\hat{f}_2$ versus Q under different frequency intervals υ when $\alpha = 1$. We only show the results of f_2 because the performance on both frequencies are similar as $\alpha = 1$. The corresponding CRLB [10] are also plotted for comparison purposes. In the case of $\upsilon = 1.1/N$, where the frequencies are slightly more than one bin ($1/N$) away, 22 iterations are enough for the RMSE of the estimate to become extremely close to CRLB. It is straightforward to find that less iterations are needed as the frequencies become further apart from each other, and only $Q = 3$ is needed in the case of $\upsilon = 2/N$. In Fig. 2 we show the RMSE of f_2 when $\alpha = 0.5$. We can find that due to lower SNR, the algorithm converges more slowly when $\upsilon = 1.1/N$, and $Q = 30$ are needed for CRLB-comparable performance. Nevertheless, similar results as in Fig. 1 can be observed for $\upsilon = 1.5/N$ and $2/N$.

Then we investigate the performance of the algorithm as a function of υ when $\alpha = 0.5$. The SNR in this test is also 30dB. In Fig. 3 we show the RMSE of $\hat{f}_1$ versus υ, which varies from $1/N$ to $32/N$. For the sake of benchmarking the performance, the novel method is compared with CRLB, Matrix Pencil [4] and the Hankel Total Least Square (HTLS) method [11], an ESPRIT-type algorithm [3] with total least square minimisation. Both methods are state-of-art time-domain estimators that are still able to outperform recently proposed methods such as [6]. The proposed algorithm is implemented under $Q = 5, 15$ and 30. For Matrix Pencil, the pencil parameter is set to $L = \lfloor N/3 \rfloor = 21$. For HTLS, the Hankel matrix size is 21×44. From the figure we find that although the proposed method has higher RMSE than the high resolution methods at $\upsilon = 1/N$ because of the unreliable maximum bin

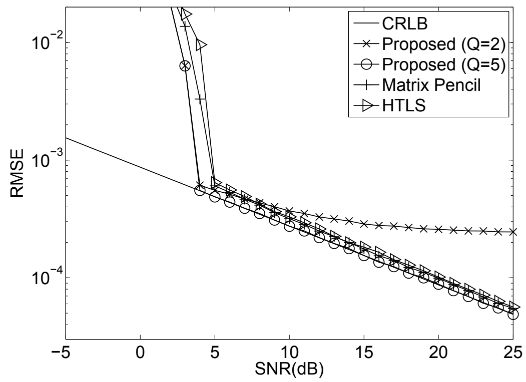

Fig. 4. RMSE of $\hat{f}_1$ versus SNR using various algorithms. 5,000 Monte Carlo runs were used.

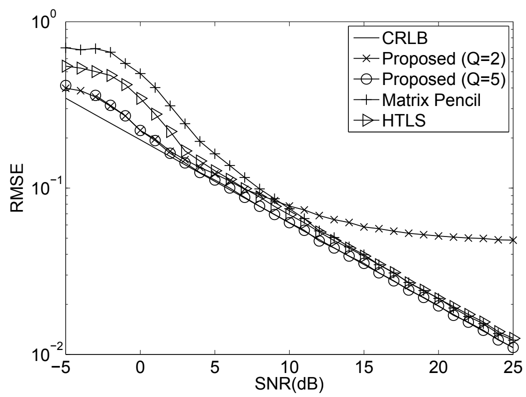

Fig. 5. RMSE of $|\hat{A}_1|$ versus SNR using various algorithms. 5,000 Monte Carlo runs were used.

search, it is capable of outperforming the other two methods at $1.1/N \leq \upsilon \leq 32/N$ using $Q \leq 30$.

Now we examine the RMSE of the parameter estimates under different SNR using the proposed method, Matrix Pencil and HTLS. The proposed algorithm is implemented using $Q = 2$ and 5 while the algorithm parameters of Matrix Pencil and HTLS are kept to be the same as the previous test. Figs. 4 to 7 show the RMSE of the parameter estimates obtained by various algorithms versus SNR when $\upsilon = 0.025$ and $\alpha = 0.5$. We find that for the proposed algorithm, the estimation bias is reduced as the iteration number increases from $Q = 2$ to $Q = 5$, and there is no observable bias at 25dB when $Q = 5$. It is also clear that the proposed algorithm can achieve smaller RMSE than both Matrix Pencil and HTLS at all SNR.

4. CONCLUSION

In this paper, we have proposed a novel method for estimating the frequencies and complex amplitudes of multiple superimposed complex exponentials in additive Gaussian noise.

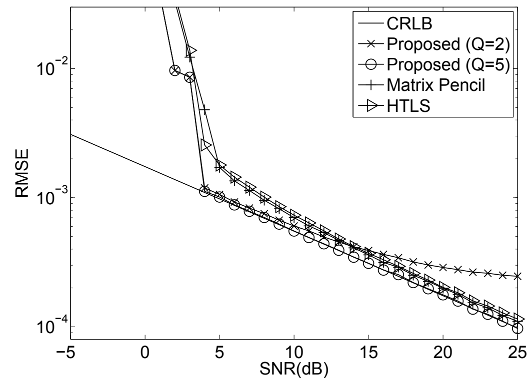

Fig. 6. RMSE of $\hat{f}_2$ versus SNR using various algorithms. 5,000 Monte Carlo runs were used.

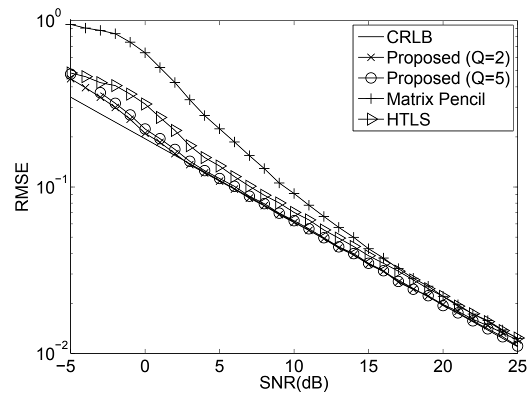

Fig. 7. RMSE of $|\hat{A}_2|$ versus SNR using various algorithms. 5,000 Monte Carlo runs were used.

The proposed algorithm utilises the A&M algorithm as the fundamental estimator of a single component combining an iterative leakage subtraction scheme. Benefit from the leakage subtraction, the error of the expected interpolated coefficients can be gradually reduced during the iterative process and so as the bias and variance of the parameter estimates. It has been verified by simulation results that the proposed algorithm can achieve accurate estimation of all the parameters with the RMSE of the estimates extremely close to the CRLB.

5. REFERENCES

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