SUBSPACE-BASED PHASE NOISE ESTIMATION IN OFDM RECEIVERS

Pramod Mathecken, Stefan Werner, Taneli Riihonen and Risto Wichman*

Aalto University School of Electrical Engineering, Department of Signal Processing and Acoustics,
P.O. Box 13000, FI-00076 Aalto, Finland, *Email: pramod.mathecken@aalto.fi

ABSTRACT

In this paper, we consider the problem of channel and phase noise estimation for an orthogonal frequency division multiplexing (OFDM) radio link. We solve this problem by first investigating the subspace in which the phase noise spectral vector lies and then exploiting this information during estimation. Building upon earlier works, the phase noise spectral estimate is obtained by minimizing a homogeneous quadratic cost function and the channel estimate in turns depends on the obtained phase noise estimate. We show that, at infinite signal-to-noise ratio, the true phase noise spectral estimate lies in the null space of the matrix associated with the cost function. We utilize this knowledge by imposing constraints that adhere to this null space when minimizing the cost function. In addition, we also propose constraints based on knowledge of the covariance matrix of the phase noise process. Through simulations, we demonstrate lower phase noise mean-square error (MSE) and consequently lower channel MSE when incorporating the subspace information.

Index Terms**—** Phase Noise, OFDM, Channel Estimation, Null Space, Optimization

1. INTRODUCTION

There are three major principal operations performed in any radio transceiver: frequency mixing, power amplification and analog-to-digital (AD) or digital-to-analog (DA) conversion. Frequency mixers are associated with *undesired phase noise* which is random perturbations in the phase of the carrier signal (generated by a local oscillator) that is used to transmit the information bearing signal. It arises due to component imperfections that make the oscillator circuitry [1–3]. In this paper, we consider an orthogonal frequency division multiplexing (OFDM) radio link impaired by only phase noise. With respect to OFDM, phase noise destroys the orthogonality between subcarriers thereby resulting in inter-carrier interference (ICI) and eventually causing severe performance degradation when left untreated. Some recent studies on the effects of phase noise can be found in [4–8]. A comprehensive overview of the subject can be found in [9]. This recognition of performance loss has spawned a plethora of estimation and compensation algorithms that revert the effects of phase noise e.g., [10–15] and references therein.

In this paper, we improve the phase noise estimation scheme proposed in [13] that is used for obtaining a channel estimate. Specifically, using a full-pilot OFDM symbol, a least-squares channel estimator is derived which depends on knowledge of the phase noise and, hence, phase noise must be estimated. Such an estimate is obtained by minimizing a homogeneous quadratic cost function subject to a linear constraint. By this approach of channel estimation, one can see that the channel mean-square error (MSE) is in direct correspondence with the phase noise MSE. The approach in [13] does not indicate or highlight as to what constraints to impose. When restricted to linear constraints, one can think of many such constraints, however, there is lack of clarity as to which one would yield the best performance. The linear constraint proposed in [13] is applicable only for small phase noise levels while poor phase noise estimates will be obtained for moderate-to-high phase noise levels.

In this work, we investigate the nature of the matrix characterizing the cost function. The purpose is to determine in which space the phase noise spectral vector lies and this information will provide an indication for designing a good constraint. Specifically, at infinite signal-to-noise ratio, we show that the phase noise spectral vector lies in the null-space of the matrix associated with the cost function. We can exploit this information, for obtaining a phase noise estimate, by restricting the search space to this particular subspace. From a practical perspective, such an approach is applicable also at medium-to-high signal-to-noise ratios. On the other hand, if the type of phase noise process is known then we can exploit its second-order statistical information. Specifically, the constraint set corresponds to the subspace spanned by the eigenvectors of the covariance matrix of the phase noise process.

2. SYSTEM MODEL

We consider an OFDM radio link wherein a (column) vector $\mathbf{s}$ of information symbols $\{s_j\}_{j=0}^{N_c-1}$ is transmitted using N_c orthogonal subcarriers [16]. In this paper, we consider only the effect of receiver phase noise while assuming a high-fidelity oscillator at the transmitter. Such a scenario can be, for example, downlink transmission where the base station is the transmitter while the receiver is a mobile terminal. Assuming sufficient timing synchronization, the received symbol vector $\mathbf{r}$, with elements $\{r_j\}_{j=0}^{N_c-1}$, is given by

$$\mathbf{r} = \mathbf{VHs} + \mathbf{n}, \tag{1}$$

where $\mathbf{H}$ is a diagonal matrix composed of elements $\{H_i\}_{i=0}^{N_c-1}$ which are the discrete Fourier transform (DFT) of the channel while the vector $\mathbf{n}$ is white Gaussian noise with diagonal covariance matrix whose diagonal values are equal to σ_n^2. The unitary matrix $\mathbf{V}$ is *column-wise circulant* with the column vector $\boldsymbol{\delta}$ whose elements are given by

$$\delta_k = \sum_{n=0}^{N_c-1} \frac{e^{\jmath\theta[n]}}{N_c} e^{-\jmath(2\pi kn)/N_c}, k = 0, 1 \ldots, N_c - 1, \tag{2}$$

where $e^{\jmath\theta[n]}$ is the complex exponential of the phase noise $\theta[n]$. The kth column of $\mathbf{V}$ is obtained by circularly shifting, $\boldsymbol{\delta}$, $k-1$ times to the bottom. Using (2), r_j can be expressed in terms of s_j as

$$r_j = \Big(\delta_0 H_j\Big) s_j + \sum_{k=0, k\neq j}^{N_c-1} \Big(\delta_{k-j} H_k\Big) s_k + n_j \tag{3}$$

When there is no phase noise, we have $\delta_0 = 1$ and $\delta_k = 0,\ k \neq 0$. From (3), we see that phase noise introduces two undesired quantities: One is rotation of the desired symbol s_j by $\delta_0 H_j$ which is known as the common phase error (CPE), and the second is to cause interference from other subcarriers, also known as inter-carrier interference (ICI), which is given by the second term in (3). In the ideal case of knowing $\boldsymbol{\delta}$, we could form the matrix $\mathbf{V}$ and perform $\mathbf{V}^\dagger \mathbf{r}$ to undo the effect of phase noise. But, in practice, we do not have this knowledge and, hence, it needs to be estimated from $\mathbf{r}$.

3. CHANNEL AND PHASE NOISE ESTIMATION

We build upon the channel and phase noise estimation method proposed in [13]. We first briefly summarize the approach after which we analyze the cost function that is minimized for obtaining a phase noise estimate.

The goal is to estimate the channel. For this purpose, a full-pilot OFDM symbol is used i.e., we have full knowledge of the vector $\mathbf{s}$ in (1). Let $\mathbf{h}$ denote the time-domain channel vector whose elements are $\{h[n]\}_{n=0}^{L-1}$ where L denotes the number of channel taps. First, we derive the least-squares estimate of the discrete-time domain channel $\mathbf{h}$. Rewriting (1) in terms of $\mathbf{h}$, we have $\mathbf{r} = \mathbf{V}\mathbf{S}\mathbf{F}_{\rm t}\mathbf{h} + \mathbf{n}$, where $\mathbf{S}$ is a diagonal matrix with elements s_j and the $N_{\rm c} \times L$ matrix $\mathbf{F}_{\rm t}$ is the truncated DFT matrix obtained from the $N_{\rm c} \times N_{\rm c}$ DFT matrix $\mathbf{F}$. The least-squares estimate of $\mathbf{h}$ follows by minimizing $||\mathbf{r} - \mathbf{V}\mathbf{S}\mathbf{F}_{\rm t}\hat{\mathbf{h}}||^2$ w.r.t. $\hat{\mathbf{h}}$ and is given by

$$\begin{aligned}\hat{\mathbf{h}} &= \left(\mathbf{F}_{\rm t}^\dagger \mathbf{S}^\dagger \mathbf{V}^\dagger \mathbf{V}\mathbf{S}\mathbf{F}_{\rm t}\right)^{-1} \mathbf{F}_{\rm t}^\dagger \mathbf{S}^\dagger \mathbf{V}^\dagger \mathbf{r} \\ &= \left(\mathbf{F}_{\rm t}^\dagger \mathbf{S}^\dagger \mathbf{S}\mathbf{F}_{\rm t}\right)^{-1} \mathbf{F}_{\rm t}^\dagger \mathbf{S}^\dagger \hat{\mathbf{V}}^\dagger \mathbf{r}, \end{aligned} \tag{4}$$

where we used the fact that $\mathbf{V}^\dagger\mathbf{V} = \mathbf{I}_{N_{\rm c}}$ with $\mathbf{I}_{N_{\rm c}}$ denoting the $N_{\rm c} \times N_{\rm c}$ identity matrix. We have also replaced $\mathbf{V}$ by $\hat{\mathbf{V}}$ which is our estimate of $\mathbf{V}$ as we do not know $\mathbf{V}$. Such an estimate can be obtained by substituting (4) back into $||\mathbf{r} - \hat{\mathbf{V}}\mathbf{S}\mathbf{F}_{\rm t}\hat{\mathbf{h}}||^2$. Expanding $||\mathbf{r} - \hat{\mathbf{V}}\mathbf{S}\mathbf{F}_{\rm t}\hat{\mathbf{h}}||^2$ while writing $\mathbf{B} = \mathbf{F}_{\rm t}^\dagger\mathbf{S}^\dagger\mathbf{S}\mathbf{F}_{\rm t}$ and $\mathbf{P}_{\rm r} = \mathbf{S}\mathbf{F}_{\rm t}\mathbf{B}^{-1}\mathbf{F}_{\rm t}^\dagger\mathbf{S}^\dagger$ (which is an orthogonal projection matrix onto an L-dimensional space), we have

$$||\mathbf{r} - \hat{\mathbf{V}}\mathbf{S}\mathbf{F}_{\rm t}\hat{\mathbf{h}}||^2 = \mathbf{r}^\dagger\mathbf{r} - \mathbf{r}^\dagger\hat{\mathbf{V}}\mathbf{P}_{\rm r}\hat{\mathbf{V}}^\dagger\mathbf{r} = \hat{\boldsymbol{\delta}}^\dagger\mathbf{M}\hat{\boldsymbol{\delta}} \tag{5}$$

where $\mathbf{M} = \left(\mathbf{R}^\dagger\mathbf{R} - \mathbf{R}^\dagger\mathbf{P}_{\rm r}\mathbf{R}\right)^{\rm T}$ is Hermitian and $\mathbf{R}$ is column-wise circulant matrix with column vector $\mathbf{r}$. The vector $\hat{\boldsymbol{\delta}}$ denotes our estimate of $\boldsymbol{\delta}$. In deriving (5), we enforced $\hat{\mathbf{V}}$ to be circulant (with column vector $\hat{\boldsymbol{\delta}}$) and unitary i.e., $\hat{\mathbf{V}}^\dagger\hat{\mathbf{V}} = \mathbf{I}_{N_{\rm c}}$ since we know the fact that $\mathbf{V}$ is a unitary column-wise circulant with $\boldsymbol{\delta}$. This simplifies the expression into a nice homogeneous quadratic cost function as seen in (5). A phase noise estimate can be obtained by minimizing (5) and, since it is homogeneous, the minimizer is the null vector of zeros. Since the actual vector $\boldsymbol{\delta}$ is non-zero, we need to impose a constraint when minimizing (5).

3.1. Nature of the $\mathbf{M}$ matrix

In this part, we show that, at infinite SNR, $\boldsymbol{\delta}$ lies in the null space of $\mathbf{M}$. Thus, if we restrict our search to this subspace by choosing constraints that correspond to this space, we will obtain good phase noise estimates and consequently good channel estimates. In practice such an approach is applicable also at high SNRs. We prove our main result in Proposition 1 but let us first make the following observations:

1. Consider the vector $\mathbf{w} = \mathbf{S}\mathbf{F}_{\rm t}\mathbf{h}$. Since the diagonal $\mathbf{S}$ matrix is of full rank and $\mathbf{F}_{\rm t}$ has full column rank of L columns, the vector $\mathbf{w}$ lies in an L-dimensional subspace. Also, the matrix $\mathbf{P}_{\rm r}$ is an orthogonal projection matrix onto the space spanned by the columns of $\mathbf{S}\mathbf{F}_{\rm t}$. This implies
$$(\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r})\mathbf{w} = \mathbf{0} \Leftrightarrow (\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r}^{\rm T})\mathbf{w}^* = \mathbf{0} \tag{6}$$
where $\mathbf{w}^*$ is the conjugate of $\mathbf{w}$ and the equivalence follows since $\mathbf{P}_{\rm r} = \mathbf{P}_{\rm r}^\dagger$.
2. The matrix $\mathbf{M} = \mathbf{R}^{\rm T}\mathbf{R}^* - \mathbf{R}^{\rm T}\mathbf{P}_{\rm r}^{\rm T}\mathbf{R}^* = \mathbf{R}^{\rm T}\left(\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r}^{\rm T}\right)\mathbf{R}^*$, where $\mathbf{R}^*$ denotes conjugate of $\mathbf{R}$, can be interpreted as the correlation between $\mathbf{R}^*$ and its projection onto the space defined by $\left(\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r}^{\rm T}\right)$. The structure of $\mathbf{R}^*$ is given by
$$\mathbf{R}^* = \begin{pmatrix} \mathbf{r}^\dagger \\ \mathbf{r}^\dagger\mathbf{P}_1 \\ \vdots \\ \mathbf{r}^\dagger\mathbf{P}_{N_{\rm c}-1} \end{pmatrix} \tag{7}$$
where $\mathbf{P}_l$ denotes the $N_{\rm c} \times N_{\rm c}$ permutation matrix and is given by $\mathbf{P}_l = (\mathbf{P}_1)^l$. The first column of $\mathbf{P}_1$ is given by the $N_{\rm c} \times 1$ vector $[0, 1, 0, \ldots, 0]^{\rm T}$ and the jth column is obtained by circularly shifting the vector $j-1$ times to the bottom.

Proposition 1. *Denote the null space of* $\mathbf{M}$ *by* $\mathcal{N}(\mathbf{M})$. *Then at infinite SNR,* $\boldsymbol{\delta} \in \mathcal{N}(\mathbf{M})$.

Proof. We need to show $\mathbf{M}\boldsymbol{\delta} = \mathbf{0}$. At infinite SNR, from (1), we have $\mathbf{r} = \mathbf{V}\mathbf{w}$ and, after substituting it in $\mathbf{R}^*$, we have

$$\mathbf{R}^*\boldsymbol{\delta} = \begin{pmatrix} \mathbf{w}^\dagger(\mathbf{V}^\dagger\boldsymbol{\delta}) \\ \mathbf{w}^\dagger(\mathbf{V}^\dagger\mathbf{P}_1\boldsymbol{\delta}) \\ \vdots \\ \mathbf{w}^\dagger(\mathbf{V}^\dagger\mathbf{P}_{N_{\rm c}-1}\boldsymbol{\delta}) \end{pmatrix} = \mathbf{w}^* \tag{8}$$

since the $N_{\rm c} \times 1$ vector $\mathbf{V}^\dagger\mathbf{P}_l\boldsymbol{\delta} = [0, 0, \ldots, 1, 0, \ldots, 0]^{\rm T}$ where the value of one occurs in the lth row. This results because $\mathbf{V}$ is unitary circulant with $\boldsymbol{\delta}$. Thus, $\mathbf{M}\boldsymbol{\delta} = \mathbf{R}^{\rm T}\left(\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r}^{\rm T}\right)\mathbf{R}^*\boldsymbol{\delta} = \mathbf{R}^{\rm T}\left(\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r}^{\rm T}\right)\mathbf{w}^* = \mathbf{0}$ after using (6). □

We conclude this section with a note on the dimensionality of $\mathcal{N}(\mathbf{M})$ as it indicates in *how big a space* the vector $\boldsymbol{\delta}$ lies. From (7), we can see that, in general, $\mathbf{R}^*$ is a full-rank matrix and, since, $\left(\mathbf{I}_{N_{\rm c}} - \mathbf{P}_{\rm r}^{\rm T}\right)$ has rank $N_{\rm c} - L$, we must have the rank of $\mathbf{M}$ also equal to $N_{\rm c} - L$ and, hence, the dimensionality of $\mathcal{N}(\mathbf{M})$ is equal to L which is the number of channel taps.

4. SUBSPACE-BASED PHASE NOISE MINIMIZATION

In this section, we present some optimization problems that exploit information on the space in which $\boldsymbol{\delta}$ lies. In general, the computational complexity in minimizing (5) is proportional to the dimension of $\hat{\boldsymbol{\delta}}$ which is equal to $N_{\rm c}$. In practical systems, $N_{\rm c}$ can be well over a few thousand and, thus, the complexity in solving (5) can be quite high. A way around this problem is to exploit the fact that most of the power in the vector $\boldsymbol{\delta}$ is confined to only a few low-frequency components because, in practice, the oscillators are designed with tolerable phase noise levels. Utilizing this information, our model of $\hat{\boldsymbol{\delta}}$ is given by

$$\hat{\boldsymbol{\delta}} = \begin{pmatrix} \mathbf{I}_{\frac{N}{2}\times\frac{N}{2}} & \mathbf{0}_{\frac{N}{2}\times\frac{N}{2}} \\ \mathbf{0}_{(N_c-N)\times\frac{N}{2}} & \mathbf{0}_{(N_c-N)\times\frac{N}{2}} \\ \mathbf{0}_{\frac{N}{2}\times\frac{N}{2}} & \mathbf{I}_{\frac{N}{2}\times\frac{N}{2}} \end{pmatrix} \underline{\hat{\boldsymbol{\delta}}} = \mathbf{L}\underline{\hat{\boldsymbol{\delta}}} \tag{9}$$

where $\underline{\hat{\boldsymbol{\delta}}}$ comprises of the $N < N_c$ non-zero components of $\hat{\boldsymbol{\delta}}$ while the rest are set to zero. Note that we estimate the top and bottom part of $\hat{\boldsymbol{\delta}}$ which corresponds to positive and negative frequencies centered around zero. We assume N to be even without any loss in generality. Substituting (9) in (5), the cost function to be minimized is given by

$$C(\underline{\hat{\boldsymbol{\delta}}}) = \underline{\hat{\boldsymbol{\delta}}}^\dagger \tilde{\mathbf{M}} \underline{\hat{\boldsymbol{\delta}}} \tag{10}$$

where $\tilde{\mathbf{M}} = \mathbf{L}^\dagger \mathbf{M} \mathbf{L}$. In the remainder of this section, we present some phase noise optimization problems and conclude with a summary of the optimization problem considered in [13] for the purpose of comparison.

4.1. NsPM: Nullspace-based Phase Noise Minimization

We would like to minimize (10) while at the same time incorporate the knowledge that $\boldsymbol{\delta} \in \mathcal{N}(\mathbf{M})$. From (9), we have $\hat{\boldsymbol{\delta}} = \mathbf{L}\underline{\hat{\boldsymbol{\delta}}}$ implying $\underline{\hat{\boldsymbol{\delta}}} = \mathbf{L}^\dagger\hat{\boldsymbol{\delta}}$ where we used the fact that $\mathbf{L}^\dagger\mathbf{L} = \mathbf{I}_N$. Let $\mathbf{N}$ denote the matrix whose columns span $\mathcal{N}(\mathbf{M})$. Thus, enforcing the constraint that $\hat{\boldsymbol{\delta}} \in \mathcal{N}(\mathbf{M})$ implies enforcing $\underline{\hat{\boldsymbol{\delta}}} \in \mathrm{span}(\mathbf{L}^\dagger\mathbf{N})$ where $\mathrm{span}(\mathbf{X})$ denotes span of the columns of the matrix $\mathbf{X}$. To put it another way, we essentially map $\mathcal{N}(\mathbf{M})$ to $\mathrm{span}(\mathbf{L}^\dagger\mathbf{N})$. Based on this rationale, we propose the following optimization problem:

$$\begin{aligned} &\text{Minimize } \ C(\underline{\hat{\boldsymbol{\delta}}}) = \underline{\hat{\boldsymbol{\delta}}}^\dagger \tilde{\mathbf{M}} \underline{\hat{\boldsymbol{\delta}}}, \\ &\qquad \text{s.t } \ \underline{\hat{\boldsymbol{\delta}}}^\dagger \underline{\hat{\boldsymbol{\delta}}} = 1, \ \underline{\hat{\boldsymbol{\delta}}} \in \mathrm{span}(\mathbf{L}^\dagger\mathbf{N}) \end{aligned} \tag{11}$$

In (11), we have enforced a unit-norm constraint on $\underline{\hat{\boldsymbol{\delta}}}$. Using Parseval's theorem, it can be easily shown that $\boldsymbol{\delta}$ has unit-norm [17] and since we assume most of the power is in $\underline{\hat{\boldsymbol{\delta}}}$ then the unit-norm constraint in (11) is reasonable. Writing $\underline{\hat{\boldsymbol{\delta}}} = \mathbf{L}^\dagger\mathbf{N}\boldsymbol{\alpha}$, the above problem expressed in terms of $\boldsymbol{\alpha}$ is given by

$$\begin{aligned} &\text{Minimize } \ C(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\dagger \left(\mathbf{N}^\dagger\mathbf{L}\tilde{\mathbf{M}}\mathbf{L}^\dagger\mathbf{N}\right)\boldsymbol{\alpha} \\ &\qquad \text{s.t } \ \boldsymbol{\alpha}^\dagger \left(\mathbf{N}^\dagger\mathbf{L}\mathbf{L}^\dagger\mathbf{N}\right)\boldsymbol{\alpha} = 1 \end{aligned} \tag{12}$$

The optimization problem (12) can be solved as follows: Let $\mathbf{N}^\dagger\mathbf{L}\mathbf{L}^\dagger\mathbf{N} = \mathbf{W}\mathbf{W}^\dagger$ be the Cholesky decomposition. Then writing $\boldsymbol{\gamma} = \mathbf{W}^\dagger\boldsymbol{\alpha}$, the minimization problem simplifies to

$$\text{Minimize } \ C(\boldsymbol{\gamma}) = \boldsymbol{\gamma}^\dagger\mathbf{Q}\boldsymbol{\gamma} \ \text{ s.t } \ \boldsymbol{\gamma}^\dagger\boldsymbol{\gamma} = 1 \tag{13}$$

where $\mathbf{Q} = (\mathbf{W}^{-1})\mathbf{N}^\dagger\mathbf{L}\tilde{\mathbf{M}}\mathbf{L}^\dagger\mathbf{N}(\mathbf{W}^\dagger)^{-1}$. The minimum value for the above problem is equal to the smallest eigenvalue of $\mathbf{Q}$ and if the eigenvalues are distinct then the minimizer corresponds to the eigenvector associated with the smallest eigenvalue.

4.2. CvPM: Covariance-based Phase Noise Minimization

If we had prior knowledge on the type of phase noise process then we can model our constraints based on the covariance matrix of the phase noise process. The motivation is as follows: For any random vector, the eigenvectors of its covariance matrix determine the space in which the vector will always be drawn from [18, Appendix C]. With this fact in mind, the optimization problem is framed as

$$\text{Minimize } C(\underline{\hat{\boldsymbol{\delta}}}) = \underline{\hat{\boldsymbol{\delta}}}^\dagger \tilde{\mathbf{M}} \underline{\hat{\boldsymbol{\delta}}}, \ \ \underline{\hat{\boldsymbol{\delta}}}^\dagger \underline{\hat{\boldsymbol{\delta}}} = 1, \ \underline{\hat{\boldsymbol{\delta}}} \in \mathrm{span}(\mathbf{U}) \tag{14}$$

where $N \times N$ unitary matrix $\mathbf{U}$ contains the eigenvectors of $\mathbf{L}^\dagger\mathbf{C}\mathbf{L}$ with $\mathbf{C}$ denoting the covariance matrix of $\boldsymbol{\delta}$. Closed-form expressions for $\mathbf{C}$ of a Wiener phase noise process as well as a PLL-type phase noise process can be found in [15]. Making a variable change by writing $\underline{\hat{\boldsymbol{\delta}}} = \mathbf{U}\alpha$ and noting that $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}_N$, we have

$$\text{Minimize } C(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\dagger(\mathbf{U}^\dagger\tilde{\mathbf{M}}\mathbf{U})\boldsymbol{\alpha}, \ \text{ s.t } \ \boldsymbol{\alpha}^\dagger\boldsymbol{\alpha} = 1 \tag{15}$$

The minimizer is equal to the eigenvector associated with the smallest eigenvalue of $(\mathbf{U}^\dagger\tilde{\mathbf{M}}\mathbf{U})$.

4.3. CoPM: Correlation-based Phase Noise Minimization [13]

We now briefly summarize the optimization problem considered in [13] for the purpose of comparison. Specifically, the optimization problem is as follows:

$$\text{Minimize } C(\underline{\hat{\boldsymbol{\delta}}}) = \underline{\hat{\boldsymbol{\delta}}}^\dagger \tilde{\mathbf{M}} \underline{\hat{\boldsymbol{\delta}}} \ \ \text{s.t } \ \underline{\hat{\boldsymbol{\delta}}}^\dagger \mathbf{e} = 1, \tag{16}$$

where $\mathbf{e} = [1\ 0, \ldots 0]^{\mathrm{T}}$ is a $N \times 1$ column vector. The constraint in (16) can be interpreted as follows: We would like our minimizer to have maximum correlation with the vector $\mathbf{e}$. In the absence of phase noise, the actual phase noise spectral vector $\boldsymbol{\delta} = \mathbf{L}\mathbf{e}$. For very small phase noise levels, we can expect $\boldsymbol{\delta}$ to be very close to $\mathbf{L}\mathbf{e}$ and, thus, the constraint in (16) is applicable in this case. However, as phase noise levels get larger, the correlation between $\boldsymbol{\delta}$ and $\mathbf{L}\mathbf{e}$ gets weaker and thereby, using the constraint in (16) will yield poor phase noise estimates. The minimizer to (16) can be easily derived and is given by $\underline{\hat{\boldsymbol{\delta}}} = \frac{\tilde{\mathbf{M}}^{-1}\mathbf{e}}{\mathbf{e}^\dagger\tilde{\mathbf{M}}^{-1}\mathbf{e}}$.

5. NUMERICAL RESULTS

We now demonstrate simulation results incorporating the proposed phase noise estimation schemes. The performance of the schemes are evaluated by the mean-square error (MSE) metric. We compute the error between our estimate $\hat{\boldsymbol{\delta}}$ and the true value given by $\boldsymbol{\delta}$. The channel estimate is given by (4) which depends on $\hat{\boldsymbol{\delta}}$ through the matrix $\hat{\mathbf{V}}$. Thus, we see that the error in our channel estimate is in direct correspondence with the phase noise estimation error. The errors are evaluated for different realizations of the OFDM preamble symbol after which they are averaged to obtain the MSE. The system parameters used in our simulations are as follows: The number of subcarriers $N_c = 512$, subcarrier spacing $f_{\mathrm{sub}} = 15$ kHz and bandwidth is equal to 7.7 MHz. For phase noise estimation, we estimate a total of $N = 7$ components of $\boldsymbol{\delta}$ while the rest are set to a value of zero. The symbol constellation is 16-QAM. The channel is Rayleigh fading with exponential power delay profile and number of taps (L) set to four i.e., $L = 4$. The coherence bandwidth is set to 800 kHz. Phase noise process used for the simulations is of the Wiener type. We denote by β as the 3-dB bandwidth of the oscillator power spectral density (PSD). A large 3-dB bandwidth implies higher phase noise levels and vice-versa. With respect to OFDM, the ratio $\rho = \frac{\beta}{f_{\mathrm{sub}}}$ determines the ICI power level and can be interpreted as the normalized phase noise 3-dB bandwidth. A large 3-dB bandwidth oscillator PSD can be compensated by having a large subcarrier spacing.

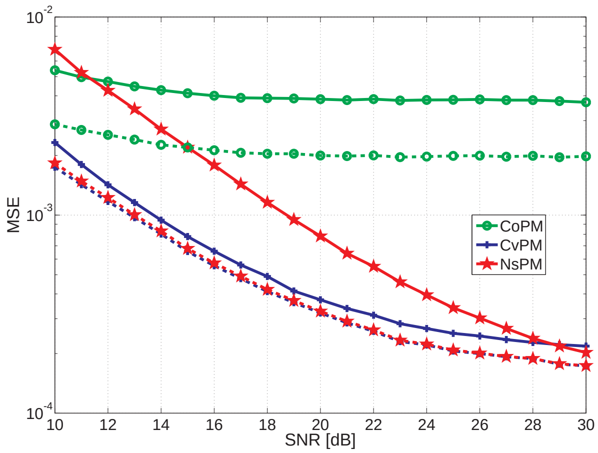

Fig. 1. Phase noise and channel MSE as a function of the signal-to-noise ratio. The phase noise MSE curves are shown by the solid lines while the dashed lines are the channel MSE curves. The value of $\rho = 0.02$.

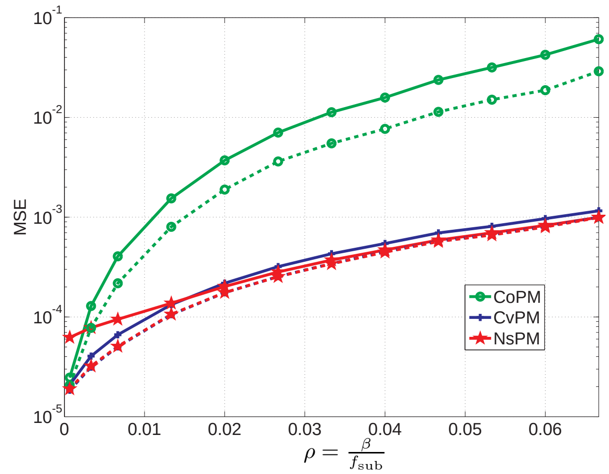

Fig. 2. Phase noise and channel MSE as a function of the ratio $\rho = \frac{\beta}{f_{\mathrm{sub}}}$. The phase noise MSE curves are shown by the solid lines while the dashed lines are the channel MSE curves. The SNR is set to 30 dB.

In Fig. 1, we plot the phase noise and channel MSE as a function of the SNR. The phase noise estimation methods shown in the figure are the proposed NsPM and CvPM minimization schemes of Section 4 and we compare them with the CoPM of [13]. We see that the NsPM and CvPM schemes provide superior MSE performance compared to the CoPM scheme. As expected, the CvPM scheme performs the best since it exploits statistical information about the phase noise process. However, the method works well only when a priori information about the type of phase noise process is available. In that respect, the NsPM does not require any knowledge of the type of phase noise process and only exploits information on which subspace the actual phase noise spectral vector lies. Theoretically at infinite SNR, from Proposition 1, this space corresponds to the null subspace of the matrix $\mathbf{M}$. Thus, at low SNR, we can expect Proposition 1 to not hold which can be indirectly inferred from the figure at low SNR points where the MSE of NsPM is higher compared to CvPM. At high SNRs, we see their performance is similar thereby verifying the fact that Proposition 1 holds well at these SNR values.

A drawback with the CoPM scheme of [13] is that the estimate is required to have maximum correlation with the unit-vector $\mathbf{e} = [1\ 0, \ldots 0]^{\mathrm{T}}$. This holds well when the phase noise level is low or, in the context of OFDM, when ρ is low. For large values of ρ, we can expect $\boldsymbol{\delta}$ to deviate well away from the unit-vector and, thus, the constraint in CoPM is not in line with the actual phase noise spectral realization. In Fig. 2, we investigate this behavior where phase noise and channel MSE are plotted as a function of the normalized 3-dB bandwidth. The figure clearly shows high MSE values for CoPM at high values of ρ while the performance of all schemes are similar for values of ρ close to zero.

6. CONCLUSION

In this paper, we provide two improved phase noise estimators that exploit information about the subspace in which the phase noise spectral vector lies. The first approach obtains this subspace information by construction of a suitable positive-definite Hermitian matrix while the second one utilizes information from the covariance matrix. The phase noise estimation problem is posed as an optimization problem where a homogeneous quadratic cost function is minimized. We show that, at infinite signal-to-noise ratio, the desired phase noise spectral vector lies in the null space of the matrix associated with the cost function. We exploit this information by imposing linear constraints when minimizing the cost function. The linear constraints correspond to this phase noise spectral subspace. The second approach exploits the subspace information available in the covariance matrix of the phase noise process. The proposed subspace based methods provide phase noise and channel estimates that result in lower mean-square error levels.

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