OPTIMIZATION METHODS FOR SEQUENCE DESIGN WITH LOW AUTOCORRELATION SIDELOBES

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ABSTRACT

Unimodular sequences with low autocorrelations are desired in many applications, especially in the area of radar and code-division multiple access (CDMA). In this paper, we propose a new algorithm to design unimodular sequences with low integrated sidelobe level (ISL), which is a widely used measure of the goodness of a sequence's correlation property. The algorithm falls into the general framework of majorization-minimization (MM) algorithms and thus shares the monotonic property of such algorithms. In addition, the algorithm can be implemented via fast Fourier transform (FFT) operations and thus is computationally efficient. Numerical experiments show that the proposed algorithm outperforms the state-of-the-art algorithm in terms of both the quality of designed sequences and the computational complexity.

***Index Terms*—** Unimodular sequences, integrated sidelobe level, autocorrelation, majorization-minimization.

1. INTRODUCTION

Since the 1950s, digital communications engineers have sought to identify sequences whose aperiodic autocorrelation sidelobes are collectively as low as possible according to some suitable measure of "goodness". Applications range from synchronization to code division multiple access (CDMA) systems and especially radar [1, 2]. Low autocorrelation can improve the detection performance of weak targets [3] in range compression radar and it is also desired for synchronization purposes in CDMA systems. Additionally, due to the limitations of sequence generation hardware components (such as the maximum signal amplitude clip of analog-to-digital converters and power amplifiers), it is desirable to transmit unimodular (i.e., constant modulus) sequences to maximize the transmitted power available in the system [4].

Owing to the practical importance and widespread applications of sequences with good correlation properties, in particular with low ISL values, a lot of effort has been devoted to identifying these sequences via either analytical construction methods or computational approaches in the literature. Some sequences with good correlation properties that can be constructed in closed-form have been proposed in the literature, such as the Frank sequence [5] and the Golomb sequence [6]. Computational approaches, such as exhaustive search [7], evolutionary algorithms [8], heuristic search [9] and stochastic optimization methods [10, 11], have also been suggested. However, these computational methods are generally not capable of designing long sequences, say of length $N \sim 10^3$ or larger, due to the increasing computational complexity. Recently, an algorithm named CAN (cyclic algorithm new) was proposed in [12] that can be used to produce unimodular sequences with low ISL of length $N \sim 10^6$ or even larger. But instead of minimizing the original ISL metric, the CAN algorithm tries to minimize another criterion which is stated to be "almost equivalent" to the ISL metric.

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In this paper, we develop a new algorithm named MISL (monotonic minimizer for integrated sidelobe level) that directly minimizes the ISL metric monotonically. MISL is derived via applying the general majorization-minimization (MM) method twice and admits a closed-form solution at every iteration. Similar to the CAN algorithm, the proposed MISL algorithm can also be implemented via fast Fourier transform (FFT) operations and is thus very efficient. But due to the nature of double majorization, the MISL algorithm may converge slow and we propose to apply an acceleration scheme to fix this issue. Numerical experiments show that compared with CAN the proposed MISL algorithm (with acceleration) can generate sequences with lower ISL and also provides a complexity saving.

Notation: $\mathbf{C}$ denotes the complex field. $\mathrm{Re}(\cdot)$ and $\arg(\cdot)$ denote the real part and the phase of a complex number, respectively. The superscripts $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ denote transpose, complex conjugate, and conjugate transpose, respectively. x_i denotes the i-th element of a vector $\mathbf{x}$. $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. $\mathrm{Diag}(\mathbf{x})$ is a diagonal matrix formed with $\mathbf{x}$ as its principal diagonal. $\mathrm{vec}(\mathbf{X})$ is a column vector consisting of all the columns of $\mathbf{X}$ stacked. $\mathbf{I}_n$ denotes an $n \times n$ identity matrix. $\mathbf{X} \succeq \mathbf{0}$ indicates that $\mathbf{X}$ is Hermitian positive semidefinite.

2. PROBLEM STATEMENT

Let $\{x_n \in \mathbf{C}\}_{n=1}^N$ be the complex unimodular (without loss of generality, we will assume the modulus to be 1) sequence to be designed and

$$r_k = \sum_{n=k+1}^{N} x_n x_{n-k}^* = r_{-k}^*,\ k = 0, \ldots, N-1 \tag{1}$$

be the autocorrelations of $\{x_n\}_{n=1}^N$. Then the integrated sidelobe level (ISL) is defined as

$$\mathrm{ISL} = \textstyle\sum_{k=1}^{N-1} |r_k|^2, \tag{2}$$

which is highly related to another important goodness measure: merit factor (MF), defined as the ratio of the central lobe energy to the total energy of all other lobes [13], i.e.,

$$\mathrm{MF} = \frac{|r_0|^2}{2\sum_{k=1}^{N-1} |r_k|^2} = \frac{N^2}{2\mathrm{ISL}}. \tag{3}$$

The problem of interest is the design of a complex unimodular sequence that minimizes the ISL metric, i.e.,

$$\begin{array}{ll} \underset{\{x_n\}_{n=1}^N}{\mathsf{minimize}} & \mathrm{ISL} \\ \mathsf{subject\ to} & |x_n| = 1,\ n = 1, \ldots, N. \end{array} \tag{4}$$

It has been shown in [12] that the ISL metric can be expressed in the frequency domain as

$$\mathrm{ISL} = \frac{1}{4N} \sum_{p=1}^{2N} \left[\left| \sum_{n=1}^{N} x_n e^{-j\omega_p (n-1)} \right|^2 - N \right]^2, \tag{5}$$

where $\omega_p = \frac{2\pi}{2N}(p-1),\ p = 1, \cdots, 2N$. By further defining

$$\begin{aligned} \mathbf{x} &= [x_1, \cdots, x_N]^T \\ \mathbf{a}_p &= \left[1, e^{j\omega_p}, \cdots, e^{j\omega_p(N-1)}\right]^T, \end{aligned}$$

the ISL minimization problem (4) can be rewritten as

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \sum_{p=1}^{2N} \left[\left| \mathbf{a}_p^H \mathbf{x} \right|^2 - N \right]^2 \\ \text{subject to} \quad & |x_n| = 1,\ n = 1, \ldots, N. \end{aligned} \tag{6}$$

Expanding the square in the objective yields

$$\sum_{p=1}^{2N} \left(\left(\mathbf{a}_p^H \mathbf{x}\mathbf{x}^H \mathbf{a}_p \right)^2 - 2N \left| \mathbf{a}_p^H \mathbf{x} \right|^2 + N^2 \right), \tag{7}$$

where the second term can be shown to be a constant (using Parseval's theorem), i.e., $\sum_{p=1}^{2N} \left| \mathbf{a}_p^H \mathbf{x} \right|^2 = 2N \|\mathbf{x}\|_2^2 = 2N^2$. Thus by ignoring the constant terms, the problem (6) can be simplified as

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \sum_{p=1}^{2N} \left(\mathbf{a}_p^H \mathbf{x}\mathbf{x}^H \mathbf{a}_p \right)^2 \\ \text{subject to} \quad & |x_n| = 1,\ n = 1, \ldots, N. \end{aligned} \tag{8}$$

The problem (8) (or (6)) is hard to tackle, due to the nonconvex unit-modulus constraints and also the quartic objective function. So, instead of dealing with the quartic ISL metric directly, Stoica et al. [12] proposed to solve the following simpler problem

$$\begin{aligned} \underset{\mathbf{x}, \{\psi_p\}_{p=1}^{2N}}{\text{minimize}} \quad & \sum_{p=1}^{2N} \left| \mathbf{a}_p^H \mathbf{x} - \sqrt{N} e^{j\psi_p} \right|^2 \\ \text{subject to} \quad & |x_n| = 1,\ n = 1, \ldots, N, \end{aligned} \tag{9}$$

whose objective is a quadratic function of $\mathbf{x}$. It was mentioned in [12] that the problem (9) is "almost equivalent" to the original problem (6) in some sense. An algorithm named CAN (cyclic algorithm new) was then derived by solving the problem (9) with respect to $\mathbf{x}$ and $\{\psi_p\}$ alternately and it was shown numerically that CAN could generate sequences with good correlation properties. Moreover, CAN is easy to implement via FFT and thus can be used to design very long sequences. But as the authors also pointed out, the two problems are not equivalent and they have different local and global minima in general. We can expect that directly solving the original problem will probably lead to better performance.

In the next section, we will develop an algorithm that directly solves the problem (8), while at the same time is computationally as efficient as CAN.

3. ISL MINIMIZATION VIA MAJORIZATION-MINIMIZATION

The majorization-minimization (MM) method is an iterative approach to solve optimization problems that are too difficult to solve directly. Interested readers may refer to [14] and references therein for more details (recent generalizations include [15, 16]).

Let $f(\mathbf{x})$ denotes the objective function of the problem (8) and $\mathcal{X} \in \mathbf{C}^N$ be the constraint set. Instead of minimizing $f(\mathbf{x})$ directly, the MM approach optimizes a sequence of approximate objective functions that majorize $f(\mathbf{x})$. Formally, a function $u(\mathbf{x}, \mathbf{x}^{(k)})$ is said to majorize the function $f(\mathbf{x})$ at the point $\mathbf{x}^{(k)}$ over $\mathcal{X}$ provided

$$u(\mathbf{x}, \mathbf{x}^{(k)}) \quad \geq \quad f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}, \tag{10}$$

$$u(\mathbf{x}^{(k)}, \mathbf{x}^{(k)}) \quad = \quad f(\mathbf{x}^{(k)}). \tag{11}$$

In other words, the majorization function $u(\mathbf{x}, \mathbf{x}^{(k)})$ is an upper bound of $f(\mathbf{x})$ over $\mathcal{X}$ and coincides with $f(\mathbf{x})$ at $\mathbf{x}^{(k)}$. To tackle the problem (8) via MM, the key point is to find a majorization function of $f(\mathbf{x})$ at each iteration such that the majorized problem is easy to solve. For that purpose we first present a simple result that will be useful when constructing the majorization functions.

Lemma 1. *Let $\mathbf{L}$ be an $n \times n$ Hermitian matrix and $\mathbf{M}$ be another $n \times n$ Hermitian matrix such that $\mathbf{M} - \mathbf{L} \succeq \mathbf{0}$. Then for any point $\mathbf{x}_0 \in \mathbf{C}^n$, the quadratic function $\mathbf{x}^H \mathbf{L} \mathbf{x}$ is majorized by $\mathbf{x}^H \mathbf{M} \mathbf{x} + 2\text{Re}\left(\mathbf{x}^H (\mathbf{L} - \mathbf{M}) \mathbf{x}_0 \right) + \mathbf{x}_0^H (\mathbf{M} - \mathbf{L}) \mathbf{x}_0$ at $\mathbf{x}_0$.*

Proof. It is easy to check that the two functions are equal at point $\mathbf{x}_0$. Since $\mathbf{M} - \mathbf{L} \succeq \mathbf{0}$, we further have

$$\begin{aligned} & \mathbf{x}^H \mathbf{L} \mathbf{x} \\ =& \mathbf{x}_0^H \mathbf{L} \mathbf{x}_0 + 2\text{Re}\left((\mathbf{x} - \mathbf{x}_0)^H \mathbf{L} \mathbf{x}_0 \right) + (\mathbf{x} - \mathbf{x}_0)^H \mathbf{L} (\mathbf{x} - \mathbf{x}_0) \\ \leq& \mathbf{x}_0^H \mathbf{L} \mathbf{x}_0 + 2\text{Re}\left((\mathbf{x} - \mathbf{x}_0)^H \mathbf{L} \mathbf{x}_0 \right) + (\mathbf{x} - \mathbf{x}_0)^H \mathbf{M} (\mathbf{x} - \mathbf{x}_0) \\ =& \mathbf{x}^H \mathbf{M} \mathbf{x} + 2\text{Re}\left(\mathbf{x}^H (\mathbf{L} - \mathbf{M}) \mathbf{x}_0 \right) + \mathbf{x}_0^H (\mathbf{M} - \mathbf{L}) \mathbf{x}_0 \end{aligned}$$

for any $\mathbf{x} \in \mathbf{C}^n$. The proof is complete. □

The objective of the problem (8) is quartic with respect to $\mathbf{x}$. To construct a majorization function via Lemma 1, some reformulations are necessary. Let us define $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ and $\mathbf{A}_p = \mathbf{a}_p \mathbf{a}_p^H$, then the problem (8) can be rewritten as

$$\begin{aligned} \underset{\mathbf{x}, \mathbf{X}}{\text{minimize}} \quad & \sum_{p=1}^{2N} \text{Tr}(\mathbf{X}\mathbf{A}_p)^2 \\ \text{subject to} \quad & \mathbf{X} = \mathbf{x}\mathbf{x}^H \\ & |x_n| = 1,\ n = 1, \ldots, N. \end{aligned} \tag{12}$$

Since $\text{Tr}(\mathbf{X}\mathbf{A}_p) = \text{vec}(\mathbf{X})^H \text{vec}(\mathbf{A}_p)$, the objective in (12) is just

$$\text{vec}(\mathbf{X})^H \boldsymbol{\Phi} \text{vec}(\mathbf{X}), \tag{13}$$

where $\boldsymbol{\Phi} = \sum_{p=1}^{2N} \text{vec}(\mathbf{A}_p) \text{vec}(\mathbf{A}_p)^H$. We can see that the objective (13) is a quadratic function of $\mathbf{X}$ now. Given $\mathbf{X}^{(k)} = \mathbf{x}^{(k)} (\mathbf{x}^{(k)})^H$ at iteration k, by choosing $\mathbf{M} = \lambda_{\max}(\boldsymbol{\Phi}) \mathbf{I}$ in Lemma 1 we know that the objective (13) is majorized by the following function at $\mathbf{X}^{(k)}$:

$$\begin{aligned} & u_1(\mathbf{X}, \mathbf{X}^{(k)}) \\ =& \lambda_{\max}(\boldsymbol{\Phi}) \text{vec}(\mathbf{X})^H \text{vec}(\mathbf{X}) \\ & + 2\text{Re}\left(\text{vec}(\mathbf{X})^H (\boldsymbol{\Phi} - \lambda_{\max}(\boldsymbol{\Phi}) \mathbf{I}) \text{vec}(\mathbf{X}^{(k)}) \right) \\ & + \text{vec}(\mathbf{X}^{(k)})^H (\lambda_{\max}(\boldsymbol{\Phi}) \mathbf{I} - \boldsymbol{\Phi}) \text{vec}(\mathbf{X}^{(k)}), \end{aligned} \tag{14}$$

where $\lambda_{\max}(\boldsymbol{\Phi})$ is the maximum eigenvalue of $\boldsymbol{\Phi}$ and can be shown to be $2N^2$. Since $\text{vec}(\mathbf{X})^H \text{vec}(\mathbf{X}) = (\mathbf{x}^H \mathbf{x})^2 = N^2$, the first term of (14) is just a constant. After ignoring the constant terms in (14), the majorized problem of (12) is given by

$$\begin{aligned} \underset{\mathbf{x}, \mathbf{X}}{\text{minimize}} \quad & \text{Re}\left(\text{vec}(\mathbf{X})^H \left(\boldsymbol{\Phi} - 2N^2 \mathbf{I} \right) \text{vec}(\mathbf{X}^{(k)}) \right) \\ \text{subject to} \quad & \mathbf{X} = \mathbf{x}\mathbf{x}^H \\ & |x_n| = 1,\ n = 1, \ldots, N, \end{aligned} \tag{15}$$

which can be rewritten as

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \sum_{p=1}^{2N}\left|\mathbf{a}_p^H\mathbf{x}^{(k)}\right|^2\mathbf{x}^H\mathbf{a}_p\mathbf{a}_p^H\mathbf{x}-2N^2\left|\mathbf{x}^H\mathbf{x}^{(k)}\right|^2\\ \text{subject to} & |x_n|=1,\ n=1,\ldots,N.\end{array}\tag{16}$$

By defining $\mathbf{A}=[\mathbf{a}_1,\ldots,\mathbf{a}_{2N}]$, we can write problem (16) more compactly as

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{x}^H\left(\mathbf{A}\mathrm{Diag}(\mathbf{p}^{(k)})\mathbf{A}^H-2N^2\mathbf{x}^{(k)}(\mathbf{x}^{(k)})^H\right)\mathbf{x}\\ \text{subject to} & |x_n|=1,\ n=1,\ldots,N,\end{array}\tag{17}$$

where $\mathbf{p}^{(k)}=\left[\left|\mathbf{a}_1^H\mathbf{x}^{(k)}\right|^2,\ldots,\left|\mathbf{a}_{2N}^H\mathbf{x}^{(k)}\right|^2\right]^T$. We can clearly see that the objective function in (17) is quadratic in $\mathbf{x}$ and by choosing $\mathbf{M}=p_{\max}^{(k)}\mathbf{A}\mathbf{A}^H$ in Lemma 1 it is majorized by the following function at $\mathbf{x}^{(k)}$:

$$\begin{aligned}&u_2(\mathbf{x},\mathbf{x}^{(k)})\\=&p_{\max}^{(k)}\mathbf{x}^H\mathbf{A}\mathbf{A}^H\mathbf{x}+2\mathrm{Re}\left(\mathbf{x}^H(\tilde{\mathbf{A}}-2N^2\mathbf{x}^{(k)}(\mathbf{x}^{(k)})^H)\mathbf{x}^{(k)}\right)\\&+(\mathbf{x}^{(k)})^H(2N^2\mathbf{x}^{(k)}(\mathbf{x}^{(k)})^H-\tilde{\mathbf{A}})\mathbf{x}^{(k)}\end{aligned}\tag{18}$$

where $\tilde{\mathbf{A}}=\mathbf{A}\left(\mathrm{Diag}(\mathbf{p}^{(k)})-p_{\max}^{(k)}\mathbf{I}\right)\mathbf{A}^H$ and $p_{\max}^{(k)}=\max_p\{p_p^{(k)}: p=1,\ldots,2N\}$. Since $\mathbf{x}^H\mathbf{A}\mathbf{A}^H\mathbf{x}=2N\mathbf{x}^H\mathbf{x}=2N^2$, the first term of (18) is a constant. By ignoring the constant terms in (18), we have the majorized problem of (17) at $\mathbf{x}^{(k)}$:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathrm{Re}\left(\mathbf{x}^H\left(\tilde{\mathbf{A}}-2N^2\mathbf{x}^{(k)}(\mathbf{x}^{(k)})^H\right)\mathbf{x}^{(k)}\right)\\ \text{subject to} & |x_n|=1,\ n=1,\ldots,N,\end{array}\tag{19}$$

which can be rewritten as

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \|\mathbf{x}-\mathbf{y}\|_2\\ \text{subject to} & |x_n|=1,\ n=1,\ldots,N,\end{array}\tag{20}$$

where

$$\begin{aligned}\mathbf{y}&=-\left(\tilde{\mathbf{A}}-2N^2\mathbf{x}^{(k)}\left(\mathbf{x}^{(k)}\right)^H\right)\mathbf{x}^{(k)}\\&=-\mathbf{A}\left(\mathrm{Diag}(\mathbf{p}^{(k)})-p_{\max}^{(k)}\mathbf{I}-N^2\mathbf{I}\right)\mathbf{A}^H\mathbf{x}^{(k)}.\end{aligned}\tag{21}$$

It is easy to see that the problem (20) has a closed form solution, which is given by

$$x_n=e^{j\arg(y_n)},\ n=1,\ldots,N.\tag{22}$$

Now we can summarize the overall algorithm and it is given in Algorithm 1. Note that although in the derivation we have majorized the objective twice, it can be viewed as directly majorizing the objective of the problem (8) at $\mathbf{x}^{(k)}$ over the constraint set by the following function:

$$u(\mathbf{x},\mathbf{x}^{(k)})=2u_2(\mathbf{x},\mathbf{x}^{(k)})+4N^4-\sum_{p=1}^{2N}\left|\mathbf{a}_p^H\mathbf{x}^{(k)}\right|^4,\tag{23}$$

where $u_2(\mathbf{x},\mathbf{x}^{(k)})$ is defined in (18). Thus, the algorithm preserves the monotonicity of the general majorization-minimization scheme and we call it MISL (Monotonic minimizer for Integrated Sidelobe Level).

From Algorithm 1, we can see that the per iteration computational complexity of MISL is dominated by two matrix vector multiplications involving $\mathbf{A}$. It is worth noting that they can be easily computed via FFT and IFFT operations. Thus the MISL algorithm is very efficient and can be used for the design of very long sequences.

Algorithm 1 MISL - Monotonic minimizer for Integrated Sidelobe Level.

Require: sequence length N
1: Set $k=0$, initialize $\mathbf{x}^{(0)}$.
2: **repeat**
3: $\quad p_p^{(k)}=\left|\mathbf{a}_p^H\mathbf{x}^{(k)}\right|^2$, $p=1,\ldots,2N$
4: $\quad p_{\max}^{(k)}=\max_p\{p_p^{(k)}: p=1,\ldots,2N\}$
5: $\quad \mathbf{y}=-\mathbf{A}\left(\mathrm{Diag}(\mathbf{p}^{(k)})-p_{\max}^{(k)}\mathbf{I}-N^2\mathbf{I}\right)\mathbf{A}^H\mathbf{x}^{(k)}$
6: $\quad x_n^{(k+1)}=e^{j\arg(y_n)}$, $n=1,\ldots,N$
7: $\quad k\leftarrow k+1$
8: **until** convergence

4. ACCELERATION VIA FIXED POINT THEORY

As described in the previous section, the derivation of MISL is based on majorization-minimization principle, and the nature of the majorization functions will dictate the convergence speed of the algorithm. Due to the double majorization scheme that we carried out in the derivation, the convergence of MISL seems to be slow, especially for large N. One option to fix this issue is to employ some acceleration schemes to accelerate the convergence of MISL, and there are various acceleration schemes available in the literature to accelerate MM algorithms.

In this section, we briefly introduce one such acceleration scheme and show how it can be used to accelerate MISL. It is the so called squared iterative method (SQUAREM), which was originally proposed in [17] to accelerate any EM algorithms. SQUAREM follows the idea of the Cauchy-Barzilai-Borwein (CBB) method [18], which combines the classical steepest descent method and the two-point step size gradient method [19], to solve the nonlinear fixed-point problem of EM. It only requires the EM updating scheme and can be readily implemented as an 'off-the-shelf' accelerator. Since MM is a generalization of EM and the update rule is also just a fixed-point iteration, SQUAREM can be readily applied to MM algorithms.

Let $\mathbf{F}_{\mathrm{MISL}}(\cdot)$ denote the nonlinear fixed-point iteration map of the MISL algorithm:

$$\mathbf{x}^{(k+1)}=\mathbf{F}_{\mathrm{MISL}}(\mathbf{x}^{(k)}),\tag{24}$$

whose form can be defined by the following equation:

$$\mathbf{x}^{(k+1)}=e^{j\arg\left(-\mathbf{A}\left(\mathrm{Diag}(\mathbf{p}^{(k)})-p_{\max}^{(k)}\mathbf{I}-N^2\mathbf{I}\right)\mathbf{A}^H\mathbf{x}^{(k)}\right)},\tag{25}$$

where $\mathbf{p}^{(k)}$ and $p_{\max}^{(k)}$ are the same as in Algorithm 1 and functions $e^{(\cdot)}$ and $\arg(\cdot)$ are applied element-wise to the vectors. With this, the steps of the accelerated-MISL based on SQUAREM are summarized in Algorithm 2. Note that we have made some changes to the original SQUAREM to deal with some potential problems. The first problem of the general SQUAREM is that it may violate the nonlinear constraints, so in Algorithm 2 the function $e^{j\arg(\cdot)}$ has been applied to project wayward points back to the feasible region. A second problem of SQUAREM is that it can violate the descent property of the original MM algorithm. To ensure the descent property, a strategy based on backtracking is adopted, which repeatedly halves the distance between α and -1:$\alpha\leftarrow(\alpha-1)/2$ until the descent property is maintained. To see why this works, we first note that $\mathbf{x}=e^{j\arg(\mathbf{x}^{(k)}-2\alpha\mathbf{r}+\alpha^2\mathbf{v})}=\mathbf{x}_2$ if $\alpha=-1$. In addition, since $\mathrm{ISL}(\mathbf{x}_2)\le\mathrm{ISL}(\mathbf{x}^{(k)})$ due to the descent property of original MM steps, $\mathrm{ISL}(\mathbf{x})\le\mathrm{ISL}(\mathbf{x}^{(k)})$ is guaranteed to hold as $\alpha\to-1$. In practice, usually only a few backtracking steps are needed to maintain the monotonicity of the algorithm.

Algorithm 2 Accelerated-MISL.

Require: sequence length N

Set $k = 0$, initialize $\mathbf{x}^{(0)}$.
repeat
  $\mathbf{x}_1 = \mathbf{F}_{\text{MISL}}(\mathbf{x}^{(k)})$
  $\mathbf{x}_2 = \mathbf{F}_{\text{MISL}}(\mathbf{x}_1)$
  $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}^{(k)}$
  $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1 - \mathbf{r}$
  Compute the step-length $\alpha = -\frac{\|\mathbf{r}\|}{\|\mathbf{v}\|}$
  $\mathbf{x} = e^{j\arg(\mathbf{x}^{(k)} - 2\alpha\mathbf{r} + \alpha^2\mathbf{v})}$
  while $\text{ISL}(\mathbf{x}) > \text{ISL}(\mathbf{x}^{(k)})$ **do**
    $\alpha \leftarrow (\alpha - 1)/2$
    $\mathbf{x} = e^{j\arg(\mathbf{x}^{(k)} - 2\alpha\mathbf{r} + \alpha^2\mathbf{v})}$
  end while
  $\mathbf{x}^{(k+1)} = \mathbf{x}$
  $k \leftarrow k + 1$
until convergence

5. NUMERICAL EXPERIMENTS

In this section, we present numerical results on applying the proposed accelerated-MISL algorithm to design unimodular sequences with low ISL and compare the performance with the state-of-the-art algorithm CAN [12]. All experiments were performed on a PC with a 3.20GHz i5-3470 CPU and 8GB RAM. The Matlab code of the benchmark algorithm, i.e., CAN, was downloaded from the website[1] of the book [4].

In the experiment, for both algorithms, the stopping criterion was $\left|\text{ISL}(\mathbf{x}^{(k+1)}) - \text{ISL}(\mathbf{x}^{(k)})\right| / \max\left(1, \text{ISL}(\mathbf{x}^{(k)})\right) \leq 10^{-5}$ and the initial sequence $\{x_n^{(0)}\}_{n=1}^N$ was chosen to be $\{e^{j2\pi\theta_n}\}_{n=1}^N$, where $\{\theta_n\}_{n=1}^N$ are independent random variables uniformly distributed in $[0, 1]$. Each algorithm was repeated 100 times for each of the following lengths: $N = 2^5, 2^6, \ldots, 2^{13}$. The average merit factor (defined in (3), the larger the better) of the output sequences and the average running time are shown in Fig. 1 and Fig. 2, respectively. From Fig. 1, we can see that the proposed accelerated-MISL algorithm can generate sequences with consistently larger merit factor (lower ISL) than the CAN algorithm for all lengths and at the same time it is several times faster, as can be seen from Fig. 2. The correlation level of two example sequences of length $N = 512$ and 4096 generated by the accelerated-MISL algorithm are shown in Fig. 3, where the correlation level is defined as

$$\text{correlation level} = 20\log_{10}\left|\frac{r_k}{r_0}\right|, k = 1, \ldots, N-1.$$

From the figure we can see that the correlation sidelobes have been suppressed a lot.

6. CONCLUSION

We have presented a new algorithm named MISL for the minimization of the ISL metric of unimodular sequences. The MISL algorithm is derived based on the general majorization-minimization framework and can be implemented via FFT operations. An acceleration scheme has also been considered to speed up MISL. Numerical results show that the proposed accelerated-MISL algorithm can generate sequences with larger merit factor (lower ISL) than the state-of-the-art method and at the same time is much faster.

[1] http://www.sal.ufl.edu/book/

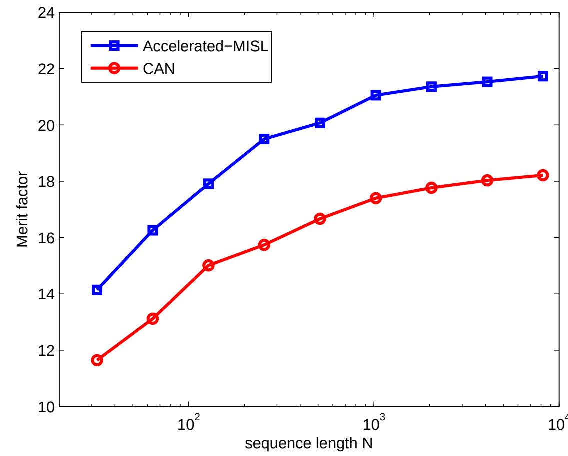

Fig. 1. Merit factor (MF) versus sequence length. Each curve is averaged over 100 random trials.

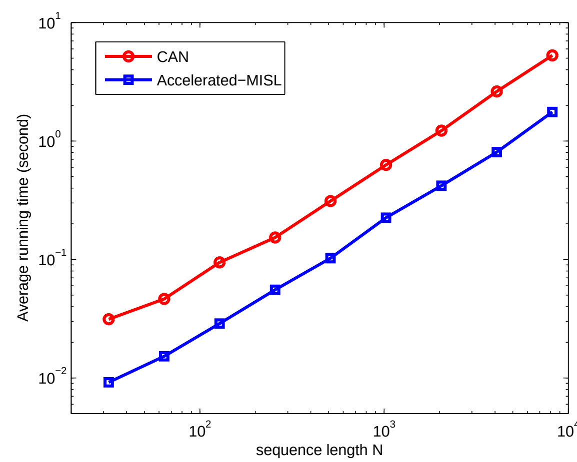

Fig. 2. Average running time versus sequence length. Each curve is averaged over 100 random trials.

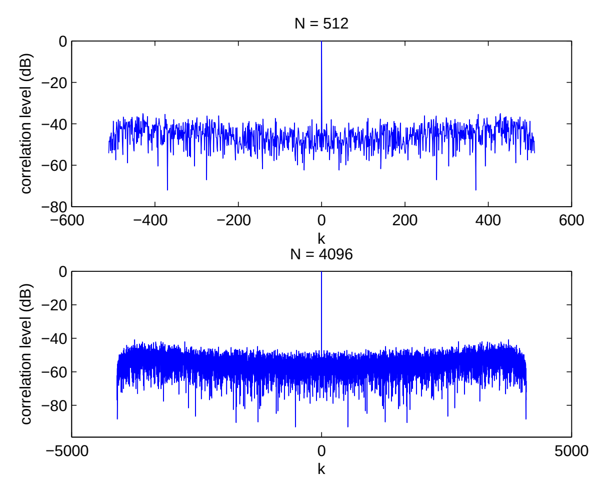

Fig. 3. Correlation level of two sequences of length $N = 512$ and $N = 4096$, generated by the accelerated-MISL algorithm.

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