AUGMENTED COVARIANCE ESTIMATION WITH A CYCLIC APPROACH IN DOA

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ABSTRACT

High resolution direction-of-arrival (DOA) estimation is an important problem in many array signal processing applications. This paper proposes an augmented covariance estimator for DOA estimation. The new method exploits the periodicity of the covariance lags when the DOAs are assumed on a discrete grid with a certain resolution. Then, it achieves twice the resolution of typical methods such as the direct augmentable approach or forward backward spatial smoothing. When the sources are not on the discrete grid, an interpolated array manifold technique is proposed to mitigate the grid mismatch error.

***Index Terms*—** Direction-of-arrival estimation, non-uniform array, minimum redundancy array, covariance matrices, MUSIC algorithm

1. INTRODUCTION

Direction of arrival (DOA) estimation has been revolutionized by the development of algorithms that exploit the structure inherit in non-uniformly spaced arrays (NUAs). In conventional approaches assuming uniform linear arrays (ULAs), with N antenna elements it is possible to estimate up to $N-1$ targets with a resolution proportional to the array aperture, which increases linearly with the number of antennas. High resolution spectral estimation algorithms like MUSIC rely on the $N \times N$ covariance matrix of the received signal. With NUAs, by minimizing the redundancies present in the co-array, an augmented covariance matrix of higher dimension from the original matrix can be build. Thus, with the same number of antennas as many as $N(N-1)/2$ targets with higher resolution can be estimated. The common goal is to reduce the cost and complexity of the array by having fewer sensors while maintaining a similar performance of that achieved with a filled ULA with the same aperture.

In [1], it was shown that there is a class of NUAs which maximizes the resolution for a given number of elements by minimizing the number of redundant spacings present in the array, minimum redundancy arrays (MRA). In [2] a superior spectral estimator for this kind of arrays was proposed allowing to detect more sources than the number of available sensors. From a N-element NUA the direct augmentable approach (DAA) is able to estimate the augmented covariance matrix of a "virtual" ULA with up to $N(N-1)/2+1$ sensors. It was shown in [3], that the DAA can provide a non positive definite covariance matrix for a finite number of snapshots, which can result in a degradation of the estimations. Several methods for constructing a proper positive definite covariance matrix were suggested [4][5]. More Recently [6], authors proposed an alternative method for augmented covariance estimation using spatial smoothing (SS) and nested arrays architectures. The SS augmented covariance estimation strategy was used in [7] and [8]. The first work consider spatial coprime sampling with two ULAs and difference inter-element spacing. In [8], authors consider the MRA based on sparse rulers [1][9] to increase the resolution and number of detectable targets. Thus, with an N-element NUA both DAA and SS can estimate an augmented covariance with dimension bounded by $N(N-1)/2+1$.

In this paper we show that under the assumption that the DOAs lie on a grid with a certain resolution, the covariance lags are periodic in the space. Using this fact, we introduce a new cyclic augmented approach (CAA) able to estimate an augmented covariance matrix up to dimension $N(N-1)+1$, twice the resolution achieved by DAA or SS. When the sources are not on the grid, we have the commonly named grid mismatch problem [10]. To mitigate this error, we propose an interpolated co-array technique that shows promising results.

2. SYSTEM MODEL

Consider the DOA estimation scenario where K narrow-band far-field sources are observed by an N-element linear array with elements at positions ξ_i, $i = 1 \dots N$. Let us assume the K narrow-band signals impinging on this array coming from directions $\theta_i, i = 1 \dots K$ and have average powers $\sigma_i^2, i = 1 \dots K$ respectively. The goal of DOA estimation is to use the data received at the array to estimate θ_i, $i = 1 \dots K$.

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The received signal is modeled as

$$\mathbf{y}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ is the matrix of steering vectors and $\mathbf{s}(t) = [s_1(t), \ldots, s_K(t)]^{\mathrm{T}}$ the t-th time snapshot of the source signal vector. Let the locations of the N receive antenna elements be normalized by half wavelength $\xi_i := \xi_i \cdot \frac{\lambda}{2}$, and $\theta := \sin(\theta)$ with $\theta \in [-1, 1)$. The steering vector corresponding to a direction θ is given by

$$\mathbf{a}(\theta) = \left[e^{j\pi\xi_1\theta}\; e^{j\pi\lambda\xi_2\theta} \ldots e^{j\pi\lambda\xi_N\theta}\right]^{\mathrm{T}}. \tag{2}$$

A1) The sources are uncorrelated from each other.

A2) The noise on the different antennas are mutually uncorrelated

The spatial covariance matrix can be written as,

$$\begin{aligned}\mathbf{R} &= \mathrm{E}\left[\mathbf{y}(t)\mathbf{y}^H(t)\right] \\ &= \sum_{i=1}^{K} \sigma_i^2 \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i) + \sigma_n^2 \mathbf{I},\end{aligned} \tag{3}$$

whose elements are all of the form

$$\begin{aligned}[\mathbf{R}]_{i,k} &= \sum_{k=1}^{K} \sigma_i^2 e^{j\pi\theta_k(\xi_i - \xi_k)} + \sigma_n^2 \delta_{ik} \quad \forall i, k \\ &= r[\xi_i - \xi_k].\end{aligned} \tag{4}$$

$\mathbf{R}$ is a $N \times N$ positive semidefinite Hermitian Toeplitz matrix characterized by the distinct covariance lags $r[m]$ with $m = (\xi_i - \xi_k)\ \forall i, k$.

3. AUGMENTED COVARIANCE ESTIMATION

The spatial covariance matrix contains information regarding the direction of propagation of any signal in the field. The philosophy of augmentable approaches is to estimate an augmented $N_a \times N_a$ covariance matrix $\hat{\mathbf{R}}_a$ of a "virtual" ULA with N_a elements from the smaller $N \times N$ sample covariance matrix $\hat{\mathbf{R}}$ of the N-element NUA. The aim is to design arrays, i.e. $\xi_{1\ldots N}$, with the minimum number of elements providing that all the second order moments needed to build $\hat{\mathbf{R}}_a$ can be estimated from $\hat{\mathbf{R}}$.

3.1. Direct Augmentable Approach

The DAA was introduced in [2]. Assuming it is possible to estimate in $\hat{\mathbf{R}}$ all contiguous correlation lags $r[m]$ from $m = 0..N_a - 1$, with $N_a \leq \frac{N(N-1)}{2} + 1$, we can build the augmented covariance matrix

$$\hat{\mathbf{R}}_a = \begin{bmatrix} r[0] & r[1] & \ldots & r[N_a - 1] \\ r^*[1] & r[0] & \ldots & r[N_a - 2] \\ \vdots & \vdots & \ddots & \vdots \\ r[N_a - 1] & r^*[1] & \ldots & r[0] \end{bmatrix}. \tag{5}$$

The DAA estimates $r[m]$ with $m = 0 \ldots N_a - 1$, by simply averaging over the set of the corresponding redundant covariance lags taken from $\hat{\mathbf{R}}$

$$r[m] = \frac{\sum_{i,k=0}^{N-1} \hat{R}_{ik}\delta_{(m,\xi_i - \xi_k)}}{\sum_{i,k=0}^{N-1} \delta_{(m,\xi_i - \xi_k)}} \;\; \xi_k > \xi_i. \tag{6}$$

When the DAA is considered, a sufficient condition for the estimation of the $N_a \times N_a$ augmented covariance matrix is that the co-array $D = \{\xi_i - \xi_k\}$ contains all the integers $\{0 \ldots N_a - 1\}$.

3.2. Spatial Smoothing Method

Alternatively to the DAA, another augmented covariance estimation technique was proposed in [6]. Forward-backward spatial smoothing is used to generate an augmented positive semi-definite spatial covariance matrix with suitable rank. The method starts by vectorizing the covariance matrix. Let us denote $\mathbf{r}_y = \mathrm{vec}\,[\mathbf{R}]$ and $\mathbf{r}_s = \mathrm{vec}\,[\mathbf{R}_s]$. Using the Khatri-Rao product we can write

$$\begin{aligned}\mathbf{r}_y &= (\mathbf{A}^* \odot \mathbf{A})\tilde{\mathbf{r}}_s + \sigma_n^2 \mathbf{e} \\ &= \mathbf{B}(\theta)\tilde{\mathbf{r}}_s + \sigma_n^2 \mathbf{e}.\end{aligned}$$

$\tilde{\mathbf{r}}_s = [\sigma_1^2 \;\; \sigma_2^2,\ \ldots,\ \sigma_k^2]^\top$ is the vector of nonzero entries in $\mathbf{r}_s$. $\mathbf{B} = [\mathbf{b}(\theta_1) \ldots \mathbf{b}(\theta_k)]$ is the $N^2 \times K$ array manifold matrix of the co-array, with columns $\mathbf{b}(\theta) = \mathbf{a}(\theta)^* \otimes \mathbf{a}(\theta)$ containing all the differences in the co-array

$$\mathbf{b}(\theta) = \left[e^{j\pi(\xi_1 - \xi_1)\theta} \quad e^{j\pi\lambda(\xi_2 - \xi_1)\theta} \quad \ldots \quad e^{j\pi\lambda(\xi_N - \xi_N)\theta}\right]^{\mathrm{T}}. \tag{7}$$

Extracting and sorting the rows from $\mathbf{B}$ corresponding to the distinct differences, a new $(2N_a - 1) \times K$ matrix $\tilde{\mathbf{B}}$ is constructed

$$\tilde{\mathbf{r}}_y = \tilde{\mathbf{B}}\tilde{\mathbf{r}}_s + \sigma_n^2 \tilde{\mathbf{e}}. \tag{8}$$

Comparing (1) to (8), we find out that the rows of $\tilde{\mathbf{B}}$ provide an array manifold matrix for a $2N_a - 1$ linear array. The equivalent source signal vector $\tilde{\mathbf{r}}_s$ consists of the powers σ_i^2 of the real sources and hence behave like fully coherent sources. Therefore, forward-backward spatial smoothing is used for rank enhancement. After an averaging process over the rows of $\tilde{\mathbf{B}}$, a new $N_a \times N_a$ covariance matrix $\hat{\mathbf{R}}_a$ is produced with proper rank. Spatial smoothing can only be applied to uniform linear arrays. Therefore, the same sufficient condition as with DAA is needed to estimate $\hat{\mathbf{R}}_a$. The co-array $D = \{\xi_i - \xi_k\}$ should contain all the integers $\{0 \ldots N_a - 1\}$.

3.3. Cyclic Augmentable Approach

In this section we consider the next assumption which is made in many DOA estimation papers [11][10][8] and commonly appears in sparse signal recovery applications:

A3) The angles of arrival lie on a grid with resolution N_a, i.e. $\theta_i = \frac{-N_a+2X_i}{N_a}$ with $X_i \in [0..N_a-1]$ for $i = 1..K$.

Under the assumption (A3), the entries of the correlation matrix $\mathbf{R}$ satisfy the following circular property $r[m] = (-1)^{pN_a} r[m+pN_a]$ with $p \in \mathbf{Z}$. .

$$\begin{aligned} r[m+pN_a] &= \sum_{i=1}^{K} \sigma_i^2 e^{j\pi\theta_i(m+pN_a)} \\ &= \sum_{i=1}^{K} \sigma_i^2 e^{j\pi\theta_i m} e^{j\pi\left(\frac{-N_a+2X_i}{N_a}\right)pN_a} \\ &= r[m] e^{j\pi pN_a} \\ &= (-1)^{pN_a} r[m] \end{aligned}$$

The relationship $r[m] = (-1)^{pN_a} r[m+pN_a]$ implies that the sufficient condition for estimation of the augmented covariance matrix $\hat{\mathbf{R}}_a$ is that the set $\tilde{D} = \{(\xi_i - \xi_j) \mod N_a\}$ contains all the numbers $\{0, ..., N_a - 1\}$. We propose Algorithm 1 to estimate all the covariance lags and build the matrix $\mathbf{R}_a$.

Definition 1. *Difference set. A subset* $\mathbf{u} = \{u_1, ..., u_k\}$ *of* $\mathcal{Z}_N$ *is called an* (N, K, λ) *difference set if the* $K(K-1)$ *differences*

$$(u_k - u_\ell) \mod N, \quad k \neq \ell \tag{9}$$

take all possible nonzero values $1, 2, ..., N-1$*, with each value exactly* λ *times.*

We can find an extensive list of difference sets with $\lambda = 1$ in [12]. Sparse arrays based on this type of difference sets don't have redundancies in the set $\tilde{D}$ and produce all the numbers $\{0..N(N-1)\}$. Using the cyclic augmentable approach (CAA) from a N-element array we can estimate a covariance matrix with dimension up to $N_a = N(N-1)+1$ instead of $N(N-1)/2+1$ with DAA or SS.

Algorithm 1 Cyclic Augmentable Approach (CAA)

Require: $\hat{\mathbf{R}}, \xi, N_a$
  $r[m] \leftarrow 0 \;\; m = 0..N_a - 1$
  $count \leftarrow 0$
  for $i = 1 : N$ **do**
    for $j = 1 : N$ **do**
      $d = (\xi_i - \xi_j)$
      $r[\mod(d, N_a)] += (-1)^{\lfloor \frac{d}{N_a} \rfloor} \hat{\mathbf{R}}(i,j)^*$
      $count[\mod(d, N_a)] ++$
    end for
  end for
  $r = r./count$
  $\hat{\mathbf{R}}_a = \text{Toeplitz}(r)$

3.4. Grid Mismatch and Interpolated Co-Arrays

When the targets are not on the discrete grid (A3) we have the so-called grid mismatch error. Grid mismatch minimization is an active research topic. Several approaches have been proposed to model and reduce the error [13][14][10]. Alternatively to CAA and to mitigate the grid mismatch error, we propose an interpolated co-array technique similar to classical array interpolation [15].

The basic assumption is that the co-array manifold of the virtual ULA $\mathbf{b}_{ula}(\theta) = [1, e^{j\pi\theta}, \dots, e^{j\pi(N_a-1)\theta}]$ can be approximated by a linear transformation of the co-array manifold of the NUA (7),

$$\mathbf{B}_{ula}(\theta) \simeq \mathbf{G}\mathbf{B}(\theta), \tag{10}$$

with $\mathbf{B}_{ula}(\theta) = [\mathbf{b_{ula}}(\theta_1), ..., \mathbf{b_{ula}}(\theta_K)]$. Given the transformation matrix $\mathbf{G}$, we can estimate all the correlation lags $\mathbf{r} = [r[0] \dots r[N_a-1]]^T$ from the covariance matrix of the real array ($\mathbf{r}_y = \text{vec}[\mathbf{R}]$)

$$\begin{aligned} \mathbf{r} &= \mathbf{B}_{ula}(\theta)\tilde{\mathbf{r}}_s \\ &\simeq \mathbf{G}\mathbf{B}(\theta)\tilde{\mathbf{r}}_s \\ &\simeq \mathbf{G}\mathbf{r}_y. \end{aligned} \tag{11}$$

From $\mathbf{r}$ we can build the augmented Toeplitz covariance matrix $\hat{\mathbf{R}}_a$ of the virtual ULA.

The accuracy of (10) directly depends on the array geometry. The transformation matrix $\mathbf{G}$ is usually computed as the least squares solution over a discrete grid of angles $\theta \in [-1, 1)$. For instance, for a difference set based array with N elements and a grid of angles with resolution $N_a = N(N-1)+1$, it is possible to find G that solves (10). However, the accuracy of the mapping decreases if we consider a more dense grid of angles.

To obtain a good approximation for the whole field of view of the array, classical array interpolation is done by dividing the angles in multiple sectors and finding one transformation matrix for each sector. In [15], a process for applying the high resolution root-Music algorithm to the set covariance matrices is described in detail. In [16], initial DOA estimates are used to remove the sector dependency and avoid possible problems with the condition number of the mapping matrix. Authors present a Wiener solution to compute the transformation matrix that has shown promising results and significantly improves the DOA estimation performance. For simplicity and to avoid the initial DOA estimation, we compute here the transformation matrix over the whole field of view, $\theta \in [-1, 1)$, as $\mathbf{G} = \mathbf{B}_{ula}(\theta)\mathbf{B}(\theta)^H \left(\mathbf{B}(\theta)\mathbf{B}(\theta)^H + \mathbf{I}\right)^{-1}$.

4. SIMULATIONS

In this section we present several simulation results. First, we compare the performance of DAA and SS with the novel CAA when the sources are on the discrete grid. DAA and

SS are used to process the data from an sparse ruler based array with $N = 10$ antennas, SR(10), with antenna locations $\xi_{SR(10)} = \{0, 1, 2, 17, 21, 24, 27, 30, 33, 35\}$ [8]. CAA is applied to the data from a difference set based array with the same number of elements, DS(10), with $\xi_{DS(10)} = \{0, 1, 3, 9, 27, 49, 56, 61, 77, 81\}$. DAA and SS provide augmented covariance matrices of dimension 36×36 while CAA 91×91. We consider $K = 5$ sources with DOAs on a grid with resolution $N_a = 91$ over the interval $\theta_i \in [-1, 1)$. Equal average power is considered for all the sources $\sigma_i^2 = 1$ with SNR=0 dB. 50 measurements are used to estimate the sample covariance. We apply Music algorithm to the augmented covariance matrices provided by each strategy. Fig.-1 shows the Music pseudo-spectrum. We clearly see CAA with the DS(10) offers better resolution than SS and DAA leading to better DOA separation. Fig.-2 shows the root mean square error (RMSE) of the Root-Music solution as a function of the number of snapshots. In this case CAA employs the same array as DAA and SS, providing a 71×71 augmented covariance matrix. $K = 5$ sources with DOAs randomly distributed on a grid [-1,1) with resolution $N_a = 71$ are considered. Finally, we evaluate the capabilities of the interpolated co-array manifold technique (IA) when targets are not on a grid. We consider $K = 5$ sources with DOA randomly distributed in the continuous interval [-1,1). Fig.-3 shows Root-Music RMSE. IA produces better DOA estimations in comparison to SS or DAA due to the higher dimensionality of the estimated augmented covariance matrix.

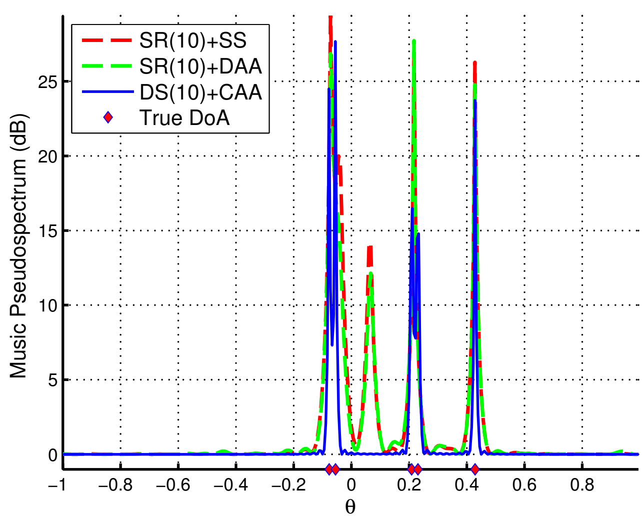

Fig. 1: Music Pseudo-spectrum in dB with $K = 5$ on-grid targets $\theta = \{-0.0769, -0.0549, 0.2088, 0.2308, 0.4286\}$.

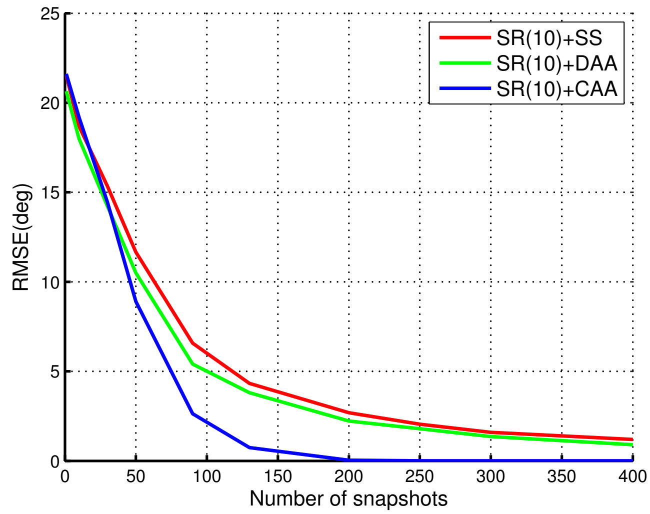

Fig. 2: RootMusic DOA estimation RMSE versus the number of snapshots for $SNR = 0$ dB and $K = 5$ on-grid targets randomly distributed in $[-1, 1)$ with resolution $N_a = 71$.

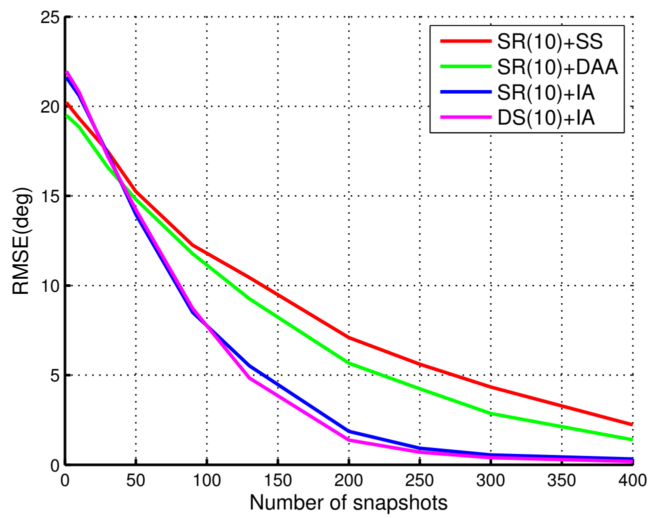

Fig. 3: RootMusic DOA estimation RMSE versus the number of snapshots for $SNR = 0$ dB and $K = 5$ targets randomly distributed in $[-1, 1)$.

5. CONCLUSION

We have introduced a novel augmentation approach for covariance estimation in DOA which achieves twice the resolution of DAA or SS when the sources are on a grid with a certain resolution. When it is applied to difference set based arrays, with N antennas CAA is able to estimate an augmented covariance matrix equivalent to that of an $N(N-1)+1$ virtual ULA. Finally, when the sources are not on the grid, we introduce an alternative interpolated co-array technique to mitigate the grid mismatch problem with promising results.

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