RANDOM MATRIX THEORY INSPIRED PASSIVE BISTATIC RADAR DETECTION WITH NOISY REFERENCE SIGNAL

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ABSTRACT

Traditional passive radar systems with a noisy reference signal use the cross-correlation statistic for detection. However, owing to the composite nature of this hypothesis testing problem, no claims can be made about the optimality of this detector. In this paper, we consider digital illuminators such that the transmitted signal in a processing interval is a weighted periodic summation of several identical pulses. The target reflectivity is assumed to change independently from one pulse to another within a processing interval. Inspired by random matrix theory, we propose a singular value decomposition (SVD) and Eigen detector for this model that significantly outperforms the conventional cross-correlation detector. We demonstrate this performance improvement through extensive numerical simulations across various surveillance and reference signal-to-noise ratio (SNR) regimes.

Index Terms— Passive radar, Random matrix theory, Phase transition, Singular value decomposition, Kolmogorov-Smirnov, Detection

1. INTRODUCTION

Passive coherent location systems have been studied for the last several decades due to the unique benefits they offer over conventional active radars by facilitating covert operation and for their low cost of implementation [1]– [3]. Passive radar systems do not possess perfect knowledge of the transmitted signal but have access to a noisy replica obtained through a dedicated reference channel. Historically, passive radar detection has been performed by computing a test statistic that is a cross correlation between received target surveillance signal and the noisy reference signal. However, no claims can be made with regards to the optimality of such a detector due to the noisy nature of the reference.

Recent Literature: Recently, [4] studied this problem and derived the GLRT detector. The model proposed in [4] has two major limitations. First, it considers the entire transmitted waveform to be *deterministic unknown* without any structure. In practice, this is never the case, as signals from terrestrial or space based communication systems are inherently periodic. In fact, for all communication systems [5], [6], the transmitted waveform contains a repetition of the same pulse several times in a processing interval. The message symbols riding on the pulse vary randomly from one symbol duration to another. Secondly, the target model in [4] assumes that the target reflectivity is constant over the entire processing interval. This assumption is seldom valid in practice. In another recent paper [7], the authors considered a random model for the transmitted signal but never consider scintillating targets in the processing interval. Furthermore, they ignore the underlying periodic nature of digital illuminators.

Contributions: In stark contrast to the aforementioned unrealistic assumptions, in this paper, we consider a composite hypothesis testing detection problem for a passive bistatic radar. The transmitted signal in a processing interval is assumed to be periodic summation of several identical pulses, and the target reflectivity is assumed to change independently from one pulse to another within a processing interval. Inspired by results from random matrix theory, we propose a simple detector by computing the SVD of the data matrix formed by concatenating the measurements from multiple pulses. This detector exploits the inherent low rank structure present in the periodic transmitted signal from a digital illuminator. Since the probability distributions of the test statistic are complicated to derive, we use the Kolmogorov-Smirnov tests to analyze the discriminating (between the two hypotheses) ability of the detector. It is seen subsequently that the proposed detector outperforms the frequently used traditional cross correlation based detector. The structure of data arising from our model lends itself to an interesting threshold behavior which is predicted by random matrix theory.

2. SIGNAL MODEL

We consider the hypothesis testing problem for the detection of a target at a given range and Doppler. In order to perform this test, the received data is shifted back by the appropriate delay and Doppler to arrive at the following testing problem

$$H_0 : \begin{cases} \boldsymbol{y}_{si} = \boldsymbol{n}_{si}, \\ \boldsymbol{y}_{ri} = \mu_{ri}\boldsymbol{u} + \boldsymbol{n}_{ri}, \end{cases} \tag{1}$$

$$H_1 : \begin{cases} \boldsymbol{y}_{si} = \mu_{si}\boldsymbol{u} + \boldsymbol{n}_{si}, \\ \boldsymbol{y}_{ri} = \mu_{ri}\boldsymbol{u} + \boldsymbol{n}_{ri}, \end{cases} \tag{2}$$

where $i \in \{1, \dots, N\}$ is the pulse index (we refer to this hereafter as snapshot) and the subscripts s and r represent the surveillance and reference channels, respectively. We assume the complex attenuation terms to be statistically independent from one pulse to the other. Without loss of generality (w.l.o.g), μ_{si} and μ_{ri} are zero mean complex Gaussian distributed with variance σ_s^2 and σ_r^2, respectively,

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This work was supported by the Air Force Office of Scientific Research under project 2311. [3] was supported by AFOSR Young Investigator Award FA9550-12-1-0266, ONR Young Investigator Award N00014-11-1-0660, US Army Research Office (ARO) under grant W911NF-11-1-0391 and NSF grant CCF-1116115. The opinions and statements in this work are the authors' own and does not constitute any explicit or implicit endorsement by the U.S. Department of Defense.

where we assume a single unit norm deterministic unknown transmit pulse $\boldsymbol{u}$ to contain M samples. The attenuation varies from pulse to pulse due to target fluctuations and also due to the random message modulations in the transmitted symbols from pulse to pulse. Note that formulating the detection problem in the above manner by distinguishing between the different pulses is possible only when there is perfect time synchronization. Most commercial illuminators afford perfect synchronization between the transmitter and receiver.

Stacking the measurement vectors, $\boldsymbol{Y}_s = [\boldsymbol{y}_{s1}, \cdots, \boldsymbol{y}_{sN}]$ and $\boldsymbol{Y}_r = [\boldsymbol{y}_{r1}, \cdots, \boldsymbol{y}_{rN}]$. Similarly, define $M \times N$ matrix $\boldsymbol{U} = [\boldsymbol{u}, \cdots, \boldsymbol{u}]$ and $\boldsymbol{\mu}_s = \operatorname{diag}\{\mu_{s1}, \ldots, \mu_{sN}\}$, $\boldsymbol{\mu}_r = \operatorname{diag}\{\mu_{r1}, \ldots, \mu_{rN}\}$. Therefore, we have

$$H_0 : \begin{cases} \boldsymbol{Y}_s = \boldsymbol{N}_s, \\ \boldsymbol{Y}_r = \boldsymbol{U}\boldsymbol{\mu}_r + \boldsymbol{N}_r, \end{cases} \tag{3}$$

$$H_1 : \begin{cases} \boldsymbol{Y}_s = \boldsymbol{U}\boldsymbol{\mu}_s + \boldsymbol{N}_s, \\ \boldsymbol{Y}_r = \boldsymbol{U}\boldsymbol{\mu}_r + \boldsymbol{N}_r. \end{cases} \tag{4}$$

We assume additive noise samples to be independent zero mean Gaussian with variance σ^2. The definition of SNR is separate under both the channels

$$\mathrm{SNR}_s = 20 \log \frac{\sigma_s}{\sqrt{M}\sigma} \mathrm{dB},$$

$$\mathrm{SNR}_r = 20 \log \frac{\sigma_r}{\sqrt{M}\sigma} \mathrm{dB}.$$

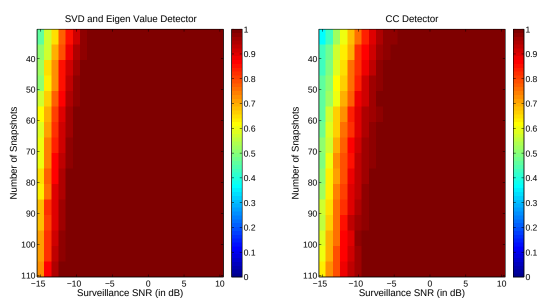

Fig. 2. Kolmogorov-Smirnov Statistic for SVD-Eigen Detector (left) and CC Detector (right), Reference Channel SNR 6dB, $M = 11$.

3. DETECTORS

The classical and frequently used approach is to form a test statistic by computing the cross correlation

$$T_{\mathrm{CC}} = \sum_{i=1}^{N} \left| \boldsymbol{y}_{si}{}^H \boldsymbol{y}_{ri} \right|^2. \tag{5}$$

Unlike an active radar system, no claims can be made about the optimality of this test statistic due to the noisy nature of the reference measurements. In particular, we observe that this detector performs a correlation on the raw data across both the channels for each of the N snapshots. It does not exploit the fact that the unit-norm transmit pulse inside each of these snapshots is the same. In other words, we would like to exploit the common rank 1 signal structure inherent to this problem. In particular, we propose

$$T_{\mathrm{SVD-Eigen}} = \left| S_s^2 \boldsymbol{v}_s{}^H \boldsymbol{v}_r \right|^2, \tag{6}$$

where $\boldsymbol{v}_s, \boldsymbol{v}_r$ denote the dominant left singular vectors of the random matrices $\boldsymbol{Y}_s$ and $\boldsymbol{Y}_r$, respectively. S_s denotes the leading singular value computed from $\boldsymbol{Y}_s$. We expect the SVD detector to perform better than the cross correlation detector because the left singular vector acts like a joint estimate of the unit-norm transmit pulse that is riding inside the measurements from all the snapshots. Hence, we compute this joint estimate from N snapshots before performing the cross correlation operation instead of doing it on the raw data from each snapshot separately. When SNR_r is very low, the estimate of the transmit pulse $\boldsymbol{u}$ is very poor and hence it is of little use to perform the cross correlation. However, due to the presence of the leading Eigenvalue from the surveillance channel, our proposed detector can still distinguish between the two hypotheses by functioning as an energy based discriminator. Further, from random matrix theory [8]– [11], SVD of these matrices have an interesting threshold behavior that can be used for asymptotic performance prediction. We will focus on this in more detail later in the paper.

For performance comparison, we consider the ideal case when all the parameters including the deterministic pulse $\boldsymbol{u}$ are known. Under this scenario, the test statistics reduce to $T_{\mathrm{SVD-Eigen}} = \left| S_s^2 \boldsymbol{v}_s{}^H \boldsymbol{u} \right|^2$ and $T_{\mathrm{CC}} = \sum_{i=1}^{N} \left| \boldsymbol{y}_{si}{}^H \boldsymbol{u} \right|^2$. In this scenario, the CC detector is optimal and is equivalent to the matched filter detector in active radar systems. We see this from the analysis below. When the transmit pulse $\boldsymbol{u}$ and the additive thermal noise variance σ^2 are known, essentially the reference signal does not carry any information that is useful for the detection problem. Therefore, the hypothesis testing problem

$$H_0 \quad : \quad \boldsymbol{y}_{si} \sim \mathcal{CN}\left(\boldsymbol{0}, \sigma^2 \boldsymbol{I}\right), i = 1, \ldots, N \tag{7}$$

$$H_1 \quad : \quad \boldsymbol{y}_{si} \sim \mathcal{CN}\left(\boldsymbol{0}, \sigma_s^2 \boldsymbol{u}\boldsymbol{u}^H + \sigma^2 \boldsymbol{I}\right), i = 1, \ldots, N. \tag{8}$$

The optimal likelihood ratio test (Clairvoyant detector) statistic

$$T_{\mathrm{LRT}} = \sum_{i=1}^{N} \left(\sigma^{-2} \boldsymbol{y}_{si}^H \boldsymbol{y}_{si} - \boldsymbol{y}_{si}^H \left(\sigma_s^2 \boldsymbol{u}\boldsymbol{u}^H + \sigma^2 \boldsymbol{I} \right)^{-1} \boldsymbol{y}_{si} \right). \tag{9}$$

Using Woodbury matrix identity, it can be easily shown that

$$T_{\mathrm{LRT}} \propto \sum_{i=1}^{N} \left| \boldsymbol{y}_{si}^H \boldsymbol{u} \right|^2. \tag{10}$$

4. STATICTICAL TECHNIQUES

In this section we describe statistical techniques used to analyze the performance of the detectors in the previous section. The probability distributions of the test statistic are complicated to derive both analytically and numerically. Hence, we use Kolmogorov-Smirnov (KS) test as a measure of separability between the two hypotheses in (1), for the detectors in the previous section. However, no claims can be made with regards to the probabilities of detection and false alarm.

4.1. Kolmogorov-Smirnov test: Not just a goodness of fitness test

Typically, the two sample KS [12]– [14] tests whether the two samples belong to a particular distribution against the alternative that they belong to different distributions. The two sample KS statistic is expressed as,

$$KS = \sup_x |\hat{F}_1(x) - \hat{G}_1(x)| \tag{11}$$

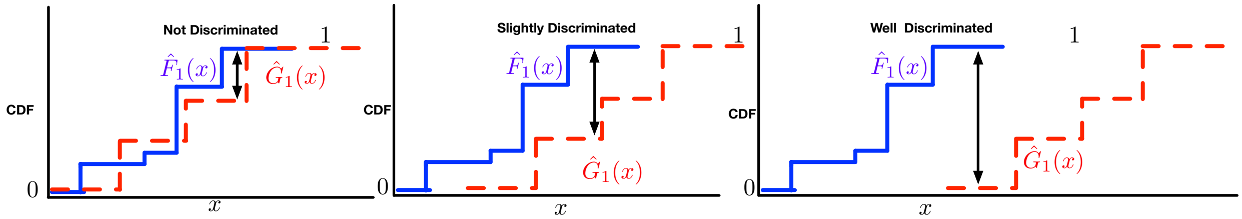

Fig. 1. Illustrative example of the discriminating capability of the KS test statistic in (11).

where $\hat{F}_1(x), \hat{G}_1(x)$ are the empirical (cumulative) distributions corresponding to the unspecified (cumulative) distributions $F(x), G(x)$, respectively. The empirical distributions are defined as

$$\hat{F}_1(x) = \begin{cases} 0 & \text{if } x < T_{(0,1)} \\ \frac{p}{P} & \text{if } T_{(0,p)} \leq x < T_{(0,p+1)}, p = 1, 2, \ldots, P-1 \\ 1 & x \geq T_{(0,P)} \end{cases}$$

$$\hat{G}_1(x) = \begin{cases} 0 & \text{if } x < T_{(1,1)} \\ \frac{p}{P} & \text{if } T_{(1,p)} \leq x < T_{(1,p+1)}, p = 1, 2, \ldots, P-1 \\ 1 & x \geq T_{(1,P)} \end{cases}$$

wherein $T_{(o,p)}, T_{(1,p)}, p = 1, 2, \ldots, P$ denote the two (ascending) ordered samples under test, and is explained subsequently. From (11), we notice immediately that if the empirical distributions are well discriminated (separated) the test statistic assumes unity, and when the distributions are not well separated the test statistic assumes smaller positive values. An illustrative example is shown in Fig. 1.

In our problem, for analyzing how the detectors discriminate between the null and alternate in (1), we first generate P measurements of the test statistic $\mathcal{T}$ under the null as in (1), and order them as $T_{(0,1)}, T_{(0,2)}, \ldots, T_{(0,P)}$ and likewise generate P measurements under the alternate hypothesis in (1) and order them as $T_{(1,1)}, T_{(1,2)}, \ldots, T_{(1,P)}$. The test statistic, $\mathcal{T}$ could be from any one of the detectors mentioned in the previous section. Next, we perform the KS test on these two samples as described above to analyze the discriminating (between the hypotheses in (1)) ability of the detectors T_{CC} and T_{SVD}. Note that this approach is employed because the probability distributions of T_{CC} and T_{SVD} are complicated to derive under H_0 and H_1 in (1), and is not the main crux of this paper. However, computing these probability distributions is a topic of our ongoing research as they are essential in computing the thresholds for target detection.

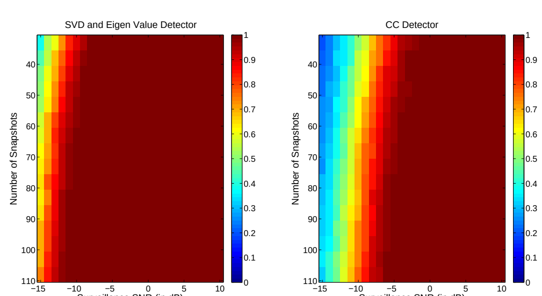

Fig. 3. Kolmogorov-Smirnov Statistic for SVD-Eigen Detector (left) and CC Detector (right), Reference Channel SNR -6dB, $M = 11$.

4.2. Phase transition thresholds

From random matrix theory [8]– [11], below a critical threshold region, the dominant eigenvalue and eigenvectors are not sufficiently representative of their true counterparts. The objective here is to provide insights on this threshold since our proposed detector in this paper is a function of the eigenvalue and eigenvectors. Consider realizations of a random vector, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ where $\mathbf{x}_i \in \mathbb{C}^M = \alpha_i \mathbf{u} + \mathbf{n}_i$. Assume that α_i is a zero mean, unit variance random variable not necessarily normally distributed but has a finite fourth order moment. The noise vector, $\mathbf{n}_i$ is zero mean, normally distributed with covariance matrix equal to $\sigma^2 \mathbf{I}$, and statistically uncorrelated from the other noise vectors, $\mathbf{n}_j, j = 1, 2, \ldots, N, j \neq i$. Denote $\mathbf{v}_1$ and $\hat{\mathbf{v}}_1$ as the true and estimated dominant eigenvectors, derived from the true covariance matrix, $\mathbf{u}\mathbf{u}^H + \sigma^2 \mathbf{I}$, and sample covariance matrix, $\frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i^H$, respectively. Likewise, denote $\hat{\lambda}_1$ as the estimated dominant eigenvalue estimated from the sample covariance matrix. Then we recall the following theorem from random matrix theory [11].

Theorem 1. *In the joint limit $M, N \to \infty$, we have,*

$$\hat{\lambda}_1 = \begin{cases} \sigma^2 \left(1 + \sqrt{\frac{M}{N}}\right)^2, & \text{if } \frac{N}{M} < \frac{\sigma^4}{||\mathbf{v}||^4} \\ (||\mathbf{v}||^2 + \sigma^2)\left(1 + \frac{M}{N} \frac{\sigma^2}{||\mathbf{v}||^2}\right), & \text{if } \frac{N}{M} \geq \frac{\sigma^4}{||\mathbf{v}||^4} \end{cases} \quad (12)$$

$$|\hat{\mathbf{v}}_1^H \mathbf{v}_1|^2 = \begin{cases} 0, & \text{if } \frac{N}{M} < \frac{\sigma^4}{||\mathbf{v}||^4} \\ \frac{(N||\mathbf{v}||^4/M\sigma^4) - 1}{(N||\mathbf{v}||^4/M\sigma^4) + (||\mathbf{v}||^2/\sigma^2)}, & \text{if } \frac{N}{M} \geq \frac{\sigma^4}{||\mathbf{v}||^4} \end{cases} \quad (13)$$

Theorem.1, (12) states that in the joint limit, the estimated eigenvalue is deterministic, and similarly, from (13) the squared magnitude of the inner products are deterministic. The threshold in both (12),(13) are identical and are given in blue. We note a somewhat surprisingly result from (13). For N/M below this threshold, the estimated eigenvector is completely orthogonal to its true counterpart, which implies that it offers no representative statistical information about the true dominant eigenvector. Although Theorem. 1 is with regard to the dominant eigenvector of the sample covariance matrix, it is readily applicable in a straightforward manner to the dominant left singular vectors of matrices $\boldsymbol{Y}_r$ and $\boldsymbol{Y}_s$ in equation (3), as well. In Section V, the threshold will be shown in the simulations for comparisons. In Section V, we observe the threshold behavior even for finite N and M.

Remark: In a passive radar problem, both the reference and surveillance SNRs must be above the threshold for the corresponding left singular vectors to not be orthogonal to each other.

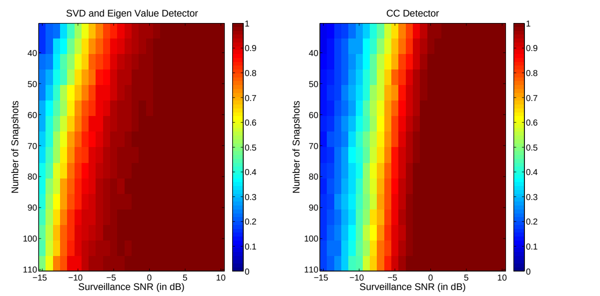

Fig. 4. Kolmogorov-Smirnov Statistic for SVD-Eigen Detector (left) and CC Detector (right), Reference Channel SNR -14dB, $M = 11$.

5. NUMERICAL EXAMPLES

A useful measure to compare the detectors is to employ the two sample Kolmogorov-Smirnov (KS) test by generating samples from both the hypotheses for varying values of N, SNR_s, and SNR_r. The KS test statistic has values varying from 0 to 1 with higher values implying greater separability between the distributions under the null and alternative hypotheses. From Figures 2 to 4, we plot the KS test statistic as a function of N and SNR_s for a fixed reference SNR_r. For these simulations, we chose transmitted pulse to be unit-norm root-raised cosine pulse to mimic a third generation wireless communications standard [6]. However, the transmit pulse could be any other unit-norm waveform. We observe that the SVD-Eigen based detector clearly outperforms the cross correlation based detector for all values of N and SNR_s when the reference SNR_r is both moderate and high. Even for very low SNR_r, CC detector outperforms SVD-Eigen only for a few values of N and SNR_s in Fig. 4. This is attributed to the numerical stability issues in MATLAB while computing the SVD at $\text{SNR}_r = -14\text{dB}$, and therefore this improvement can be discounted.

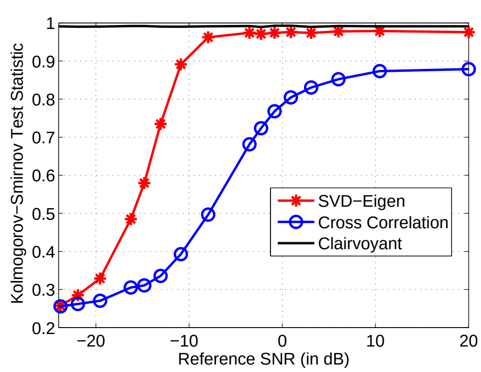

Fig. 5. Kolmogorov-Smirnov Statistic as a function of SNR_r for $\text{SNR}_s = -10\text{dB}$, $N = 50$, $M = 11$. (Clairvoyant detector does not depend on SNR_r since $\boldsymbol{u}$ is known)

In Fig. 5, we fix the number of snapshots $N = 50$ and the surveillance $\text{SNR}_s = -10\text{dB}$. We plot the KS test statistic as a function of SNR_r. We clearly observe the significant improvement in performance offered by the SVD-Eigen detector over the CC detector for all SNR_r above -20dB. Moreover we also notice that the SVD-Eigen performance is close to the clairvoyant detector (known transmit pulse $\boldsymbol{u}$) for SNR_r above -5dB. Note that the performance of the clairvoyant detector does not depend on the reference SNR since the transmit pulse is already known. Next, we fix $\text{SNR}_r = -10\text{dB}$ and plot as a function of SNR_s in Fig. 6. We notice significant improvement in performance for the SVD-Eigen detector when compared with the CC detector. Furthermore, the SVD-Eigen detector is also very close to its clairvoyant counterpart even for a relatively small $N = 50$, $M = 11$. In Fig. 7 we plot the theoretical phase transition threshold computed from equation (13), shown as a solid white line. We observe from that even for relatively small value of $M = 11$, the asymptotic thresholds for SNR_s computed using (12) and (13) very accurately capture the performance transition at $\text{SNR}_r = 0\text{dB}$ which is above the reference phase transition threshold.

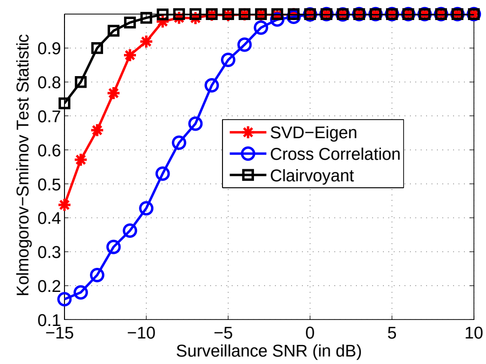

Fig. 6. Kolmogorov-Smirnov Statistic as a function of SNR_s for $\text{SNR}_r = -10\text{dB}$, $N = 50$, $M = 11$.

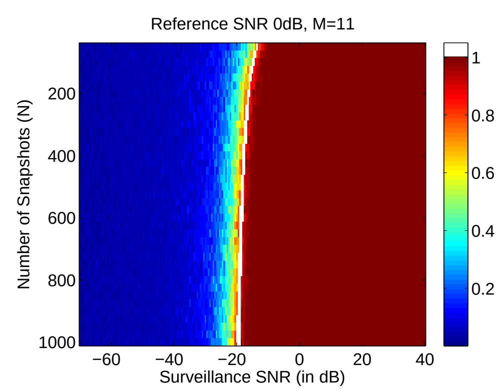

Fig. 7. Kolmogorov-Smirnov Statistic of the SVD-Eigen detector for $\text{SNR}_r = 0\text{dB}$, $M = 11$, $K = 1$.

6. CONCLUDING REMARKS

To test the presence of a fluctuating target at a given range / Dopper cell, we formulated a composite hypothesis test under the assumption that the reference channel is noisy. Inspired by recent developments in random matrix theory, and exploiting the inherent low rank common transmitted signal in the radar processing interval, a simple but powerful SVD detector was proposed. It was shown using statistical techniques and extensive Monte-Carlo testing that this detector outperforms the frequently used cross correlation detector.

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