MISMATCHED FILTER DESIGN FOR RADAR WAVEFORMS BY SEMIDEFINITE RELAXATION

Tuomas Aittomäki and Visa Koivunen

Department of Signal Processing and Acoustics, SMARAD CoE
Aalto University
PO Box 13000, FI-00076 Aalto, Finland
tuomas.aittomaki@aalto.fi, visa.koivunen@aalto.fi

ABSTRACT

Radar systems commonly require use of waveforms with low sidelobes and also low cross-correlation if multiple waveforms are being used. It is possible to decrease the apparent peak sidelobe and cross-correlation levels at the receiver by employing a mismatched filter. In this paper, we propose mismatched filter design method that minimizes the peak sidelobe and cross-correlation levels for all Doppler frequencies. The proposed design method is formulated as an optimization problem employing sum of squares representation of nonnegative polynomials and solved using semidefinite relaxation.

Index Terms— radar, MIMO radar, mismatched filter, filter design, semidefinite relaxation, trigonometric polynomial constraint

1. INTRODUCTION

Efficient operation of a radar system typically requires that the transmitted waveform has low sidelobes. If multiple waveforms are being used as in a MIMO radar, for example, it is also necessary that the cross-correlation of the waveforms is low. While it is well-known that using a matched filter at the receiver is optimal in additive white Gaussian noise, once the waveforms are fixed, nothing can be done to reduce the peak sidelobe (PSL) and peak cross-correlation (PCC) levels. However, if a mismatched filter [1] is used, it is possible to decrease the PSL and PCC levels. Naturally, the improved PSL and PCC levels of a mismatched filter come at the cost of a decrease in SNR. Thus, it is essential that this SNR loss is constrained when designing mismatched filters.

Mismatched filtering can be especially beneficial in MIMO radars, in which multiple waveforms are transmitted simultaneously [2]. Optimal target detection and parameter estimation in a MIMO radar requires that the waveforms can be separated at the receiver, which typically requires that the waveforms are orthogonal. However, it is not possible to have waveforms that are orthogonal for all time delays and Doppler shifts [3]. Using mismatched filters at the receiver end, it is possible to improve the PSL and the PCC of the waveforms so that they would be closer to being orthogonal.

Previously, clutter rejection with mismatched filtering for binary sequences was proposed in [4]. Mismatched filter with Pareto-optimal integrated sidelobe level and the peak sidelobe level was developed in [5]. Mismatched filterbank design for MIMO radars was considered in [6] for limiting the peak autocorrelation sidelobe and cross-correlation levels. Interference and jamming power was minimized in [7] while maintaining desired autocorrelation sidelobe and cross-correlation levels. However, these studies did not take Doppler shift into account. In this paper, we develop a mismatched filter design that minimizes the peak sidelobe and cross-correlation levels while keeping SNR loss below a desired value. Furthermore, power of jamming and interfering signals can also be controlled if the second order statistics of the interference can be estimated.

The mismatched filter design method proposed in this paper is based on converting trigonometric polynomial constraints into positive-semidefinite matrix constraints. This conversion, presented in [8], relies on the sum-of-squares representation of nonnegative polynomials and can be applied in various signal processing problems. It was used in radar waveform design for optimizing the worst case SNR with respect to the received Doppler in [9] and in [10] for detecting signals with uncertainty in the angle of arrival, for example.

The problem formulation using the positive-semidefinite matrix constraints includes nonconvex quadratic equality constraints. We convert this problem into a convex one by applying semidefinite relaxation [11]. If the solution of the relaxed problem is a rank-one matrix, the filter coefficient are obtained from the eigenvector of the solution. Otherwise, randomization can be used to find the actual filter coefficients [11]. We propose a projection method to obtain randomized filter coefficients that meet the unit response and maximum SNR loss constraints.

This work was supported by the Academy of Finland, Center of Excellence program

This paper is organized as follows: The filter design problem is described in Section 2. The proposed method to solve the filter design problem is explained in Section 3. Numerical results are be provided in Section 4, and Section 5 gives the concluding remarks.

2. PROBLEM FORMULATION

The goal in the filter design for a radar receiver is a filter that would minimize the peak sidelobe (PSL) and the peak cross-correlation (PCC) levels for the transmitted waveforms while adhering to constraints on SNR loss and interference power. The receiver filter for each waveform can be optimized independent of the other filters. The design method is applicable to conventional radars as well as both distributed and colocated MIMO radar configurations.

Let $\mathbf{w}$ denote the $L \times 1$ column vector of filter coefficients. It is assumed that the waveforms are modulated pulse trains, where the baseband samples of the modulating symbols of the ith waveform at delay m are given in $L \times 1$ vectors $\mathbf{s}_i(m)$. Without loss of generality, we can scale the vectors containing the symbols at zero delay to have unit norm such that $\|\mathbf{s}_i(0)\| = 1$. Each received waveform can be written as

$$a_i \mathbf{s}_i(m) \odot \mathbf{u}(f) + \mathbf{v}, \tag{1}$$

where a_i is a complex amplitude parameter that takes into account propagation losses, scattering, antenna gains etc., m is the delay, $\odot$ denotes element-wise multiplication, $\mathbf{u}$ is a Doppler phase vector, and $\mathbf{v}$ is a vector of random noise and interference. The Doppler phase vector is defined as

$$\mathbf{u}(f) = \begin{bmatrix} 1 & e^{-j2\pi f} & \dots & e^{-j2\pi(L-1)f} \end{bmatrix}^T, \tag{2}$$

where j is the imaginary unit and f is the normalized Doppler frequency.

The peak sidelobe and the peak cross-correlation that need to be minimized are defined for the kth waveform as

$$\text{PSL} = \max_{f,m} \left|\mathbf{w}^H[\mathbf{s}_k(m) \odot \mathbf{u}(f)]\right|^2, \quad |f| \geq \delta_{0m} f_0 \tag{3}$$

$$\text{PCC} = \max_{f,i,m} \left|\mathbf{w}^H[\mathbf{s}_i(m) \odot \mathbf{u}(f)]\right|^2, \quad i \neq k, \tag{4}$$

where δ_{ij} is the Kronecker delta notation, f_0 is the half-width of the main peak in frequency, $m = -L+1, \dots, L-1$ and $f \in [-\frac{1}{2}, \frac{1}{2}]$. We have assumed without loss of generality that the main peak of the autocorrelation function is scaled to one, which is equivalent to the assumption $\|\mathbf{s}_i(0)\| = 1$ stated previously. The SNR loss of the filter is given by

$$\frac{|\mathbf{w}^H \mathbf{s}_k(0)|^2}{|\mathbf{s}_k^H(0)\mathbf{s}_k(0)|^2} = \|\mathbf{w}\|^2. \tag{5}$$

The interference plus noise power at the filter output is given by

$$\mathrm{E}[|\mathbf{w}^H \mathbf{v}|^2] = \mathbf{w}^H \mathbf{R}_\mathbf{v} \mathbf{w}, \tag{6}$$

where $\mathbf{R}_\mathbf{v}$ is the covariance matrix of interference plus noise.

The filter design problem minimizing the PSL and PCC can now be formulated as

$$\min_{\mathbf{w},\alpha} \alpha \tag{7a}$$

$$\text{s.t} \; |\mathbf{w}^H[\mathbf{s}_i(m) \odot \mathbf{u}(f)]|^2 \leq \alpha, \quad |f| \geq \delta_{ik}\delta_{0m} f_0 \tag{7b}$$

$$\mathbf{w}^H \mathbf{w} \leq \beta, \tag{7c}$$

$$\mathbf{w}^H \mathbf{R}_\mathbf{v} \mathbf{w} \leq \gamma, \tag{7d}$$

$$\mathbf{w}^H \mathbf{s}_k(0) = 1, \tag{7e}$$

where α is the PSL and PCC level, β is the maximum allowed SNR loss in linear scale, and γ is the maximum interference plus noise power of the output. Equation (7e) is needed to maintain the unit gain to the waveform of interest at zero delay.

Due to the continuous nature of the Doppler frequency f, we cannot solve the filter design problem as an ordinary quadratically constrained program, as infinite number of constraints would be required to satisfy (7b). The method proposed in this paper for solving this problem type is explained in the next section.

3. RELAXED PROBLEM

In order to tackle the receiver filter design problem in (7), we first note that

$$|\mathbf{w}^H[\mathbf{s}_i(m) \odot \mathbf{u}(f)]|^2 = \mathbf{u}^H(f)[\mathbf{w}\mathbf{w}^H \odot \mathbf{s}_i^*(m)\mathbf{s}_i^T(m)]\mathbf{u}(f). \tag{8}$$

This is in fact a real-valued trigonometric polynomial that can be written as

$$p(\omega) = x_0 + 2\mathrm{Re}\Big\{\sum_{n=1}^{L-1} x_n e^{-j\omega n}\Big\}. \tag{9}$$

In this particular case,

$$x_k = \sum_{n=1}^{L-k} w_{k+n} w_n^* (\mathbf{s}_i(m))_{k+n}^* (\mathbf{s}_i(m))_n \tag{10}$$

and $\omega = 2\pi f$.

Next, we leverage Theorem 1 of [8], which states that a trigonometric polynomial of the form given in (9) is nonnegative on the interval $[0, 2\pi]$ *if and only if* there is an $L \times L$ positive-semidefinite matrix $\mathbf{X}$ such that

$$\mathbf{x} = \mathbf{F}^H \mathrm{diag}(\mathbf{F}\mathbf{X}\mathbf{F}^H), \tag{11}$$

where $\mathbf{x} = \begin{bmatrix} x_0 & \dots & x_{L-1} \end{bmatrix}^T$, $\mathrm{diag}(\cdot)$ denotes a vector consisting of the elements of the main diagonal of a matrix, $\mathbf{F}$ is a matrix consisting of the L first columns of an $M \times M$ DFT matrix, i.e.

$$(\mathbf{F})_{mn} = e^{-j2\pi(m-1)(n-1)/M}, m = 1 \dots M, n = 1 \dots L, \tag{12}$$

and $M \geq 2L-1$.

In order to deal with the main peak, we apply Theorem 2 provided in [8] stating that a trigonometric polynomial of the form given in (9) is nonnegative on the interval $[\omega_0 - \Delta, \omega_0 + \Delta]$ *if and only if* there are an $L \times L$ positive-semidefinite matrix $\mathbf{X}$ and an $(L-1)\times(L-1)$ positive-semidefinite matrix $\mathbf{Y}$ such that

$$\mathbf{x} = \mathbf{F}^H[\mathrm{diag}(\mathbf{F}\mathbf{X}\mathbf{F}^H) + \mathbf{c} \odot \mathrm{diag}(\mathbf{F}_1\mathbf{Y}\mathbf{F}_1^H)], \qquad (13)$$

where the elements of the vector $\mathbf{c}$ are

$$c_n = \cos(2\pi(n-1)/M - \omega_0) - cos(\Delta),\ n = 1, \ldots, M. \qquad (14)$$

The matrix $\mathbf{F}_1$ consists of the $L-1$ first columns of an $M \times M$ DFT matrix. Letting $\omega_0 = \pi$ and $\Delta = \pi - 2\pi f_0$ we achieve the desired interval. Since the ambiguity function is periodic in f with a period of one, the same result is achieved regardless of doing the optimization on $[-\frac{1}{2}, \frac{1}{2}]$ or $[0, 1]$.

Equation (10) is a quadratic equality constraint, which is nonconvex. In order to obtain a convex problem, we apply semidefinite relaxation and replace (10) with

$$x_k = \sum_{n=1}^{L-k} (\mathbf{W})_{k+n,n} (\mathbf{s}_i(m))^*_{k+n} (\mathbf{s}_i(m))_n, \qquad (15)$$

where $\mathbf{W}$ is an $L \times L$ positive-semidefinite matrix that is the new optimization variable in place of $\mathbf{w}$. Consequently, we obtain a semidefinite problem

$$\min \alpha \qquad (16a)$$

$$\text{s.t } (\mathbf{x}_{i,m})_j = \delta_{0j}\alpha - \sum_{n=1}^{L-j} (\mathbf{W})_{j+n,n} (\mathbf{s}_i(m))^*_{j+n} (\mathbf{s}_i(m))_n \qquad (16b)$$

$$\mathbf{x}_{k,0} = \mathbf{F}^H[\mathrm{diag}(\mathbf{F}\mathbf{X}_{k,0}\mathbf{F}^H) + \mathbf{c} \odot \mathrm{diag}(\mathbf{F}_1\mathbf{Y}_0\mathbf{F}_1^H)], \qquad (16c)$$

$$\mathbf{x}_{i,m} = \mathbf{F}^H \mathrm{diag}(\mathbf{F}\mathbf{X}_{i,m}\mathbf{F}^H),\ (i,m) \neq (k,0) \qquad (16d)$$

$$\mathrm{tr}(\mathbf{W}) \leq \beta \qquad (16e)$$

$$\mathrm{tr}(\mathbf{W}\mathbf{R}_\mathbf{v}) \leq \gamma \qquad (16f)$$

$$\mathrm{tr}(\mathbf{W}\mathbf{s}_k(0)\mathbf{s}_k^H(0)) = 1 \qquad (16g)$$

$$\mathbf{W} \succeq 0, \mathbf{X}_{i,m} \succeq 0, \mathbf{Y}_0 \succeq 0, \qquad (16h)$$

where $\mathbf{X} \succeq 0$ means that matrix $\mathbf{X}$ is positive semidefinite and $\mathbf{c}$, $\mathbf{F}$ as well as $\mathbf{F}_1$ have been defined previously. This is a convex optimization problem so the global optimum can be found efficiently.

Let the global optimum of the relaxed problem (16) be $\mathbf{W}_o$. If the rank of $\mathbf{W}_o$ is equal to one, we can write $\mathbf{W}_o = \mathbf{w}\mathbf{w}^H$, where the filter coefficient vector $\mathbf{w}$ is in fact the globally optimal solution to the non-relaxed problem [11]. If the rank of the solution to the relaxed problem is larger than one, we can use the randomization approach in order to obtain the filter coefficient vector $\mathbf{w}$. This is done by generating random solution candidates from the complex normal distribution,

$$\tilde{\mathbf{w}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{W}_o). \qquad (17)$$

However, the randomized candidates will never satisfy the unit gain constraint to the waveform of interest in (7e). Therefore, we project $\tilde{\mathbf{w}}$ to the space orthogonal to $\mathbf{s}_k(0)$ such that

$$\mathbf{z} = [\mathbf{I} - \mathbf{s}_k(0)\mathbf{s}_k^H(0)]\tilde{\mathbf{w}}. \qquad (18)$$

We can also satisfy the SNR loss constraint (7c) by scaling $\mathbf{z}$ so that its norm does not exceed $\sqrt{\beta - 1}$. Thus, a filter coefficient vector

$$\mathbf{w} = \mathbf{s}_k(0) + \min\left(1, \frac{\sqrt{\beta-1}}{\|\mathbf{z}\|}\right)\mathbf{z} \qquad (19)$$

will satisfy both (7c) and (7e). This filter consists of the mismatched filter and a orthogonal projection $\mathbf{z}$ of the randomized solution modifying the mismatched filter for decreasing the PSL and PCC. Once the SDR problem in (16) has been solved, producing the filter coefficients in (19) is computationally simple. We can generate multiple such coefficient vectors and choose the one that provides the lowest PSL and PCC levels.

4. NUMERICAL EXAMPLES

In this section, we show numerical results of the filter design using the proposed design method. The waveforms are two randomly generated amplitude-only modulated signals with twenty symbols drawn from the normal distribution independently. The resulting waveforms suffered from high sidelobes and cross-correlation with the PSL approximately -4.92dB for the first and -5.82dB for the second sequence, while the PCC is approximately -6.28dB.

The goal was then to design mismatched filters minimizing the peak sidelobe level and cross-correlation with thirty complex-valued filter coefficients for both waveforms. The maximum allowed SNR loss was 2dB and no interference or jamming was present in addition to additive white Gaussian noise. The semidefinite relaxation of the filter design problem in (16) was then solved for both waveforms separately with CVX, a package for solving convex programs [12, 13], using SeDuMi [14] as the solver. The obtained solutions were rank-one matrices for both filters, so the globally optimum filters in the original, non-relaxed problem were obtained. The PSL and PCC were approximately -7.84 dB for the first filter and -8.04 dB for the second. The SNR losses were approximately 1.25 dB and 1.17 dB, respectively.

Figure 1 shows the maximum of the ambiguity function of the first waveform over all delays for the matched filter and the proposed mismatched filter design. To be precise, the quantity $\max_m 20\log_{10}\left|\mathbf{w}^H[\mathbf{s}_1(m) \odot \mathbf{u}(f)]\right|$, where m is the

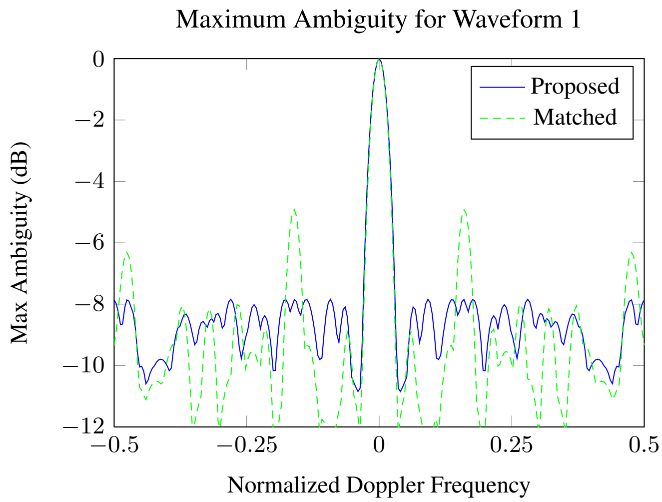

Fig. 1. Maximum of the ambiguity function of the first waveform. The proposed mismatched filter has significantly lower peak sidelobe and there is less variation in the sidelobe heights.

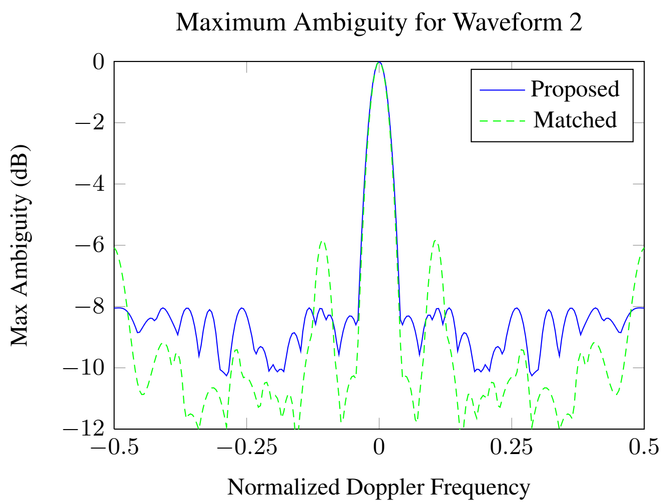

Fig. 2. Maximum of the ambiguity function of the second waveform. The proposed mismatched filter has significantly lower peak sidelobe and there is less variation in the sidelobe heights.

delay, is plotted as a function of the normalized Doppler frequency f. The proposed mismatched filter has significantly lower peak sidelobe level and there is less variation in the sidelobe heights. The maximum ambiguity for the second waveform in Fig.2 shows similar results.

The maximum cross-ambiguity functions are shown in Fig.3. Since there are only two waveforms, there is only one cross-ambiguity function for the matched filter. The mismatched filters are generally distinct, so the cross-ambiguity functions are shown for both filters. The cross-ambiguity

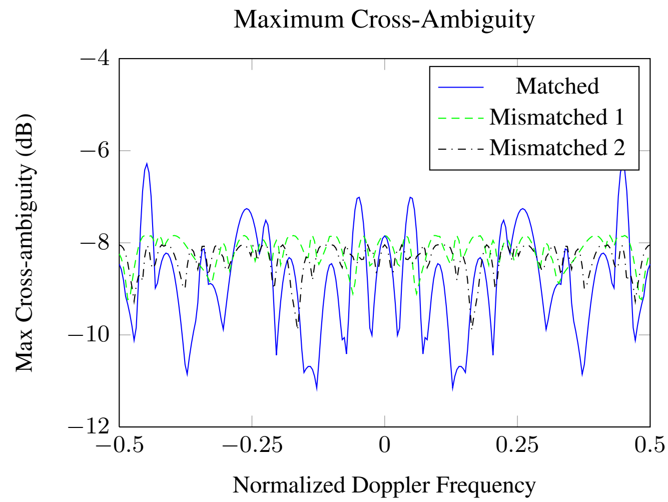

Fig. 3. Maximum of the cross-ambiguity function for the matched and the proposed mismatched filter designs. The mismatched filters provide much lower PCC than the matched filter.

functions of the mismatched filters display yet again being smoother and having lower peak level than the counterpart of the matched filter.

5. CONCLUSIONS

We proposed a method for designing mismatched filters for radar systems. The design method minimizes the peak sidelobe and cross-correlation levels over all delays and Doppler frequencies while satisfying the constraint on maximum SNR loss. The method works by replacing quadratic inequality constraints on a continuous interval with positive-semidefinite matrix constraints. Semidefinite relaxation can then be used to obtain a convex optimization problem that can be solved to global optimality. If the solution is not of rank-one, a randomization method can be applied to find the filter coefficients. We proposed a projection method that guarantees that the randomized filter coefficients fulfill the unit response and SNR loss constraints. The semidefinite program forming the mismatched filter design problem can be solved with any general-purpose SDP solver, but the solution could be obtained faster with a specific solver.

6. REFERENCES

[1] Martin H. Ackroyd and F. Ghani, "Optimum mismatched filters for sidelobe suppression," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-9, no. 2, pp. 214–218, 1973.

[2] Jian Li and Petre Stoica, Eds., *MIMO Radar Signal Processing*, John Wiley & Sons, 2009.

[3] B.M. Keel and T.H. Heath, “A comprehensive review of quasi-orthogonal waveforms,” in *IEEE Radar Conference*, 2007, pp. 122–127.

[4] Petre Stoica, Jian Li, and Ming Xue, “On binary probing signals and instrumental variables receivers for radar,” *IEEE Transactions on Information Theory*, vol. 54, no. 8, pp. 3820–3825, Aug 2008.

[5] A. De Maio, M. Piezzo, S. Iommelli, and A. Farina, “Design of pareto-optimal radar receive filters,” *International Journal of Electronics and Telecommunications*, vol. 57, no. 4, pp. 477–481, 2011.

[6] Liangbing Hu, Hongwei Liu, Da zheng Feng, Bo Jiu, Xu Wang, and Shunjun Wu, “Optimal mismatched filter bank design for MIMO radar via convex optimization,” in *International Waveform Diversity and Design Conference (WDD)*, aug 2010, pp. 126–131.

[7] T. Aittomaki and V. Koivunen, “MIMO radar filterbank design for interference mitigation,” in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2014, pp. 5297–5301.

[8] T. Roh and L. Vandenberghe, “Discrete transforms, semidefinite programming, and sum-of-squares representations of nonnegative polynomials,” *SIAM Journal on Optimization*, vol. 16, no. 4, pp. 939–964, 2006.

[9] Antonio De Maio, Yongwei Huang, M. Piezzo, Shuzhong Zhang, and Alfonso Farina, “Design of optimized radar codes with a peak to average power ratio constraint,” *IEEE Transactions on Signal Processing*, vol. 59, no. 6, pp. 2683–2697, June 2011.

[10] A. De Maio, S. De Nicola, A. Farina, and S. Iommelli, “Adaptive detection of a signal with angle uncertainty,” *Radar, Sonar Navigation, IET*, vol. 4, no. 4, pp. 537–547, August 2010.

[11] Zhi-Quan Luo, Wing-Kin Ma, A.M.-C. So, Yinyu Ye, and Shuzhong Zhang, “Semidefinite relaxation of quadratic optimization problems,” *IEEE Signal Processing Magazine*, vol. 27, no. 3, pp. 20–34, May 2010.

[12] Inc. CVX Research, “CVX: Matlab software for disciplined convex programming, version 2.0,” http://cvxr.com/cvx, Aug. 2012.

[13] M. Grant and S. Boyd, “Graph implementations for nonsmooth convex programs,” in *Recent Advances in Learning and Control*, V. Blondel, S. Boyd, and H. Kimura, Eds., Lecture Notes in Control and Information Sciences, pp. 95–110. Springer-Verlag Limited, 2008.

[14] Jos F. Sturm, “Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones,” *Optimization Methods and Software*, vol. 11, no. 1-4, pp. 625–653, 1999.