NARROW-RANGE FREQUENCY ESTIMATION BASED ON COMPREHENSIVE OPTIMIZATION OF DFT AND INTERPOLATION

Byeong Yong Kong and In-Cheol Park

Korea Advanced Institute of Science and Technology (KAIST), Republic of Korea

ABSTRACT

An efficient procedure for frequency estimation is proposed in this paper to alleviate the computational complexity. Grounded on the fact that the frequency of a target signal usually lies in a known range in practical applications, two fundamental steps in the frequency estimation, i.e., the discrete Fourier transform (DFT) and the interpolation of the DFT samples, are modified accordingly. Unlike the previous works focusing on either the DFT or the interpolation, this paper does not decouple the two steps but optimizes the whole procedure comprehensively by considering the interrelationship between the two steps. As a result, the number of operations required for the estimation is remarkably diminished while the performance remains competitive with the recent works.

***Index Terms*—** Discrete Fourier transform (DFT), frequency estimation, interpolation, radar signal processing, resonant-stylus pressure-sensing

1. INTRODUCTION

Frequency estimation is to guess the frequency of a single-tone signal from a set of its time-domain samples. It is essential in many applications such as resonant-stylus pressure-sensing and radar signal processing [1]. The conventional procedure for the frequency estimation consists of two successive steps: discrete Fourier transform (DFT) and interpolation of the DFT samples. As both steps are fundamental signal processing problems that have been studied for decades, the estimation has been conventionally achieved by employing one of DFT and one of interpolation techniques in the literature [2]–[7].

Although a combination of the algorithms selected in such a way may work well in terms of performance, it may not be efficient in the viewpoint of computational complexity due to the following two reasons. First of all, the algorithms usually do not take into account particular characteristics of the aforementioned applications: the frequency of the signal does not deviate from the narrow range known a priori. For example, the resonant frequency of a touch-screen stylus varies only about tens of kHz while the sampling frequency exceeds several MHz. In such a case, we do not need to evaluate the whole DFT samples. Secondly, the interrelationship between the two steps has never been considered seriously in selecting an algorithm for each step, although there may exist many chances to optimize the associated formulae.

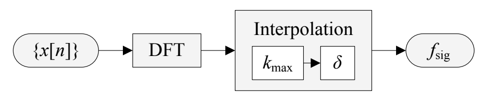

Fig. 1. Conventional frequency estimation.

To overcome such drawbacks, in this paper, we deliberate the narrowness of the actual frequency range in the DFT as well as in the interpolation steps. Moreover, we do not decouple the two steps but comprehensively optimize the whole procedure to maximize the efficiency. As a result, we successfully alleviate the computational complexity compared to the previous methods while sustaining the performance.

2. FREQUENCY ESTIMATION

As shown in Fig. 1, the frequency estimation is to guess the frequency of a single-tone signal f_{sig} from a set of its time-domain samples $\{x[n]\}$, where $n = 0, 1, \ldots, N-1$ and N is the number of the samples. In the first step of the conventional procedure, the DFT sample $X[k]$ is computed as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}} \tag{1}$$

for $k = 0, 1, \ldots, N-1$. In recent applications, the Goertzel algorithm [7]–[8] is actively replacing the fast Fourier transform (FFT) as it is a technique suitable to evaluating only a few DFT samples rather than all the N samples. Theoretically, it is equivalent to (1) and computationally more efficient than the FFT when the number of DFT samples to evaluate is smaller than $\log_2(N)$ [8]. According to the following formulae, the Goertzel algorithm first calculates $g_k[n]$ in a recurrent manner and then computes $X[k]$:

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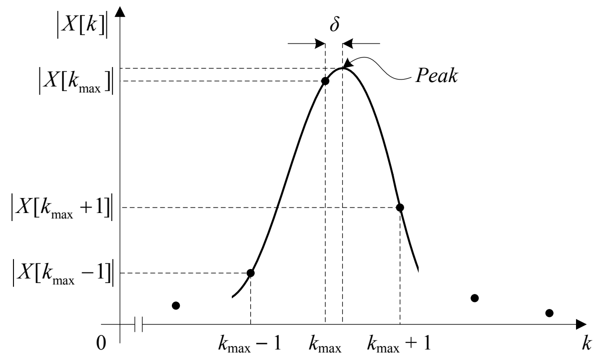

Fig. 2. The magnitude spectrum interpolated by DFT samples.

$$g_k[n] = 2\cos\frac{2\pi k}{N} g_k[n-1] - g_k[n-2] + x[n],$$
$$X[k] = \left(g_k[N] - \cos\frac{2\pi k}{N} g_k[N-1]\right) + j\left(\sin\frac{2\pi k}{N} g_k[N-1]\right), \quad (2)$$

where $g_k[-1] = g_k[-2] = 0$.

The second step that computes f_{sig} from the DFT samples can be described as follows. First of all, the index of the DFT sample associated with the largest magnitude, k_{max}, is determined. More precisely,

$$k_{max} = \underset{k\in\{0,1,\ldots,N-1\}}{\arg\max} \left|X[k]\right|. \quad (3)$$

Subsequently, $X[k_{max}]$ and the DFT samples adjacent to $X[k_{max}]$ are utilized to determine the compensation value δ that is shown in Fig. 2. The peak of the interpolated spectrum is designated by $k_{max} + \delta$, and f_{sig} is calculated as

$$f_{sig} = \frac{(k_{max} + \delta) f_s}{N} \quad (4)$$

where f_s is the sampling frequency. In the Jacobsen method [4], which is one of the most computationally efficient methods, δ is computed as

$$\delta = \mathrm{Re}\left(\frac{X[k_{max}-1] - X[k_{max}+1]}{2X[k_{max}] - X[k_{max}-1] - X[k_{max}+1]}\right). \quad (5)$$

3. PROPOSED ESTIMATION PROCEDURE

We now present a frequency estimation procedure composed of the two steps that are optimized comprehensively by taking into account the narrowness of the frequency. The derivation of the proposed procedure starts from the conventional procedure that employs the Goertzel algorithm and the Jacobsen for the DFT and the interpolation, respectively. Thus, (2), (3), and (5) constitutes the initial procedure.

A set of modifications are introduced to those formulae by considering the narrowness of the frequency band. Let us consider $k_1, k_2 \in [k_{first}, k_{last}]$ where k_{first} and k_{last} are the first and the last indices of the DFT samples that are within the narrow range of f_{sig}, respectively. By virtue of the narrowness, $X[k_1]$ and $X[k_2]$ are close to each other, i.e., $(k_{last} - k_{first}) << N$, and we may assume that

$$\arg(X[k_1]) = \arg(X[k_2]). \quad (6)$$

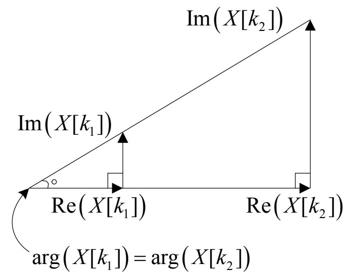

Fig. 3. Similarity of two right triangles formed by two adjacent DFT samples.

Grounded on (6), as illustrated in Fig. 3, two right triangles formed by the real and imaginary components of $X[k_1]$ and $X[k_2]$ are similar to each other. We exploit this similarity to modify some equations, and such modifications as well as the assumption of (6) will be empirically justified in Section 4 by measuring the performance.

3.1. Modification of the Metric

In the conventional procedure, the exact magnitude of a DFT sample, i.e.,

$$\left|X[k]\right| = \sqrt{\mathrm{Re}(X[k])^2 + \mathrm{Im}(X[k])^2} \quad (7)$$

is used as the metric in (3). However, we do not have to adhere to it if there exists any other metric $|\tilde{X}[k]|$ such that

$$k_{max} = \underset{k\in[k_{first},k_{last}]}{\arg\max} \left|X[k]\right| = \underset{k\in[k_{first},k_{last}]}{\arg\max} \left|\tilde{X}[k]\right|. \quad (8)$$

The validity of (8) is sufficiently guaranteed if $|\tilde{X}[k]|$ maintains the same order as arranged by (7). More precisely, $\forall\, k_1, k_2 \in [k_{first}, k_{last}]$,

$$\left|X[k_1]\right| \le \left|X[k_2]\right| \to \left|\tilde{X}[k_1]\right| \le \left|\tilde{X}[k_2]\right|. \quad (9)$$

To replace (7) with a computationally more efficient one, we define

$$\tilde{X}[k] = \mathrm{Re}(X[k]) + \mathrm{Im}(X[k]). \quad (10)$$

Note that $|\tilde{X}[k]|$ looks similar to the l^1-norm that is free from square operations, whereas (7) is the l^2-norm that requires square operations. Let us now prove that $|\tilde{X}[k]|$ satisfies (9).

Proposition 1. Let $|\tilde{X}[k]| = |\mathrm{Re}(X[k]) + \mathrm{Im}(X[k])|$. Then, $\forall$ $k_1, k_2 \in [k_{first}, k_{last}]$,

$$\left|X[k_1]\right| \le \left|X[k_2]\right| \rightarrow \left|\tilde{X}[k_1]\right| \le \left|\tilde{X}[k_2]\right|. \quad (11)$$

Proof: By virtue of the similarity between the triangles formed by $X[k_1]$ and $X[k_2]$, the ratio of real terms is equal to that of imaginary terms, i.e.,

$$u = \frac{\mathrm{Re}\left(X[k_1]\right)}{\mathrm{Re}\left(X[k_2]\right)} = \frac{\mathrm{Im}\left(X[k_1]\right)}{\mathrm{Im}\left(X[k_2]\right)}. \quad (12)$$

Without loss of generality, we assume

$$\begin{gathered} \left|X[k_1]\right| \le \left|X[k_2]\right| \\ \Downarrow \\ \sqrt{u^2\,\mathrm{Re}\left(X[k_2]\right)^2 + u^2\,\mathrm{Im}\left(X[k_2]\right)^2} = |u|\left|X[k_2]\right| \le \left|X[k_2]\right| \\ \Downarrow \\ |u| \le 1. \end{gathered} \quad (13)$$

In such a case, $\left|\tilde{X}[k_1]\right| \le \left|\tilde{X}[k_2]\right|$ since

$$\left|\tilde{X}[k_1]\right| = \left|u\,\mathrm{Re}\left(X[k_2]\right) + u\,\mathrm{Im}\left(X[k_2]\right)\right| = |u|\left|\tilde{X}[k_2]\right| \le \left|\tilde{X}[k_2]\right|. \quad (14)$$

Thus, $\left|X[k_1]\right| \le \left|X[k_2]\right| \rightarrow \left|\tilde{X}[k_1]\right| \le \left|\tilde{X}[k_2]\right|$. ■

By virtue of Proposition 1, we can substitute $\left|\tilde{X}[k]\right|$ for (7). (An appendix regarding the applicability of this metric is at the end of this paper.)

3.2. Modification of the Interpolation

Let us prove Proposition 2 prior to manipulating (5).

Proposition 2. Let a, b, c, d denote the followings:

$$\begin{aligned} a &= \mathrm{Re}\left(X[k_{\max}-1]\right) - \mathrm{Re}\left(X[k_{\max}+1]\right), \\ b &= \mathrm{Im}\left(X[k_{\max}-1]\right) - \mathrm{Im}\left(X[k_{\max}+1]\right), \\ c &= \mathrm{Re}\left(2X[k_{\max}]\right) - \mathrm{Re}\left(X[k_{\max}-1]\right) - \mathrm{Re}\left(X[k_{\max}+1]\right), \\ d &= \mathrm{Im}\left(2X[k_{\max}]\right) - \mathrm{Im}\left(X[k_{\max}-1]\right) - \mathrm{Im}\left(X[k_{\max}+1]\right). \end{aligned} \quad (15)$$

Then, $ad - bc = 0$.

Proof: Let u and v denote the following ratios grounded on the similarity of the triangles formed by the adjacent DFT samples:

$$\begin{aligned} u &= \frac{\mathrm{Re}\left(X[k_{\max-1}]\right)}{\mathrm{Re}\left(X[k_{\max}]\right)} = \frac{\mathrm{Im}\left(X[k_{\max-1}]\right)}{\mathrm{Im}\left(X[k_{\max}]\right)}, \\ v &= \frac{\mathrm{Re}\left(X[k_{\max+1}]\right)}{\mathrm{Re}\left(X[k_{\max}]\right)} = \frac{\mathrm{Im}\left(X[k_{\max+1}]\right)}{\mathrm{Im}\left(X[k_{\max}]\right)}. \end{aligned} \quad (16)$$

Then, (15) can be rewritten as

$$\begin{aligned} a &= (u-v)\,\mathrm{Re}\left(X[k_{\max}]\right), \\ b &= (u-v)\,\mathrm{Im}\left(X[k_{\max}]\right), \\ c &= (2-u-v)\,\mathrm{Re}\left(X[k_{\max}]\right), \\ d &= (2-u-v)\,\mathrm{Im}\left(X[k_{\max}]\right), \end{aligned} \quad (17)$$

and $ad - bc = 0$. ■

Grounded on Proposition 2, we manipulate (5) as

$$\begin{aligned} \delta &= \mathrm{Re}\left(\frac{X[k_{\max}-1] - X[k_{\max}+1]}{2X[k_{\max}] - X[k_{\max}-1] - X[k_{\max}+1]}\right) \\ &= \mathrm{Re}\left(\frac{a+jb}{c+jd}\right) = \frac{ac+bd}{c^2+d^2} \\ &= \frac{a+b}{c+d} + \frac{(c-d)(ad-bc)}{\left(c^2+d^2\right)(c+d)} \\ &= \frac{a+b}{c+d}. \end{aligned} \quad (18)$$

Note that the addend at the third line is equal to 0 because $ad - bc = 0$, and square operations and multiplications are completely removed.

3.3. Optimization by the Interrelationship of $\tilde{X}[k]$ and δ

Most of the previous interpolation methods [2]–[5] include costly complex-valued operations demanding both $\mathrm{Re}\left(X[k]\right)$ and $\mathrm{Im}\left(X[k]\right)$ from the DFT step. On the contrary, the proposed interpolation does not. Note that (18), a formula associated with the interpolation step, can be simply expressed in terms of (10), a formula associated with the DFT step. In other words,

$$\delta = \frac{a+b}{c+d} = \frac{\tilde{X}_{k-1} - \tilde{X}_{k+1}}{2\tilde{X}_k - \tilde{X}_{k-1} - \tilde{X}_{k+1}}. \quad (19)$$

By virtue of this interrelationship between the two successive steps, the proposed interpolation can be constituted by only a few real-valued additions associated with $\tilde{X}[k]$. As a result, it becomes unnecessary for the proposed procedure to keep $\mathrm{Re}\left(X[k]\right)$ and $\mathrm{Im}\left(X[k]\right)$ separately, and we can merge the two terms in (10) by utilizing a trigonometric formula as follows:

$$\begin{aligned} \tilde{X}[k] &= \mathrm{Re}\left(X[k]\right) + \mathrm{Im}\left(X[k]\right) \\ &= g_k[N] - \cos\frac{2\pi k}{N}\,g_k[N-1] + \sin\frac{2\pi k}{N}\,g_k[N-1] \\ &= g_k[N] - \sqrt{2}\cos\left(\frac{2\pi k}{N} + \frac{\pi}{4}\right)g_k[N-1]. \end{aligned} \quad (20)$$

It is worth noting that the number of trigonometric values required for the computations remains the same. For all $k \in [k_{\text{first}}, k_{\text{last}}]$, the original Goertzel algorithm needs $\cos(2\pi k/N)$ and $\sin(2\pi k/N)$. In case of the proposed one, $\sqrt{2}\cos(2\pi k/N + \pi/4)$ is newly required while the need for $\sin(2\pi k/N)$ is completely eliminated.

To sum up, the proposed procedure calculates $g_k[n]$ as expressed in (2). Instead of X_k, it computes $\tilde{X}[k]$ as

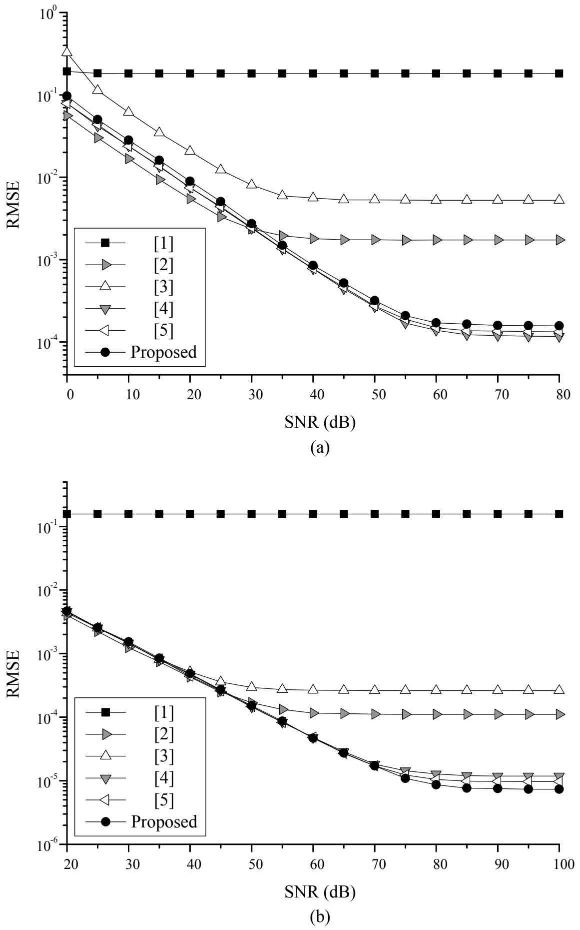

Fig. 4. RMSEs when (a) $N = 256$ and (b) $N = 512$.

described in (20). Then, it determines $k_{\max}$ by considering $\left|\tilde{X}[k]\right|$ and computes δ as (19).

4. EVALUATION

The proposed procedure is evaluated by comparing it to 5 interpolation methods [1]–[5]. Since there is no work that deals with the whole estimation procedure, we assume that the Goertzel algorithm precedes the interpolation schemes. Note that [4] accompanied with the Goertzel algorithm is the same as the initial procedure from which the proposed algorithm is derived.

4.1. Performance

Considering specifications of a commercial touch-screen stylus, we assume that f_{sig} varies from 350 kHz to 370 kHz and f_s = 4MHz. In Fig. 4, root mean square errors (RMSEs) of δ when $N = 256$ and 512 are measured in the presence of the additive white Gaussian noise.

In case of [1]–[3], the RMSE floor occurs at far lower signal-to-noise ratios (SNRs) than the case when either [4], [5], or the proposed procedure is employed. There are only negligible differences between the RMSEs of [4], [5], and the proposed procedure. The similar performances of [4] and the proposed procedure empirically corroborate that the modifications made in Section 3 are justifiable.

TABLE I
COMPUTATIONAL COMPLEXITIES

	Operation	[1]	[2]	[3]	[4]	[5]	Proposed
$\tilde{X}_k$	×	2	2	2	2	2	1
	+	1	1	1	1	1	1
	$(\,)^2$	2	2	2	2	2	-
δ	×	-	4	4	2	3	-
	+	3	7	6	8	8	3
	$(\,)^2$	-	4	1	2	2	-
	$\sqrt{\;}$	3	-	1	-	-	-
	/	1	2	2	1	1	1

4.2. Computational Complexity

The numbers of operations required for the two main tasks, i.e., the calculation of $\tilde{X}[k]$ from $g_k[N]$ and $g_k[N-1]$, and the evaluation of δ, are summarized in Table I. All the operations in the table are counted in terms of real-valued ones, implying that the complex-valued operations are converted to the real-valued counterparts.

As indicated in Table I, the proposed procedure requires a significantly less number of operations than the other techniques. In particular, there is completely no need for the computationally intensive operations such as squares and square roots. It is therefore highly probable that the proposed procedure will result in an efficient hardware when it is implemented.

5. CONCLUSION

By exploiting the property originated from the narrowness of the frequency range, the key equations in the procedure of the frequency estimation are simplified. Furthermore, the procedure is optimized based on the interrelationship of the metric and the interpolation. As a result, the computational complexity of the associated formulae is greatly reduced while the performance remains competitive with the recent works.

6. APPENDIX: THE APPLICABILITY OF THE PROPOSED PROCEDURE

The proposed procedure may not be appropriate if $\arg(X[k]) = -\pi/4$ or $3\pi/4$ where $k \in [k_{\text{first}}, k_{\text{last}}]$. In such a case, since $\text{Re}\left(X[k]\right) + \text{Im}\left(X[k]\right) = 0$, the metric defined in Section 3.1 becomes 0 and may not designate $k_{\max}$ correctly. However, such a limit does not impair the contribution of this paper severely since the problematic case can be regarded as a rare one and can be easily avoided by slightly modifying f_s or N.

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