ASYMPTOTIC PROPERTIES OF THE ROBUST ANMF

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ABSTRACT

This paper presents two approaches to derive an asymptotic distribution of the robust Adaptive Normalized Matched Filter (ANMF). More precisely, the ANMF has originally been derived under the assumption of Gaussian distributed noise where the variance is different between the observation under test and the set of secondary data. We propose in this work to relax the Gaussian hypothesis: we analyze the ANMF built with robust estimators, namely the M-estimators and the Tyler's estimator, under the Complex Elliptically Symmetric (CES) distributions framework. In this context, we derive two asymptotic distributions for this robust ANMF. Firstly, we combine the asymptotic properties of the robust estimators and the Gaussian-based distribution of the ANMF at finite distance. Secondly, we directly derive the asymptotic distribution of the robust ANMF. Then, Monte-Carlo simulations show the good approximation provided by the proposed methods. Moreover, for a non-asymptotic regime, the simulations provide very promising results.

***Index Terms*—** Adaptive Normalized Match Filter, M-estimators, Tyler's estimator, Complex Elliptically Symmetric distributions, non-Gaussian detection, robust estimation theory.

1. INTRODUCTION

In the general statistical signal processing area, the detection problem is an important topic of research. For instance, one can cite the works in radar processing [1, 2, 3, 4]. Since in practice, the noise parameters are unknown, an estimation step is required leading to the so-called adaptive detection processes. Among these unknown parameters, the noise Covariance Matrix (CM) is probably one of the most important since the performance of main adaptive detectors relies on the estimation accuracy of this CM. This is the case for the Adaptive Matched Filter (AMF) [5], the Kelly's test [6] and the Adaptive Normalized Matched Filter (ANMF) [1]. Generally, the CM is estimated by the Sample Covariance Matrix (SCM). Although this estimator is simple and provides the optimal performance under a Gaussian noise, the resulting adaptive detector performance can strongly be degraded when the noise turned to be non-Gaussian, heterogeneous or when it contains outliers/jammers.

To fill these gaps, a general framework on robust estimation theory has been extensively studied in the statistical community in the 1970s following the seminal works of Huber and Maronna [7, 8]. The multivariate real case has been recently extended to the complex case [9, 10, 11] more adapted for signal processing applications. Under this robust theory framework, most of recent works in CM estimation considers the broader class of Complex Elliptically Symmetric (CES) distributions. A complete review on CES applied to array processing can be found in [9].

In this CES framework, the so-called M-estimators [8] and the Tyler's estimator [12, 11] are alternatives to the Gaussian-based SCM. Although these robust estimators provide good results in practice [10], the statistical analysis of the resulting adaptive detectors is a difficult point. This is mainly due to the non explicit form of these estimators, defined through fixed point equations. However, their asymptotic properties have been recently derived in [9, 10]. Following these works, the aim of this paper is to derive the asymptotic properties of the ANMF built with these estimators, namely the M-estimators and the Tyler's estimator. The interest of such an analysis is to provide a better statistical characterization of the ANMF than the one based on the NMF ([13]).

The paper is organized as follows. The next section provides the background of this work while Section IIII contains the main theoretical contribution, the asymptotic distribution of the ANMF built with robust estimators. Then, Section IV validates the interest of this results through Monte Carlo simulations. Finally, some conclusions and perspectives are drawn in the last section.

The following convention is adopted: italic indicates a scalar quantity, lower (resp. upper) case boldface indicates a vector (resp. matrix) quantity and upper case boldface a matrix. ^{T} and ^{H} represent respectively the transpose and the transpose conjugate operators, Tr(.) denotes the trace operator, vec the vec operator and $\mathcal{CN}$ (resp $\mathcal{N}$) stands for the complex (resp. real) Gaussian distribution while $\mathcal{CES}$ stands for the Complex Elliptically Symmetric distribution.

2. BACKGROUND

2.1. The Adaptive Normalized Matched Filter (ANMF)

Detecting a complex signal corrupted by an additive Gaussian noise $\mathbf{c} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\,\mathbf{M})$ in a m-dimensional complex observation vector $\mathbf{y}$ can be stated as the following binary hypothesis test:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \dots, N \\ H_1 : \mathbf{y} = \alpha\,\mathbf{p} + \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \dots, N \end{cases}, \tag{1}$$

where $\mathbf{p}$ is a perfectly known complex steering vector, α is the unknown signal amplitude and where the $\mathbf{c}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{M})$ are N signal-free independent measurements, traditionally called the secondary data, used to estimate the background covariance matrix $\mathbf{M}$. When the variance σ^2 is unknown, this binary hypothesis test is solved by the Generalized Likelihood Ratio Test (GLRT) theory leading to a well-known Normalized Matched Filter [13] denoted

$H(.)$ and defined in $[0,1]$ by

$$H(\mathbf{M}) = \frac{|\mathbf{p}^H \mathbf{M}^{-1} \mathbf{y}|^2}{(\mathbf{p}^H \mathbf{M}^{-1} \mathbf{p}^H)(\mathbf{y}^H \mathbf{M}^{-1} \mathbf{y})}. \quad (2)$$

Under H_0, $H(\mathbf{M})$ follows a beta distribution $\beta(1, m-1)$ whose PDF is

$$p_\beta(u) = (m-1)\,(1-u)^{m-2}\,\mathbb{1}_{[0,1]}(u)\,, \quad (3)$$

where $\mathbb{1}_{[0,1]}(.)$ is the indicator function on $[0,1]$. The theoretical relationship between the detection threshold λ and the Probability of False Alarm (PFA) is defined as: $P_{fa} = \mathbb{P}(H(\mathbf{M}) > \lambda | H_0) = (1-\lambda)^{m-1}$. This last relation will serve as a benchmark as it characterizes a perfectly known covariance matrix for the detection test.

When an estimate is plugged into the NMF (two-step GLRT), this detector is called the ANMF or ACE (Adaptive Coherence Estimator) [1, 3]. Assuming that the SCM, defined as $\widehat{\mathbf{M}}_{SCM} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{c}_k\,\mathbf{c}_k^H$ is used, the resulting PDF $f_{H(\widehat{\mathbf{M}}_{SCM})}$ of $H(\widehat{\mathbf{M}}_{SCM})$ is given by [14]

$$f_{H(\widehat{\mathbf{M}}_{SCM})}(u) = K\,(1-u)^{a-2}\;{}_2F_1(a;a;b;u)\,\mathbb{1}_{[0,1]}(u)\,. \quad (4)$$

where $K = \frac{(a-1)(m-1)}{(N+1)}$, $a = N - m + 2, b = N + 2$ and ${}_2F_1(.)$ is the hypergeometric function [15].

2.2. M-estimators, Tyler's estimator and asymptotic properties

The purpose of this paper is to use robust alternatives to the SCM. This section presents the M-estimators, the Tyler's estimator as well as their asymptotic properties. Details of the following results can be found in [10, 9] for M-estimators and in [12, 11, 16] for the Tyler's estimator.

In the literature of radar detection and estimation, Spherically Invariant Random Vector (SIRV) modeling and Complex Elliptical Symmetric distributions (CES), originally introduced by Kelker in [17], have been considered and studied for their good statistical properties and for their good fitting to experimental non-Gaussian radar data. They provide a multivariate location-scale family of distributions that primarily serve as long tailed alternatives to the multivariate Gaussian model. A good review on these distributions can be found in [9, 18]. Let $\mathbf{c}$ be a m-dimensional complex random vector. $\mathbf{c}$ follows a CES distribution if its PDF can be written as

$$g_{\mathbf{c}}(\mathbf{c}) = |\boldsymbol{\Sigma}^{-1}|\,h_{\mathbf{c}}((\mathbf{c}-\boldsymbol{\mu})^H \boldsymbol{\Sigma}^{-1}(\mathbf{c}-\boldsymbol{\mu})), \quad (5)$$

where $h_{\mathbf{c}} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is any function such that (5) defines a PDF, $\boldsymbol{\mu}$ is the statistical mean and $\boldsymbol{\Sigma}$ is a scatter matrix. $\boldsymbol{\Sigma}$ reflects the structure of the covariance matrix of $\mathbf{c}$, i.e. the covariance matrix is equal to $\boldsymbol{\Sigma}$ up to a scale factor. One can notice that the Gaussian distribution is a particular case of CES. In this paper, we will assume that $\boldsymbol{\mu} = \mathbf{0}$ and without loss of generality, the scatter matrix will be taken equal to the covariance matrix $\mathbf{M}$.

Now, let $(\mathbf{c}_1, ..., \mathbf{c}_N)$ be an N-sample of m-dimensional complex independent vectors with $\mathbf{c}_k \sim \mathcal{CES}(\mathbf{0}, \mathbf{M})$, $k = 1, \ldots, N$. The M-estimators are defined as the unique solution of the following equation

$$\widehat{\mathbf{M}} = \frac{1}{N}\sum_{k=1}^{N} u\left(\mathbf{c}_k^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{c}_k\right)\;\mathbf{c}_k\,\mathbf{c}_k^H\,, \quad (6)$$

where u stands for any real-valued function that satisfies a set of general assumptions (see [10, 9]), mainly for ensuring the existence, uniqueness and convergence of the previous equation. Note that MLEs are a particular solution of the previous equation.

An attractive and powerful estimator, independent of the CES distribution, is the Tyler's estimator also called the Fixed Point and defined as the solution of

$$\widehat{\mathbf{M}} = \frac{m}{N}\sum_{k=1}^{N} \frac{\mathbf{c}_k\,\mathbf{c}_k^H}{\mathbf{c}_k^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{c}_k}. \quad (7)$$

For all M-estimator $\widehat{\mathbf{M}}$ which verifies equation (6), one has the important asymptotical statistical behaviour:

$$\sqrt{N}\left(\text{vec}(\widehat{\mathbf{M}} - \mathbf{M})\right) \xrightarrow{d} \mathcal{GCN}\left(\mathbf{0}_{m^2,1}, \boldsymbol{\Sigma}_M, \boldsymbol{\Omega}_M\right), \quad (8)$$

where $\mathbf{M}$ is the consistent limit of $\widehat{\mathbf{M}}$ and $\mathcal{GCN}(\mathbf{0}, \boldsymbol{\Sigma}_M, \boldsymbol{\Omega}_M)$ denotes the Generalized Complex Normal distribution with $\boldsymbol{\Sigma}_M$ the covariance matrix and $\boldsymbol{\Omega}_M$ the pseudo-covariance matrix defined as

$$\begin{aligned} \boldsymbol{\Sigma}_M &= \nu_1\,\mathbf{M}^T \otimes \mathbf{M} + \nu_2\,\text{vec}(\mathbf{M})\,\text{vec}(\mathbf{M})^H, \\ \boldsymbol{\Omega}_M &= \nu_1\,(\mathbf{M}^T \otimes \mathbf{M})\,\mathbf{K} + \nu_2\,\text{vec}(\mathbf{M})\,\text{vec}(\mathbf{M})^T, \end{aligned} \quad (9)$$

where $\mathbf{K}$ is the commutation matrix which transforms $\text{vec}(\mathbf{A})$ into $\text{vec}(\mathbf{A}^T)$, ν_1 and ν_2 are real scalars relying on the CES distribution and given in [19, 9].

It is important to notice that the previous result is also valid for the SCM when the observations are Gaussian ($\nu_1 = 1$ and $\nu_2 = 0$, see e.g. [20]) and for the Tyler's estimator for CES-distributed observations ($\nu_1 = (m+1)/m$ and $\nu_2 = -(m+1)/m^2$, see e.g. [16]). This shows that, asymptotically, the behaviour of all these estimators is similar. More precisely, the M-estimators and the Tyler's estimator behaves asymptotically as the SCM, it differs only from the quantities ν_1 and ν_2.

2.3. Asymptotic properties of the ANMF built with M-estimates

The asymptotic behaviour of all the presented estimators can then be extended to the ANMF thanks to the following result.

Let $H(.)$ be a r-dimensional multivariate function on the set of $m \times m$ positive-definite symmetric matrices with continuous first partial derivatives and such as $H(\mathbf{M}) = H(\alpha\,\mathbf{M})$ for all $\alpha > 0$. For all $\widehat{\mathbf{M}}$ that verifies equation (8), one has the following result, derived in [19, 9]:

$$\sqrt{N}\left(H(\widehat{\mathbf{M}}) - H(\mathbf{M})\right) \xrightarrow{d} \mathcal{GCN}\left(\mathbf{0}_{r,1}, \boldsymbol{\Sigma}_H, \boldsymbol{\Omega}_H\right), \quad (10)$$

where $\boldsymbol{\Sigma}_H$ and $\boldsymbol{\Omega}_H$ are defined as

$$\begin{aligned} \boldsymbol{\Sigma}_H &= \nu_1\,H'(\mathbf{M})\,(\mathbf{M}^T \otimes \mathbf{M})\,H'(\mathbf{M})^H, \\ \boldsymbol{\Omega}_H &= \nu_1\,H'(\mathbf{M})\,(\mathbf{M}^T \otimes \mathbf{M})\,\mathbf{K}\,H'(\mathbf{M})^T, \end{aligned} \quad (11)$$

and $H'(\mathbf{M}) = \frac{\partial H(\mathbf{M})}{\partial \text{vec}(\mathbf{M})} = (h'_{ij})$ with $h'_{ij} = \frac{\partial h_i}{\partial m_j}$ and m_j's denote the elements of $\text{vec}(\mathbf{M})$, for $j = 1, ..., m^2$.

When comparing to the asymptotical behavior of any function H with SCM argument $\mathbf{M}$, we obtain $\nu_1 = 1$. For any function H with Tyler's argument $\mathbf{M}$, we obtain $\nu_1 = (m+1)/m$. This explains that any function H of M-estimators has the same asymptotic distribution than those of a Wishart matrix (SCM) with N/ν_1 degrees of freedom. It also means that under Gaussian assumption, M-estimates need ν_1 times more secondary data than for SCM estimates to reach the same performances.

3. ASYMPTOTIC BEHAVIOR OF THE ANMF TEST

The goal of the this section is to propose two ways of deriving an approximate distribution of the test $H(\widehat{\mathbf{M}})$. The first approach consists in using the asymptotic distribution presented in section 2.3 of the different estimators while the second approach is to compute analytically the parameters $\mathbf{\Sigma}_H$ and $\mathbf{\Omega}_H$ characterizing the asymptotic distribution of the ANMF, equation (8).

3.1. Correction of the degrees of freedom, compared to the Gaussian-based SCM

Let us first consider the PDF given by (4) under a Gaussian hypothesis for the noise. Note that equation (4) provides the exact distribution of $H(\widehat{\mathbf{M}}_{SCM})$ when the observations $\mathbf{y}, \mathbf{y}_1, ..., \mathbf{y}_N$ are Gaussian distributed. Now, for N sufficiently large, equation (8) states that a M-estimator built with $\nu_1 N$ observations behaves like the SCM built with N observations. Consequently, combining this result with equation (4) leads to the following approximate distribution for $H(\widehat{\mathbf{M}})$ where $\widehat{\mathbf{M}}$ stands for any M-estimator or for the Tyler's estimator:

$$f_{H(\widehat{\mathbf{M}})}(x) = K\,(1-x)^{a-2}\;{}_2F_1(a;a;b;x)\,\mathbb{1}_{[0,1]}(x)\,, \qquad (12)$$

where $K = \dfrac{(a-1)(m-1)}{(N\,\nu_1+1)}$, $a = N\,\nu_1 - m + 2$ and $b = N\,\nu_1 + 2$ and the theoretical relationship between the detection threshold λ and the PFA $P_{fa} = \mathbb{P}(H(\widehat{\mathbf{M}}) > \lambda | H_0)$ is therefore given by:

$$P_{fa} = (1-\lambda)^{a-1}\,{}_2F_1(a, a-1; b-1; \lambda)\,. \qquad (13)$$

As illustrated in the simulations and although no rigorous proof is given, the previous result provides a very accurate PDF for $H(\widehat{\mathbf{M}})$ even for small N.

3.2. Asymptotic covariance of the ANMF

Let us now turn to the asymptotic distribution of the ANMF for any CM estimator.

Proposition 3.1 *Let us consider the ANMF test defined by*

$$H(\widehat{\mathbf{M}}) = \frac{|\mathbf{p}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{y}|^2}{(\mathbf{p}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{p}^H)(\mathbf{y}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{y})}\,. \qquad (14)$$

For any estimator $\widehat{\mathbf{M}}$ satisfying equation (8)*, one has*

$$\sqrt{N}\left(H(\widehat{\mathbf{M}}) - H(\mathbf{M})\right) \xrightarrow{d} \mathcal{N}\left(0, \mathbf{\Sigma}_H\right), \qquad (15)$$

where the asymptotic variance $\mathbf{\Sigma}_H$ of the ANMF statistic is given by

$$\mathbf{\Sigma}_H = 2\,\nu_1\,H(\mathbf{M})\,(H(\mathbf{M})-1)^2\,. \qquad (16)$$

Proof 3.1 *First, when proving* (10)*, one can easily show that if H is a real-valued function, one has $\mathbf{\Sigma}_H = \mathbf{\Omega}_H$. Consequently, let us focus on $\mathbf{\Sigma}_H$. For simplicity matters and without loss of generality, let us consider $\mathbf{M}$ as a real symmetric positive-definite matrix and let us derive $H'(\mathbf{M}) = \dfrac{\partial H(\mathbf{M})}{\partial \text{vec}(\mathbf{M})}$.*

To find this term, the following results are required [21]:

$$\begin{aligned}
\partial\,\text{Tr}\,(\mathbf{X}) &= \text{Tr}\,(\partial\mathbf{X})\,, && (17a)\\
\partial\text{vec}\,(\mathbf{X}) &= \text{vec}\,(\partial\mathbf{X})\,, && (17b)\\
\partial\mathbf{A}^{-1} &= -\mathbf{A}^{-1}\partial\mathbf{A}\mathbf{A}^{-1}, && (17c)\\
\partial(\mathbf{A}\&\mathbf{B}) &= \partial(\mathbf{A})\&\mathbf{B} + \mathbf{A}\&\partial(\mathbf{B}), && (17d)\\
\textit{where}\ \ \& &\ \ \textit{stands for}\ \times\ \textit{or}\otimes, &&\\
\text{Tr}\,(\mathbf{AB}) &= \text{Tr}\,(\mathbf{BA})\,, && (17e)\\
\mathbf{M}^{-1}\otimes\mathbf{M}^{-1} &= (\mathbf{M}\otimes\mathbf{M})^{-1}\,, && (17f)\\
\text{Tr}\left(\mathbf{A}^H\mathbf{B}\right) &= \text{vec}^H\,(\mathbf{A})\,\text{vec}\,(\mathbf{B})\,, && (17g)\\
\text{vec}\,(\mathbf{ABC}) &= \left(\mathbf{C}^T\otimes\mathbf{A}\right)\text{vec}\,(\mathbf{B})\,. && (17h)
\end{aligned}$$

The first step is to derive $\partial(\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b})$. For that purpose, let us set $u = \mathbf{a}^H\mathbf{M}^{-1}\mathbf{b}$, then one has:

$$\begin{aligned}
\partial u &= \partial\,\text{Tr}(\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b})\\
&= \text{Tr}(\partial(\mathbf{a}^H\partial(\mathbf{M}^{-1})\mathbf{b}))\ \textit{from (17a)}\\
&= \text{Tr}(\mathbf{a}^H\mathbf{M}^{-1}\partial(\mathbf{M})\mathbf{M}^{-1}\mathbf{b})\ \textit{from (17c)}\\
&= -\,\text{Tr}(\partial(\mathbf{M})\mathbf{M}^{-1}\mathbf{b}\mathbf{a}^H\mathbf{M}^{-1})\ \textit{from (17e)}\\
&= -\text{vec}^H(\partial\mathbf{M})\text{vec}(\mathbf{M}^{-1}\mathbf{b}\mathbf{a}^H\mathbf{M}^{-1})\ \textit{from (17g)}
\end{aligned}$$

Thus, from (17b), (17h) (17f)*, one has $\forall\,\mathbf{a}, \mathbf{b} \in \mathbb{C}^m$,*

$$\partial(\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b}) = -\partial(\text{vec}^H(\mathbf{M}))\,(\mathbf{M}^T\otimes\mathbf{M})^{-1}\text{vec}(\mathbf{b}\mathbf{a}^H). \quad (18)$$

Moreover, $\forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{C}^m$, one has

$$\begin{aligned}
&\frac{\text{vec}^H(\mathbf{a}\mathbf{b}^H)}{\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b}}(\mathbf{M}^T\otimes\mathbf{M})^{-1}\,\frac{\text{vec}(\mathbf{c}\mathbf{d}^H)}{\mathbf{d}^H\mathbf{M}^{-1}\mathbf{c}}\\
&= \frac{\text{vec}^H(\mathbf{a}\mathbf{b}^H)\text{vec}((\mathbf{M}^{-1}\mathbf{c}\mathbf{d}^H\mathbf{M}^{-1})}{\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b}\,\mathbf{d}^H\mathbf{M}^{-1}\mathbf{c}}\\
&= \frac{\text{Tr}(\mathbf{b}\mathbf{a}^H\mathbf{M}^{-1}\mathbf{c}\mathbf{d}^H\mathbf{M}^{-1})}{\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b}\,\mathbf{d}^H\mathbf{M}^{-1}\mathbf{c}} = \frac{\mathbf{a}^H\mathbf{M}^{-1}\mathbf{c}\,\mathbf{d}^H\mathbf{M}^{-1}\mathbf{b}}{\mathbf{a}^H\mathbf{M}^{-1}\mathbf{b}\,\mathbf{d}^H\mathbf{M}^{-1}\mathbf{c}}
\end{aligned}$$

Moreover, since that $\partial(uv) = \partial u\,v + u\,\partial v$ and $\partial\dfrac{u}{v} = \dfrac{\partial u\,v - u\,\partial v}{v^2}$, one can derive $\partial H(\mathbf{M})$ as follows

$$\begin{aligned}
\partial H(\mathbf{M}) &= \frac{\partial(\mathbf{y}^H\mathbf{M}^{-1}\mathbf{p})\mathbf{p}^H\mathbf{M}^{-1}\mathbf{y} + \partial(\mathbf{p}^H\mathbf{M}^{-1}\mathbf{y})\mathbf{y}^H\mathbf{M}^{-1}\mathbf{p}}{(\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}^H)(\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y})}\\
&- \frac{H(\mathbf{M})\left(\partial(\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y})\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p} + \partial(\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p})\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y}\right)}{(\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}^H)(\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y})}
\end{aligned}$$

Now, applying equation (18) *to the derivatives of the quadratic forms in previous equation leads to*

$$\begin{aligned}
&\partial H(\mathbf{M}) = -\partial(\text{vec}^H(\mathbf{M}))\,(\mathbf{M}^T\otimes\mathbf{M})^{-1}H(\mathbf{M})\\
&\times\left[\frac{\text{vec}(\mathbf{p}\mathbf{y}^H)}{\mathbf{y}^H\mathbf{M}^{-1}\mathbf{p}} + \frac{\text{vec}(\mathbf{y}\mathbf{p}^H)}{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{y}} - \frac{\text{vec}(\mathbf{p}\mathbf{p}^H)}{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}} - \frac{\text{vec}(\mathbf{y}\mathbf{y}^H)}{\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y}}\right] \quad (19)
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
H'(\mathbf{M}) = -H(\mathbf{M})&\left[\frac{\text{vec}^H(\mathbf{p}\mathbf{y}^H)}{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{y}} + \frac{\text{vec}^H(\mathbf{y}\mathbf{p}^H)}{\mathbf{y}^H\mathbf{M}^{-1}\mathbf{p}}\right.\\
&\left.-\frac{\text{vec}^H(\mathbf{p}\mathbf{p}^H)}{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}} - \frac{\text{vec}^H(\mathbf{y}\mathbf{y}^H)}{\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y}}\right](\mathbf{M}^T\otimes\mathbf{M})^{-1}
\end{aligned}$$

since $\mathbf{M}$ *is a real symmetric matrix. Then, replacing this result in equation* (11) *and using equation* (19) *leads to*

$$\boldsymbol{\Sigma}_H = \nu_1\, H(\mathbf{M})^2 \left[\frac{2}{H(\mathbf{M})} + 2H(\mathbf{M}) - 4\right], \qquad (20)$$

which can be rewritten as $\boldsymbol{\Sigma}_H = 2\,\nu_1\, H(\mathbf{M})\,(H(\mathbf{M})-1)^2$.

Notice that, contrary to the first approach, the previous asymptotic distribution is a distribution conditional to the observation $\mathbf{y}$ that appears in $H(\mathbf{M})$. Consequently, a supplementary step is required to obtain the asymptotic distribution of $H(\widehat{\mathbf{M}})$. Let us rewrite the result of Proposition 3.1 as

$$H(\widehat{\mathbf{M}}) \xrightarrow{d} \mathcal{N}\left(H(\mathbf{M}), \frac{2\,\nu_1}{N} H(\mathbf{M})\,(H(\mathbf{M})-1)^2\right). \qquad (21)$$

For N large enough, considering that $H(\widehat{\mathbf{M}}) \sim \mathcal{N}\left(X, \sigma_X^2\right)$ where $\sigma_X^2 = \frac{2\,\nu_1}{N}\, X\,(X-1)^2$ and $X = H(\mathbf{M}) \sim \beta(1, m-1)$, one can obtain the asymptotic distribution $f^a_{H(\widehat{\mathbf{M}})}$ of $H(\widehat{\mathbf{M}})$ as follows

$$f^a_{H(\widehat{\mathbf{M}})}(u) = \int_0^1 \frac{\sqrt{N}\,\exp\left(-\dfrac{N\,(u-x)^2}{4\,\nu_1\,x\,(x-1)^2}\right)}{\sqrt{4\,\pi\,\nu_1\,x\,(x-1)^2}} p_\beta(x)\,dx\,. \quad (22)$$

where $p_\beta(.)$ denotes the density of the beta distribution.

Now, if we denote $\Phi(.)$ the cumulative distribution of the Normal distribution, one obtains the corresponding asymptotical P_{fa}-λ relationship:

$$P_{fa} = 1 - (m-1)\int_0^1 (1-x)^{m-2}\,\Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right) dx \qquad (23)$$

4. SIMULATIONS

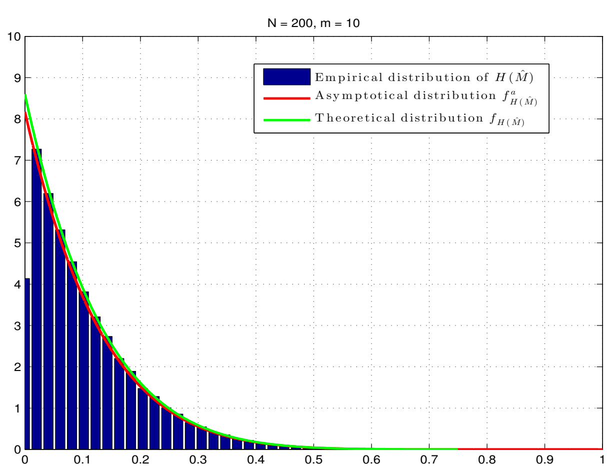

Fig. 1. Histogram distribution of $H(\widehat{\mathbf{M}})$ versus $f^a_{H(\widehat{\mathbf{M}})}$ (Eq. 22) in red and versus $f_{H(\widehat{\mathbf{M}})}$ (Eq. 12) in blue where $\widehat{\mathbf{M}}$ is the Tyler's estimator, $m = 10$, $N = 200$, $\mathbf{p} = [1, \ldots, 1]^T$, $y \sim K_\nu$ where K_ν is a K-distribution with shape $\nu = 0.1$.

In this section, the ANMF is built with the Tyler's estimator. Since this estimator is distribution-free [22], the data data are simulated according to a zero-mean complex Gaussian distribution with a covariance matrix $\mathbf{M}$ whose entries are defined as $M_{ij} = \rho^{|i-j|}$. $\mathbf{M}$ is Toeplitz and is only defined through a correlation coefficient ρ. In this section, ρ is set to 0.5.

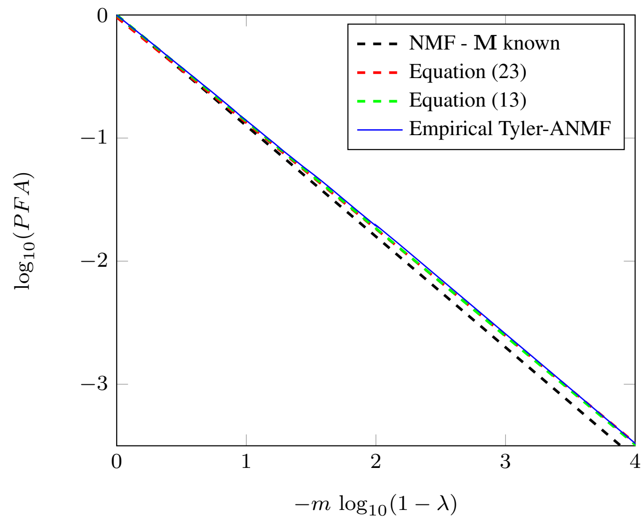

Fig. 2. Comparison between PFA-threshold relationships for the ANMF built with the Tyler's estimator, $m = 10$, $N = 200$, $\mathbf{p} = [1, \ldots, 1]^T$, $y \sim K_\nu$ where K_ν is a K-distribution with shape $\nu = 0.1$.

Figure 1 presents the empirical distribution of the ANMF built with the Tyler's estimator and the two corresponding distributions proposed in equations (12) and (22) for K-distributed secondary data ($N = 200$, $m = 10$) with shape parameter $\nu = 0.1$. The results concordance corroborates the use of these approximate distributions. Figure 2 shows the PFA-threshold relationships for the NMF ($\mathbf{M}$ is known), the first approximate distribution of the ANMF built with Tyler's estimator (equation (13)), the asymptotic expression derived in equation (23) for the Tyler's estimator and the empirical PFA for the Tyler-ANMF, for $N = 200$ and $m = 10$, for K-distributed secondary data with shape parameter $\nu = 0.1$. First, the asymptotic regime is achieved and we can observe a good match between the two asymptotic distributions derived in this paper. Moreover, this shows that these two approximations provide a very good characterization of the Tyler-ANMF behaviour (plain blue curve).

5. CONCLUSION

In the context of robust detection in Gaussian or non-Gaussian noise, two asymptotic distributions of the ANMF have been proposed. More precisely, using robust CM estimators such as the M-estimators or the Tyler's estimator, two asymptotic approximations of the corresponding ANMF distribution have been derived following different approaches. First, we have combined the exact distribution of the ANMF built with the SCM under Gaussian noise and the asymptotic properties of the robust estimators. Finally, we have directly derived the asymptotic distribution of the robust ANMF under CES environment. These results provide a very good approximation of the ANMF distribution even for a small number of observations and have been applied to theoretically regulate the false alarm probability.

6. REFERENCES

[1] E. Conte and M. Lops and G. Ricci, "Asymptotically Optimum Radar Detection in Compound-Gaussian Clutter," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 31, no. 2, pp. 617–625, 1995.

[2] F. Gini, "Sub-Optimum Coherent Radar Detection in a Mixture of K-Distributed and Gaussian Clutter," *IEE Proc. on Radar, Sonar Navig.*, vol. 144, pp. 39–48, February 1997.

[3] S. Kraut and L. L. Scharf, "The cfar adaptive subspace detector is a scale-invariant glrt," *Signal Processing, IEEE Transactions on*, vol. 47, no. 9, pp. 2538–2541, 1999.

[4] S. Kraut, L. Scharf, and L. M. Whorter, "Adaptive Subspace Detector," *IEEE Trans. on SP*, vol. 49, pp. 1–16, January 2001.

[5] F. C. Robey, D. R. Fuhrmann, E. J. Kelly, and R. Nitzberg, "A CFAR adaptive matched filter detector," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 28, no. 1, pp. 208–216, January 1992.

[6] E. J. Kelly, "An adaptive detection algorithm," *Aerospace and Electronic Systems, IEEE Transactions on*, no. 2, pp. 115–127, 1986.

[7] P. J. Huber, "Robust estimation of a location parameter," *The Annals of Mathematical Statistics*, vol. 35, no. 1, pp. 73–101, 1964.

[8] R. A. Maronna, "Robust M-estimators of multivariate location and scatter," *Annals of Statistics*, vol. 4, no. 1, pp. 51–67, January 1976.

[9] E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, "Complex elliptically symmetric distributions: Survey, new results and applications," *Signal Processing, IEEE Transactions on*, vol. 60, no. 11, pp. 5597 –5625, nov. 2012.

[10] M. Mahot, F. Pascal, P. Forster, and J.-P. Ovarlez, "Asymptotic properties of robust complex covariance matrix estimates," *Signal Processing, IEEE Transactions on*, vol. 61, no. 13, pp. 3348–3356, July 2013.

[11] F. Pascal, Y. Chitour, J. Ovarlez, P. Forster, and P. Larzabal, "Covariance structure maximum-likelihood estimates in compound Gaussian noise: existence and algorithm analysis," *Signal Processing, IEEE Transactions on*, vol. 56, no. 1, pp. 34–48, Jan. 2008.

[12] D. Tyler, "A distribution-free m-estimator of multivariate scatter," *The Annals of Statistics*, vol. 15, no. 1, pp. 234–251, 1987.

[13] L. L. Scharf and D. W. Lytle, "Signal detection in Gaussian noise of unknown level: an invariance application," *Information Theory, IEEE Transactions on*, vol. 17, pp. 404–411, July 1971.

[14] F. Pascal, J.-P. Ovarlez, P. Forster, and P. Larzabal, "Constant false alarm rate detection in spherically invariant random processes," in *Proc. of the European Signal Processing Conf.*, Vienna, September 2004, pp. 2143–2146.

[15] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. National Bureau of Standard, AMS 55, June 1964.

[16] F. Pascal, P. Forster, J. Ovarlez, and P. Larzabal, "Performance analysis of covariance matrix estimates in impulsive noise," *Signal Processing, IEEE Transactions on*, vol. 56, no. 6, pp. 2206–2217, Jun. 2008.

[17] D. Kelker, "Distribution theory of spherical distributions and a location-scale parameter generalization," *Sankhyā: The Indian Journal of Statistics, Series A*, vol. 32, no. 4, pp. 419–430, 1970.

[18] G. Frahm, "Generalized elliptical distributions: Theory and applications," Ph.D. dissertation, Universität zu Köln, 2004.

[19] M. Mahot, F. Pascal, P. Forster, and J.-P. Ovarlez, "Asymptotic properties of robust covariance matrix estimates," *Signal Processing, IEEE Transactions on*, vol. 61, no. 13, pp. 3348–3356, 2013.

[20] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, 1st ed. John Wiley & Sons, New York, 1958.

[21] K. B. Petersen and M. S. Pedersen, "The matrix cookbook," Feb. 2008, available at http://matrixcookbook.com.

[22] E. Ollila and D. E. Tyler, "Distribution-free detection under complex elliptically symmetric clutter distribution," in *Sensor Array and Multichannel Signal Processing Workshop (SAM), 2012 IEEE 7th*. IEEE, 2012, pp. 413–416.