ASSESSING RANGE ACCURACY FOR BEARINGS-ONLY GEOLOCATION USING OPTIMAL LOGARITHMIC SPIRAL SENSOR PATH TRAJECTORIES

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ABSTRACT

In this paper, we develop a procedure for evaluating the performance of a single moving sensor system to estimate the range to a stationary emitter based on the discrete-time collection of bearing measurements along the trajectory travelled. We describe a numerical procedure to calculate the range p.d.f. from a chosen trajectory assuming a constant quality for the bearing measurements and use this procedure to evaluate the range root mean-square error as a function of the distance travelled. The logarithmic spiral family of trajectories is of particular interest, both from the standpoint of optimal control where such paths are derived, and from experimental biology, where large birds of prey are observed to travel in such paths in their search for food. Our performance analysis of these scenarios indicates why pitch angles less than 45° are to be preferred when a balanced range estimation performance throughout the trajectory is desired.

Index Terms— antenna arrays, azimuthal angle, path planning, parameter estimation, spirals, statistical distributions

1. INTRODUCTION

Bearings-only geolocation algorithms have numerous indoor and outdoor applications, including emergency response, healthcare, military, and commercial applications. In two-dimensional bearings-only geolocation with a single antenna array, a wireless emitter at position $\{x_0, y_0\}$ is passively-located by its emissions using a moving sensor array collecting noisy bearing measurements $\widehat{\theta}_i$ at positions $\{x_i, y_i\}$, $i = \{1, 2, 3, \ldots\}$, where

$$\widehat{\theta}_i(x_0, y_0) = \theta_i(x_0, y_0) + \eta_i \quad (1)$$

$$\theta_i(x_0, y_0) = \arctan\left(\frac{y_0 - y_i}{x_0 - x_i}\right) \quad (2)$$

and η_i is an uncorrelated zero-mean bearing noise sequence. The goal is to reduce the root-mean-square-error (root-MSE) associated with the position estimate $\{\widehat{x}_{0,k}, \widehat{y}_{0,k}\}$, where k is the number of measurements used. In this problem statement, we assume a stationary emitter and a mobile antenna array, where the accuracy of the bearings is fixed such that η_i has a constant standard deviation δ.

An important question arises from such a formulation: What are the parameters of an optimal trajectory for a single moving sensor array to passively-locate a transmitter from a fixed starting range? Such a choice is motivated by practical concerns for several reasons, such as cost and convenience of implementation. A single moving sensor array means that no additional communication is required for data collection, and no coordination between multiple sensor systems is required. The approach has the primary constraints that the emitter to be located is both stationary and persistent. Additionally, it is assumed that the array collects measurements at a regular timing intervals, and the movement between collection points is constant. This choice is reasonable for both airborne and ground vehicles and allows the widest possible use of the results.

Past work on this problem can be found in [2, 3, 4, 17]. In [2], an optimal discrete-time control problem is formulated, and the optimal trajectory based on this formulation is shown to be the logarithmic spiral, with a constant angle of attack to the emitter position. However, no particular parameter values or numerical studies are indicated showing how performance varies according to the parameters of the problem such as sensor speed and bearing accuracy. In [3], two different approaches to the optimal trajectory path are considered, including the logarithmic spiral family, and analytical results are presented depicting error ellipses based on the trajectories calculated. However, discrete-time effects are not taken into account, as the bearings are assumed to be measured continuously. In [4], optimal trajectory design is formulated as a discrete-time optimal control problem with constraints, and both unconstrained and constrained paths are considered. The constant angle of attack case, however, is not considred in their formulation. In [17], optimal trajectory is estimated by minimizing the mean square error of predicted emitter position, estimated using the extended Kalman filter.

In this paper, we explore the performance of the logarithmic spiral path in estimating the root-mean-square-error (root-MSE) of the range of a stationary emitter with respect to the starting position for various trajectory parameters including incident angle, velocity, bearing rate, and bearing accuracy. Our focus on the logarithmic spiral is motivated not only by the work in [2], but also by work in experimental biology, where it has been observed that large birds of prey such as peregrine falcons initially fly in a path similar to that of a logarithmic spiral in search of food [10, 11, 12]. Our past studies of position root-MSE indicate that range error dominates the position root-MSE for a single moving sensor for even moderate bearing accuracies, and thus range root-MSE represents a simpler criterion from which to compare performance. The method used to evaluate the range root-MSE is a variation of the numerical procedure in [9] originally designed to estimate the achievable root-MSE of the position independent of the numerical procedure used for position estimation. Our numerical studies indicate the performance relationships between the various parameters and enable one to determine which combinations result in best performance as a function of observation time and distance.

2. LOGARITHMIC SPIRAL AND DISCRETE APPROXIMATIONS

The logarithmic spiral centered at (x_0, y_0) is defined in (x_i, y_i) coordinates by the parametric relations

$$x_i = x_0 + R\exp(-bt)\cos(\omega t) \quad (3)$$

$$y_i = y_0 + R\exp(-bt)\sin(\omega t) \quad (4)$$

where R is the initial range, ω is angular velocity, and b is an angle of incidence parameter. When $b = 0$, the logarithmic spiral becomes a circular path, and $b \to \infty$ results in a straight-line path. The logarithmic spiral has the mathematical property that the tangent to the spiral at any position (x_i, y_i) makes a constant angle or *pitch* of

$$\phi = \arctan\frac{1}{b} \quad (5)$$

with respect to the radial line to the emitter position (x_0, y_0). The logarithmic spiral is a self-similar curve that is exhibited in many natural phenomena such as the collections of stars in spinning galaxies and biological structures such as shells [16].

For position estimation in localization, the logarithmic spiral represents a particular advantage in terms of sensor array design. Once an approximate direction of the emitter has been found, a sensor array need only look in a small angle about the pitch ϕ to continue to collect bearing measurements. Thus, the system can be designed to have a highly-accurate bearing sensor system for a small angular arc, thereby reducing sensor density elsewhere along with amount of sensor calibration required. This efficiency in sensor placement has been argued as the reason for the optical characteristics of the eyes of birds of prey [12]. The spatial resolution of the foveal extent of such birds allows for the spotting of prey from distances of over 1 km. Then, due to the 40-degree pitch of the eye, these birds use only one of their eyes to direct their flight path through the air in an approximate logarithmic spiral, all the while keeping their heads pointing forward to minimize air drag [11]. Extensive observations of the spatial flight patterns of the peregrine falcon indicate this type of hunting behavior in the wild.

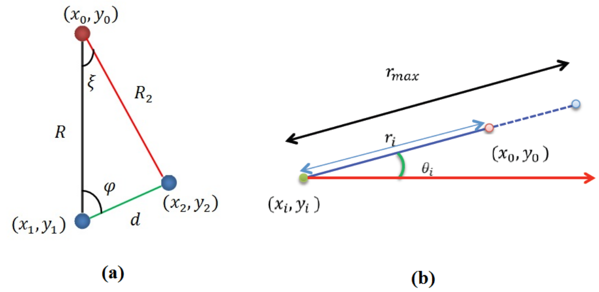

Fig. 1. Emitter - receiver geometry.

In our evaluations of the logarithmic spiral for localization, we require a discretized approximation that encompasses the physical nature of a sampled measurement system. We now describe how our trajectories are computed. Fig. 1 (a) illustrates the 2D geometry of emitter position (x_0, y_0) and two receiver positions (x_1, y_1) and (x_2, y_2) at ranges R and R_2, respectively, in a candidate logarithmic spiral path. This path is parametrized by the distance d travelled between successive points as well as the pitch ϕ. Without loss of generality, assume that $(x_0, y_0) = (0, 0)$. Using the law of cosines,

$${R_2}^2 = R^2 + d^2 - 2Rd\cos\phi \quad (6)$$

$$\sin\xi = \frac{d\sin\phi}{R_2} \quad (7)$$

$$\cos\xi = \sqrt{1 - \frac{d^2\sin^2\phi}{{R_2}^2}} \quad (8)$$

Thus, considering the receiver's second position as $(x_2, y_2) = (0, -R_2)$, the receiver's initial position is defined as

$$y_1 = \frac{d^2 - R^2 - R_2^2}{2R_2} \quad (9)$$

$$x_1 = \sqrt{R^2 - y_1^2}. \quad (10)$$

This process is repeated for successive positions to construct the spatially-discrete trajectory. Fig. 2 shows example paths computed from this procedure as used in the simulations for various pitch angles.

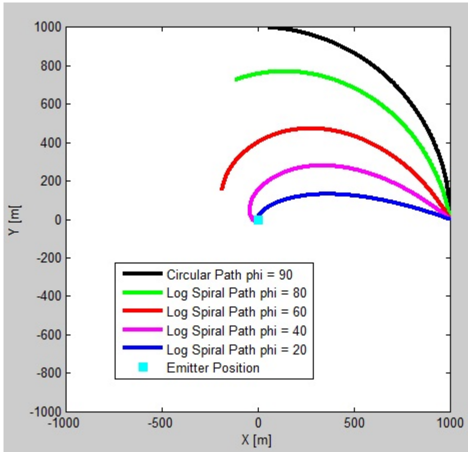

Fig. 2. Receiver trajectories used in the simulations.

3. NUMERICAL EVALUATION OF RANGE ROOT-MSE

We now describe a procedure for evaluating the range accuracy obtainable by a moving sensor platform. Our procedure employs a closed-form p.d.f. of the emitter location that has been used in a numerical procedure for evaluating of the root-MSE geolocation performance of a moving sensor array [9]. The location p.d.f. is calculated numerically assuming that the noise corrupting the Angle of arrival(AOA) measurements is zero-mean, statistically-independent from bearing to bearing, and Gaussian-distributed with variance σ^2. Past experience in the exploration of the root-MSE for such systems indicates that the primary uncertainty is in the range to the emitter. Thus, in this paper, we simplify the numerical calculation to only estimate the range root-MSE based on the projected position of sensor (x_i, y_i) on the line-of-bearing between the emitter and the initial receiver position using the p.d.f. of the measurements $\widehat{\theta}_i$ which is assumed to be known. This calculation is independent of the geolocation algorithm methodology and therefore can be used to evaluate the measurement scenario itself. In this paper, we use the range root-MSE to assess geolocation performance for different path types so that the effects of bearing accuracy σ, the constant angle-of-attack ϕ, and distance between bearing measurements d can be assessed.

The range p.d.f. is evaluated for the random variable $z \in [-R, R]$ with coordinates $(x_z, y_z) = (z \sin\xi, z \cos\xi)$ along the line between the emitter and receiver initial position. The probability density function of the location of the emitter given ith AOA measurement in Cartesian coordinates is evaluated in [9] given the model for noisy AOA measurements in (1)–(2). Fig. 1 (b) illustrates this geometry, where r_i is the distance between the emitter and the sensor array at measurement position (x_i, y_i). Assume that the measurement errors η_i are i.i.d. Gaussian with zero mean and variance σ^2.

Then,

$$f_Z(z) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma r_{max}^2} \exp\left(-\frac{(\theta_i(x_z, y_z) - \theta_i)^2}{2\sigma^2}\right) \quad (11)$$

where θ_i is the true AOA measurement, $r_{max} = R$ is the maximum value for the range coordinate, and $\theta_i(x, y)$ is defined as

$$\theta_i(x,y) = \begin{cases} \frac{\pi}{2} - \tan^{-1}\left(\frac{y-y_i}{x-x_i}\right), & (x - x_i) \geq 0, \\ \left[\frac{3\pi}{2} - \tan^{-1}\left(\frac{y-y_i}{x-x_i}\right)\right] \bmod 2\pi, & (x - x_i) < 0. \end{cases} \quad (12)$$

In this expression, (x_z, y_z) are the coordinates of the points along the line between the emitter and initial receiver position.

For a moving sensor system, the bearing measurements form a discrete-time sequence that is statistically-independent. As such, the evaluation of the range probability density function given n such measurements t consists of the product of n individual p.d.f. calculation at each time. The joint range p.d.f. of these n measurements is

$$f_{\{Z_i\}_1^n}(z) = \frac{\prod_{i=1}^n f_{Z_i}(z)}{\int_{\mathcal{R}} \prod_{i=1}^n f_{Z_i}(\xi) d\xi}, \quad (13)$$

where $\mathcal{R} = [-R, R]$ is a set of all points over the line between the emitter and receiver initial position The probability density function $f_Z(z)$ calculated in (11) is used to construct this joint p.d.f.

The probability density function in (13) is then used to evaluate the range root-MSE for a sequence of measurements along the path, given by

$$D_{rMSE}(n) = \left(\int_{\mathcal{R}} (z^2) f_{\{Z_i\}_1^n}(z) dz\right)^{1/2} \quad (14)$$

This quantity provides a lower bound on the range performance of any AOA geolocation algorithm, since this calculation is dependent on the measurements characteristics and not the geolocation methodology. Thus, it can be used to explore performance issues associated with the choise problem scenario as well as the performance of any particular algorithm applied to this scenario.

4. NUMERICAL EVALUATIONS

In this section, we explore the geolocation performance of a sensor array moving on a discretized-approximation to a logarithmic spiral using our range root-MSE evaluation procedure. In all of our examples, the emitter is assumed to be located at the origin $(x_0, y_0) = (0, 0)$ without loss of generality, and the receiver begins at a position of $R = 1000$ m from the emitter. We consider logarithmic spiral trajectories with incident angles drawn from $\phi \in \{20°, 35°, 40°, 45°, 60°, 80°, 90°\}$ where the latter corresponds to a circle of radius R, constant distances between

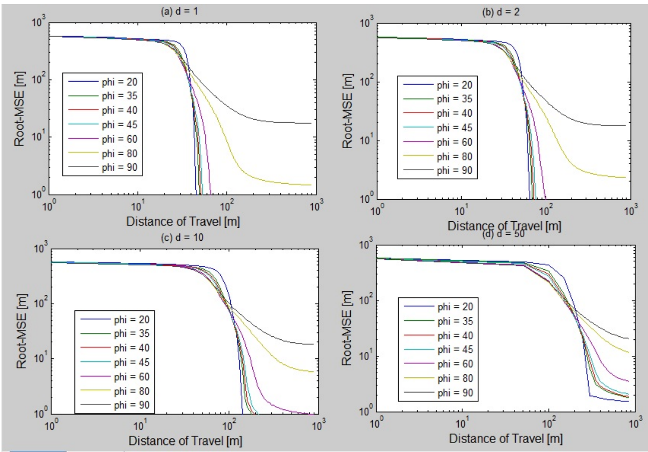

Fig. 3. Root-MSE for trajectories with incident angle $[20°, 35°, 40°, 45°, 60°, 80°, 90°]$, $\sigma = 1°$.

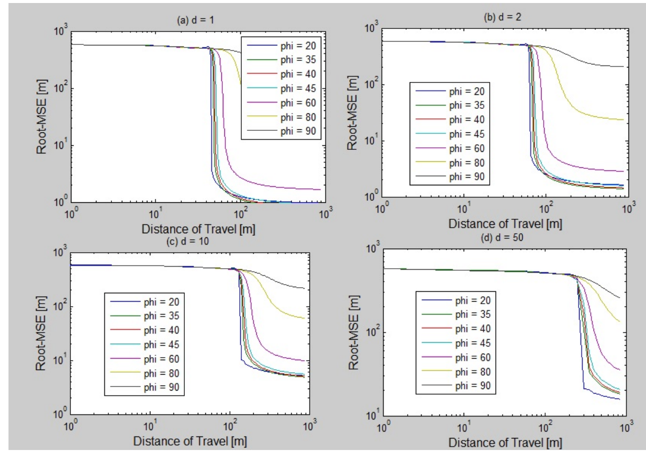

Fig. 4. Root-MSE for trajectories with incident angle $[20°, 35°, 40°, 60°, 80°, 90°]$, $\sigma = 10°$.

bearing measurements of $d \in \{1, 2, 10, 50\}$ m, and angular accuracies of $\sigma \in \{1°, 10°\}$. We express the range root-MSE in terms of the total distance travelled, which for n bearings is equal to nd. We use total distance travelled as the independent variable in order to compare with existing geolocation performance predictions for the logarithmic spiral based on a continuous-time optimal control framework [3]. In [3], it is shown that for "continuously-measured" bearings, each value of pitch angle ϕ corresponds to an optimal distance travelled as a function of the initial range R. Table 1 lists these optimal distances from a control theory standpoint as a function of the values of ϕ considered.

Table 1. Optimal values of total distance travelled for specific pitch angles ϕ based on Eqn. (27) of [3].

Incident Angle	Distance of Travel
20°	855.9510
35°	667.9105
40°	603.7851
45°	540.1815
60°	354.2487
80°	116.5517
90°	0.0000

Fig. 3 shows the range root-MSE for a geolocation system for different logarithmic spiral paths and a constant bearing accuracy of $\sigma = 1°$ for four different inter-bearing distance values d. As can be seen from these plots, larger values of d yield slower convergence of the range root-MSE for a given distance travelled, which is to be expected. In this case of more-accurate bearing measurements, paths with high pitch angles – those close to a circular path – provide superior range estimation performance for short travel distances, but their abilities to estimate range is ultimately limited by the fact that the approach to the emitter is slowed. For logarithmic spiral paths with an aggressive pitch angle closer to a direct path, the ability to estimate range is initially poor but ultimately superior as the sensor system continues to move, due to the increased angular accuracy provided by a smaller range to the emitter over time. Interestingly, there seems to be a saturation of performance in this regard for $\phi \leq 45°$, as the curves tend to collapse onto one another. From these plots, it would appear that a choice of $\phi = 40°$ corresponding to the approximate path taken by birds of prey [11] provides a balanced performance between reasonable initial reductions in range root-MSE and an accurate estimate of range for longer distances travelled, even for different rates of travel (different d values). Note that these results do not follow the predictions of Table 1, likely because of the real-world discrete-time nature of the bearing measurement process.

Fig. 4 shows the range root-MSE for an identical set of trajectories and inter-bearing distances for a constant bearing accuracy of $\sigma = 10°$, corresponding to less-accurate bearing measurements. In this case, all of the trajectories provide similar initial performances, and the best performance is ultimately obtained by the most-direct logarithmic spiral of $\phi = 20°$. These results indicate that, when the bearing measurements are inaccurate, movement around the emitter is statistically-inefficient, and the best initial strategy is to approach the emitter to make better use of the poor bearing information being collected.

5. CONCLUSIONS

In this paper, we introduce a method for calculating the range root-MSE for a moving sensor array based on the trajectory it traverses in observing bearings from a stationary emitter. The range root-MSE depends only on the measurements characteristics and receiver trajectory, and thus it is useful for studying the fundamental performance limits of a particular chosen path. We explore the performance of the logarithmic spiral in this task, and show that performance depends on bearing measurement rate as well as the pitch angle and distance travelled.

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