REFERENCE-DISTANCE ESTIMATION APPROACH FOR TDOA-BASED SOURCE AND SENSOR LOCALIZATION

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ABSTRACT

In this paper, we present a new method to find solutions to the *time difference of arrival* (TDOA)-based source and sensor localization problem. This paper is a continuation of [1], in which sources and sensors are localized on the basis of *time of arrival* (TOA) measurements. Generally, the TOA is known if the TDOA and *reference-distances* with the sound velocity are given, where the reference-distances are defined as the distances from the first (reference) sensor to the sources. We show that when the numbers of sources and sensors are at least six and eight, respectively, the reference-distances can be computed directly from TDOA measurements. This means that in such cases, the positions of the sources and sensors can be directly estimated in closed-form solutions, except for one reference-distance, which is estimated by a grid search. The validity of our algorithm is evaluated by synthetic experiments in noise-free and noisy cases.

Index Terms— Time Difference of Arrival, Time of Arrival, Reference-distance, Source and Sensor Localization

1. INTRODUCTION

Not only source localization but also sensor localization is important in a wide range of problems involving array signal processing. For example, in the *ad-hoc microphone array* problem [2, 3, 4, 5, 6, 7, 8], sensor localization has received significant attention. In this paper, we study source and sensor localization based on *time difference of arrival* (TDOA) measurements. This is a continuation of [1], in which this problem was studied on the basis of *time of arrival* (TOA) measurements. The TOA is known if the TDOA and *reference distances* with the sound velocity are given, where the *reference-distances* are defined as the distances from the first (reference) sensor to the sources. Thus, this paper focuses on estimating the reference-distances from TDOA measurements to obtain TOA values, and applying the algorithm proposed in [1] to determine the positions of sources and sensors.

Several solutions to localization based on the TDOA have been proposed. Some are iterative methods based on least-squares criteria [4, 9, 10, 11, 12] or a maximum likelihood principle [3, 13, 14, 15], and some are non-iterative methods [16, 17]. Generally, since the cost functions used in the iterative methods are nonlinear and nonconvex, they can be easily trapped at local minima. In the localization, local minima are very far from true solutions. Therefore, determining the closed-form solution for source and sensor positions has attracted considerable attention.

Recently, by using a *small rank constraint* for the distance matrix, as discussed in [18, 19, 20], Kuang et al. [16, 17] proposed novel methods to estimate source and sensor positions on the basis of TOA and TDOA measurements. Our works are similar to their works on solving TOA-based localization and applying TDOA-based localization after estimating the reference-distances. Kuang et al. showed that the positions of sources and sensors can be computed using parameters that are the solutions of multivariate quartic and cubic equations. They then used the Gröbner basis method [21] and Macaulay2 software [22] to obtain formulae for these unknown parameters and also formulae for the source and sensor positions. However, the Gröbner basis method and Macaulay2 software are not familiar to non-mathematicians and are difficult for general use.

By carefully studying the properties of the TOA and TDOA-based localizations, we prove that when the numbers of sources and sensors are at least six and eight, respectively, the positions of the sources and sensors can be computed using parameters that are the solutions of univariate quartic equations. Since there are closed-form solutions of univariate quartic equations, our method is *simple*, *accurate*, and *stable*.

2. TDOA-BASED SOURCE AND SENSOR LOCALIZATION

Let us consider M sources and N sensors, and let the positions of the sources and sensors in $\mathbb{R}^3$ be $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and $\mathbf{y}_1, \ldots, \mathbf{y}_N$, respectively. For simplicity, we hereafter refer to their positions as the $\mathbf{x}$-group and $\mathbf{y}$-group. *Source and sensor localization* is considered on the basis of the following two problems:

1. Given *distance matrix* $\mathbf{D} = (d_{mn})_{M\times N}$, where $d_{mn} =$

$\|\mathbf{x}_m - \mathbf{y}_n\|_2$, the determination of $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and $\mathbf{y}_1, \ldots, \mathbf{y}_N$.

2. Given *distance-difference matrix* $\mathbf{\Delta} = (\delta_{mn})_{M\times(N-1)}$, where $\delta_{mn} = \|\mathbf{x}_m - \mathbf{y}_{n+1}\|_2 - \|\mathbf{x}_m - \mathbf{y}_1\|_2$, the determination of $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and $\mathbf{y}_1, \ldots, \mathbf{y}_N$,

where $\|\cdot\|_2$ is the Euclidean distance. It is clear that the second problem is more general than the first problem, and we can consider the first problem as a part of the second problem.

In acoustic signal processing problem, the distance matrix $\mathbf{D}$ can be determined from the sound velocity and traveling times of sounds from each source to each sensor (TOA), and the distance-difference matrix $\mathbf{\Delta}$ can be determined from the differences between these traveling times (TDOA). Thus, in acoustic applications, the first problem is named *TOA-based source and sensor localization*, which is to determine the positions of sources and sensors from TOA measurements, and the second problem is named *TDOA-based source and sensor localization*, which is to determine the positions of sources and sensors from TDOA measurements.

Generally, TOAs are estimated when the sources and sensors are synchronous (i.e., all sources and sensors have a common clock), and TDOAs are estimated when only the sensors are synchronous (i.e., all sensors have a common clock). Therefore, the applicability of TDOA-based localization is much wider than that of TOA-based localization.

Formulae for the positions in the TOA-based localization problem have previously been found by us [1] for some cases. As a continuation of this work, in this paper we solve the TDOA-based localization problem on the basis of the following two steps: (i) we find formulae for the *reference-distances* $\|\mathbf{x}_m - \mathbf{y}_1\|_2$ using the distance-difference matrix $\mathbf{\Delta}$, and (ii) we find formulae for the $\mathbf{x}$- and $\mathbf{y}$-groups using the distance matrix $\mathbf{D} = (d_{mn})_{M\times N}$, where $d_{m1} = \|\mathbf{x}_m - \mathbf{y}_1\|_2$ and $d_{mn} = \delta_{m,n-1} + \|\mathbf{x}_m - \mathbf{y}_1\|_2$ for $n \geqslant 2$. Note that $\mathbf{D}$ and $\mathbf{\Delta}$ are invariant under reflection, translation, and rotation. To remove this ambiguity, we assume $\mathbf{x}_1 \equiv (0,0,0)^T, \mathbf{x}_2 \equiv (0,0,\alpha)^T$, and $\mathbf{y}_1 \equiv (0,\beta,\gamma)^T$ $(\alpha,\beta \geqslant 0)$.

3. REFERENCE-DISTANCE ESTIMATION

3.1. Rank constraint for distance matrix

We denote $\alpha_m = \|\mathbf{x}_m - \mathbf{y}_1\|_2$, $m = 1, \ldots, M$, as the reference-distances to be estimated. The distance matrix $\mathbf{D}$ is determined from the distance-difference matrix $\mathbf{\Delta}$ and the reference-distances as: $d_{m1} = \alpha_m$ and $d_{mn} = \delta_{m,n-1} + \alpha_m$ for $n \geqslant 2$. Note that since $d_{mn}^2 - d_{m1}^2 - d_{1n}^2 + d_{11}^2 = -2(\mathbf{x}_m - \mathbf{x}_1)^T(\mathbf{y}_n - \mathbf{y}_1)$, we have $\delta_{mn}^2 - \delta_{1n}^2 + 2\delta_{mn}\alpha_m - 2\delta_{1n}\alpha_1 = -2(\mathbf{x}_m - \mathbf{x}_1)^T(\mathbf{y}_{n+1} - \mathbf{y}_1)$. We set $\mathbf{X} = (\mathbf{x}_2 - \mathbf{x}_1, \ldots, \mathbf{x}_M - \mathbf{x}_1)$, $\mathbf{Y} = -2(\mathbf{y}_2 - \mathbf{y}_1, \ldots, \mathbf{y}_N - \mathbf{y}_1)$, and matrix $\mathbf{\Lambda} = (\lambda_{mn})_{(M-1)\times(N-1)}$, where

$$\lambda_{mn} = \delta_{m+1,n}^2 - \delta_{1n}^2 + 2\delta_{m+1,n}\alpha_{m+1} - 2\delta_{1n}\alpha_1, \qquad (1)$$

and we also have $\mathbf{\Lambda} = \mathbf{X}^T\mathbf{Y}$. It can be verified that the ranks of matrices $\mathbf{X}$ and $\mathbf{Y}$ are at most three, so the rank of $\mathbf{\Lambda}$ is at most three. Thus, the following polynomial equations are obtained for all $1 \leqslant m_1 < m_2 < m_3 < m_4 \leqslant M-1$ and $1 \leqslant n_1 < n_2 < n_3 < n_4 \leqslant N-1$:

$$\det\begin{pmatrix} \lambda_{m_1n_1} & \lambda_{m_1n_2} & \lambda_{m_1n_3} & \lambda_{m_1n_4} \\ \lambda_{m_2n_1} & \lambda_{m_2n_2} & \lambda_{m_2n_3} & \lambda_{m_2n_4} \\ \lambda_{m_3n_1} & \lambda_{m_3n_2} & \lambda_{m_3n_3} & \lambda_{m_3n_4} \\ \lambda_{m_4n_1} & \lambda_{m_4n_2} & \lambda_{m_4n_3} & \lambda_{m_4n_4} \end{pmatrix} = 0. \qquad (2)$$

Given m_1, m_2, m_3, and m_4, there are $(N-1)(N-2)(N-3)(N-4)/24$ polynomial equations expressed in (2), and each polynomial has five variables α_1, α_{m_1+1}, α_{m_2+1}, α_{m_3+1}, and α_{m_4+1}. In this paper, we assume that α_1 is a known value and determine closed-form solutions of α_{m_1+1}, α_{m_2+1}, α_{m_3+1}, and α_{m_4+1} in terms of the value of α_1 and the polynomial equations given in (2). Then, we can simply apply a grid search to determine α_1. We prove that when $M \geqslant 5$ and $N \geqslant 8$, the variables α_{m_1+1}, α_{m_2+1}, α_{m_3+1}, and α_{m_4+1} can be determined from $\mathbf{\Delta}$ and α_1 by simple closed-form solutions. Further more, when $M \geqslant 6$, we also prove that the variable α_1 can be determined on the basis of differences, for example, the difference in α_2 when it is estimated in the cases of $(m_1, m_2, m_3, m_4) \equiv (1,2,3,4)$ and $(m_1, m_2, m_3, m_4) \equiv (1,2,3,5)$.

3.2. Linear method of solving polynomial equations

Assume that α_1 is known. Letting $a_{mn} = \delta_{m+1,n}^2 - \delta_{1n}^2 - 2\delta_{1n}\alpha_1$ and $b_{mn} = 2\delta_{m+1,n}$, (1) implies that $\lambda_{mn} = a_{mn} + b_{mn}\alpha_{m+1}$. Given m_1, m_2, m_3, m_4, for each $1 \leqslant n_1 < n_2 < n_3 < n_4 \leqslant N-1$, (2) gives a polynomial equation in four variables α_{m_1+1}, α_{m_2+1}, α_{m_3+1}, α_{m_4+1} corresponding to 16 monomials $\mathbf{T} = \{z_1z_2z_3z_4\ ,\ z_1z_2z_3\ ,\ z_1z_2z_4\ ,\ z_1z_3z_4\ ,\ z_2z_3z_4\ ,\ z_1z_2\ ,\ z_1z_3\ ,\ z_1z_4\ ,\ z_2z_3\ ,\ z_2z_4\ ,\ z_3z_4\ ,\ z_1\ ,\ z_2\ ,\ z_3\ ,\ z_4\ ,\ 1\}$, where $z_1 = \alpha_{m_1+1}$, $z_2 = \alpha_{m_2+1}$, $z_3 = \alpha_{m_3+1}$, $z_4 = \alpha_{m_4+1}$. Let $B_{n_1n_2n_3n_4}$ be the vector of coefficients of the polynomial corresponding to (n_1, n_2, n_3, n_4) and the order of the monomials $\mathbf{T}$. The vector $B_{n_1n_2n_3n_4}$ is given in Table 1 and is simply expressed in terms of $a_{m_in_j}$ and $b_{m_in_j}$ with $1 \leqslant i,j \leqslant 4$, and $\mathbf{T}B_{n_1n_2n_3n_4}^T = 0$. Let $\mathbf{B}_{m_1m_2m_3m_4}$ be the matrix whose rows are $B_{n_1n_2n_3n_4}$ for all $n_1, \ldots, n_4$. $\mathbf{B}_{m_1m_2m_3m_4}$ has a size of $[(N-1)(N-2)(N-3)(N-4)/24] \times 16$ and

$$\mathbf{T}\mathbf{B}_{m_1m_2m_3m_4}^T = \mathbf{0}, \qquad (3)$$

where $\mathbf{0}$ is the zero vector. The solvability of (3) depends on the rank of $\mathbf{B}_{m_1m_2m_3m_4}$, which is given by the following lemma.

Lemma 1. *If $N \geqslant 8$ and the $\mathbf{x}$-group is full rank, i.e., none of the points in the $\mathbf{x}$-group lie on the same plane, and the $\mathbf{y}$-group is also full rank, we have* $rank(\mathbf{B}_{m_1m_2m_3m_4}) = 15$.

Table 1. Formula for $B_{n_1n_2n_3n_4}$, given m_1, m_2, m_3, and m_4.

$$
\begin{aligned}
B_{n_1n_2n_3n_4} &= A_{4321} + A_{4213} + A_{4132} - A_{4123} - A_{4312} - A_{4231} + A_{1423} + A_{2431} + A_{3412} - A_{3421} - A_{2413} - A_{1432} \\
&\quad + A_{3241} + A_{2143} + A_{1342} - A_{1243} - A_{2341} - A_{3142} + A_{1234} + A_{2314} + A_{3124} - A_{3214} - A_{2134} - A_{1324} \\
A_{i_1i_2i_3i_4} &= (e_{i_1i_2i_3i_4}\,,\, e_{\bar{i_4}i_1i_2i_3}\,,\, e_{\bar{i_3}i_1i_2i_4}\,,\, e_{\bar{i_2}i_1i_3i_4}\,,\, e_{\bar{i_1}i_2i_3i_4}\,,\, e_{\bar{i_3}\bar{i_4}i_1i_2}\,,\, e_{\bar{i_2}\bar{i_4}i_1i_3}\,,\, e_{\bar{i_2}\bar{i_3}i_1i_4}\,,\, e_{\bar{i_1}\bar{i_4}i_2i_3}\,,\, e_{\bar{i_1}\bar{i_3}i_2i_4}\,, \\
&\quad e_{\bar{i_1}\bar{i_2}i_3i_4}\,,\, e_{\bar{i_2}\bar{i_3}\bar{i_4}i_1}\,,\, e_{\bar{i_1}\bar{i_3}\bar{i_4}i_2}\,,\, e_{\bar{i_1}\bar{i_2}\bar{i_4}i_3}\,,\, e_{\bar{i_1}\bar{i_2}\bar{i_3}i_4}\,,\, e_{\bar{i_1}\bar{i_2}\bar{i_3}\bar{i_4}}),
\end{aligned}
$$

where $e_{j_1j_2j_3j_4} = c_{m_1n_{j_1}}c_{m_2n_{j_2}}c_{m_3n_{j_3}}c_{m_4n_{j_4}}$ with $c_{m_kn_{j_h}} = a_{m_kn_{j_h}}$ if $j_h = i_h$, and $c_{m_kn_{j_h}} = b_{m_kn_{j_h}}$ if $j_h = \bar{i_h}$.

Since the length of the paper is limited, we do not give the proof of the lemma here. A *linear method of solving the polynomial equations* in (3) is explained as follows: Let $\mathbf{Q}$ and $\mathbf{R}$ be the *QR-factorization* of $\mathbf{B}_{m_1m_2m_3m_4}$, where $\mathbf{Q}$ is a unitary matrix and $\mathbf{R}$ is an upper triangular matrix [23]. Then we have the following lemma.

Lemma 2. *If* $rank(\mathbf{B}_{m_1m_2m_3m_4}) = 15$ *and* α_{m_1+1}, α_{m_2+1}, α_{m_3+1}, α_{m_4+1} *are solutions of* (3), *it can be verified that*

$$
\begin{aligned}
\alpha_{m_4+1} &= -\frac{R_{15,16}}{R_{15,15}}\,,\ \alpha_{m_3+1} = -\frac{R_{14,16}}{R_{14,14}} - \frac{R_{14,15}}{R_{14,14}}\alpha_{m_4+1} \\
\alpha_{m_2+1} &= -\frac{R_{13,16}}{R_{13,13}} - \frac{R_{13,15}}{R_{13,13}}\alpha_{m_4+1} - \frac{R_{13,14}}{R_{13,13}}\alpha_{m_3+1} \\
\alpha_{m_1+1} &= -\frac{R_{12,16}}{R_{12,12}} - \frac{R_{12,15}}{R_{12,12}}\alpha_{m_4+1} - \frac{R_{12,14}}{R_{12,12}}\alpha_{m_3+1} \\
&\quad - \frac{R_{12,13}}{R_{12,12}}\alpha_{m_2+1},
\end{aligned} \tag{4}
$$

where R_{ij} *denotes the* (i,j)*-element of* $\mathbf{R}$.

Proof. Because the solutions of (3) are the solutions of $\mathbf{TR}^T = \mathbf{0}$, $\mathbf{R}$ is an upper triangular matrix with 16 columns, and $rank(\mathbf{R}) = rank(\mathbf{B}_{m_1m_2m_3m_4}) = 15$, it can be verified that $R_{i,i} \neq 0,\ R_{j,i} = 0\ (1 \leqslant j < i \leqslant 15)$ and (4) is satisfied. □

3.3. Reference-distances estimation algorithm

In this subsection, we propose a method to estimate the reference-distances $\alpha_m, 1 \leqslant m \leqslant M$ for the cases $M \geqslant 6$ and $N \geqslant 8$. In our problem, we assume that the $\mathbf{x}$-group and $\mathbf{y}$-group are *full rank* and that the distances between points in the $\mathbf{x}$-group and points in the $\mathbf{y}$-group are finite.

3.3.1. Estimation of α_m ($m \geqslant 2$) by averaging of all combinations

Lemma 1 and Lemma 2 confirm that using any value of $\hat{\alpha}_1$ to estimate α_1 and $m \geqslant 2$, for each $1 \leqslant m_1 < m_2 < m_3 \leqslant M-1$, $m_1, m_2, m_3 \neq m-1$, (4) gives an estimate of α_m that is denoted by $\hat{\alpha}_{(m|m_1+1,m_2+1,m_3+1)}$. This is a function of $\hat{\alpha}_1$. Generally, a different triplet (m_1, m_2, m_3) gives a different estimate of α_m. Thus, α_m is naturally estimated by the following formula:

$$
\hat{\alpha}_m = \sum_{\substack{1 \leqslant m_1 < m_2 < m_3 \leqslant M-1 \\ m_1, m_2, m_3 \neq m-1}} \frac{6\hat{\alpha}_{(m|m_1+1,m_2+1,m_3+1)}}{(M-2)(M-3)(M-4)}. \tag{5}
$$

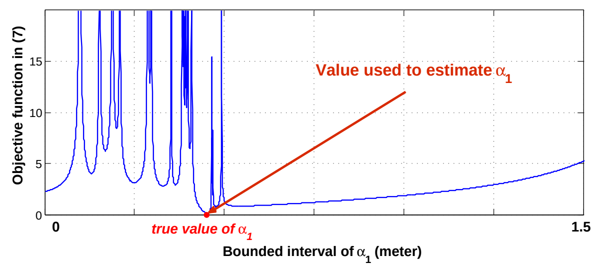

Fig. 1. Example of determining α_1 for $M = 6, N = 8$. The red points are minimum points.

The error in the estimation of α_m is given by

$$
\mathcal{E}(\alpha_m) = \sum_{\substack{1 \leqslant m_1 < m_2 < m_3 \leqslant M-1 \\ m_1, m_2, m_3 \neq m-1}} \frac{6\left(\hat{\alpha}_m - \hat{\alpha}_{(m|m_1+1,m_2+1,m_3+1)}\right)^2}{(M-2)(M-3)(M-4)}. \tag{6}
$$

3.3.2. Grid search for α_1

The condition $M \geqslant 6$ infers $(M-2)(M-3)(M-4)/6 > 1$, and $\mathcal{E}(\alpha_m)$ is positive if the values $\hat{\alpha}_{(m|m_1+1,m_2+1,m_3+1)}$ are not equal. It can be verified that if $\hat{\alpha}_1 = \alpha_1$, the error value $\mathcal{E}(\alpha_m)$ should be zero. Thus, we propose

$$
\alpha_1 = \arg\min_{\hat{\alpha}_1} \sum_{m=2}^{M} \mathcal{E}(\alpha_m). \tag{7}
$$

To solve (7), an upper bound of α_1 is needed. Fortunately, in most acoustic applications, we generally have an upper bound of α_1. When the upper bound of α_1 is given, the value of α_1 can be found by (7) if we consider $\hat{\alpha}_1$ as a running parameter on the bounded interval of α_1. An example of the solution of (7) is given in Figure 1. In this example, $M + N$ points in the $\mathbf{x}$- and $\mathbf{y}$-groups are chosen as independently and uniformly distributed points inside a cube of side 1 m. The upper bound of α_1 is 1.5 m. When α_1 is determined, the parameters α_m ($m \geqslant 2$) are determined from (4) and (5).

4. SOURCE AND SENSOR LOCALIZATION BASED ON ESTIMATED DISTANCE-MATRIX

In the previous section, we confirmed that the reference-distances $\alpha_m = \|\mathbf{x}_m - \mathbf{y}_1\|_2$ can be estimated from the distance-difference matrix $\mathbf{\Delta}$. In such a case, the distance matrix $\mathbf{D}$ can be determined as follows: $d_{m1} = \alpha_m$ and

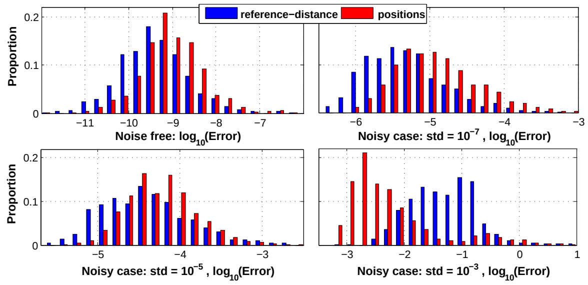

Fig. 2. Histograms of $\log_{10}(\mathcal{E}_{ref})$ and $\log_{10}(\mathcal{E}_{pos})$ for 1000 independent experiments for noise-free and noisy cases.

$d_{mn} = \alpha_m + \delta_{m,n-1}$ for $n \geqslant 2$. Now, we focus on the problem of how to estimate $\mathbf{x}_1, \ldots, \mathbf{x}_M$ and $\mathbf{y}_1, \ldots, \mathbf{y}_N$ from $\mathbf{D}$. In [1], we completely solved this problem for the cases of $N \geqslant 8$. The main ideas obtaining this result are briefly presented as follows: From subsection 3.1, we know that $\mathbf{\Lambda} = \mathbf{X}^T\mathbf{Y}$. The full-rank property of the $\mathbf{x}$-group and $\mathbf{y}$-group infers that the rank of $\mathbf{\Lambda}$ is three. By factorizing $\mathbf{\Lambda}$ using, for example, singular value decomposition, we have two matrices $\mathbf{U}$ and $\mathbf{V}$, i.e., $\mathbf{\Lambda} = \mathbf{U}^T\mathbf{V}$. $\mathbf{U}$ and $\mathbf{V}$ are used to determine $\mathbf{X}$ and $\mathbf{Y}$, respectively, in the sense that there exists an invertible (3×3) matrix $\mathbf{L}$ such that $\mathbf{X} = \mathbf{L}^{-T}\mathbf{U}$ and $\mathbf{Y} = \mathbf{L}\mathbf{V}$. Thus, if $\mathbf{L}$, $\mathbf{x}_1$, and $\mathbf{y}_1$ are known, then $\mathbf{x}_m$ and $\mathbf{y}_n$ are known for all m, n. As our assumption, $\mathbf{x}_1$ is known. $\mathbf{L}$ and $\mathbf{y}_1$ are determined by the following proposition.

Proposition 1. *$\mathbf{L}$ and $\mathbf{y}_1$ are computed directly by (i) $\alpha = \|\mathbf{x}_1 - \mathbf{x}_2\|_2$, which is a solution of a quartic equation whose coefficients are given by $\mathbf{D}$ ($N \geqslant 9$), or (ii) a solution of another quartic equation whose coefficients are given by α and $\mathbf{D}$ ($N = 8$).*

Proposition 1 is a combination of Proposition 1 and Proposition 2 in [1].

5. EVALUATION AND CONCLUSION

To evaluate our formulae, we consider the error $\mathcal{E}_{ref} = \left(\frac{1}{M}\sum_{m=1}^{M}(\hat{\alpha}_m - \alpha_m)^2\right)^{1/2}$ in the estimation of the reference-distances and the error $\mathcal{E}_{pos} = \left(\frac{1}{M+N}\sum_{m=1}^{M}\|\hat{\mathbf{x}}_m - \mathbf{x}_m\|_2^2 + \sum_{n=1}^{N}\|\hat{\mathbf{y}}_n - \mathbf{y}_n\|_2^2\right)^{1/2}$ in the estimation of positions. These errors are evaluated in 1000 independent experiments, in each of which M is chosen uniformly from $\{6, \ldots, 50\}$, N is chosen uniformly from $\{8, \ldots, 50\}$, and the source and sensor positions are simulated as independently and uniformly distributed points inside a cube of size 1 m. Many different levels of independent Gaussian noise are added to the distance-difference matrix, i.e., $\mathbf{\Delta}_{std} \leftarrow \mathbf{\Delta} + std * \mathcal{N}(0, 1)$, where std denotes a different level (*meter*) and $\mathcal{N}(0, 1)$ denotes an $(M, N-1)$ Gaussian matrix with zero mean and identity covariance to study a *noisy environment*. The errors obtained in the experiments are shown in Figure 2.

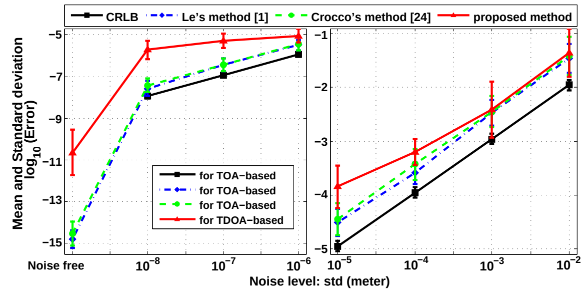

Fig. 3. *Mean* and *standard deviation* of $\log_{10}(\mathcal{E}_{pos})$ compared with CRLB in 1000 independent synthetic experiments.

For the case of estimating source and sensor positions, Figure 3 presents a comparison of results obtained using (i) the proposed method (using TDOA measurement), (ii) Le's method ([1], using TOA measurement), (iii) the closed-form solution of Crocco's method ([24], using TOA measurement), and (iv) *Cramér-Rao lower bound* (CRLB) for TOA-based localization estimation [3]. Since the information of TOA measurement contains the information of TDOA measurement, the above comparison is valid for evaluating the estimation accuracy of our method for TDOA-based localization. Because the closed-form solution of Crocco's method only works for $M, N \geqslant 7$ and $\mathbf{x}_1 \equiv \mathbf{y}_1$, the synthetic experiments whose results are shown in Figure 3 are set up assuming these conditions. For noisy cases, independent Gaussian noises are added to distance matrices in the TOA-based method, and distance-difference matrices are computed from noisy distance matrices in the TDOA-based method. The *mean* and *standard deviation* of $\log_{10}(\mathcal{E}_{pos})$ in 1000 independent experiments are computed.

From the results given in Figure 2 and Figure 3, we conclude that our estimations for the TOA-based [1] and TDOA-based (this paper) localizations are accurate and stable. Moreover, since our formulae have a closed-form with an unknown *grid-search* parameter, i.e., $\alpha = \|\mathbf{x}_2 - \mathbf{x}_1\|_2$ for TOA-based localization and $\alpha_1 = \|\mathbf{y}_1 - \mathbf{x}_1\|_2$ for reference-distance estimation, our algorithms are simple.

In summary, on the basis of [1], we propose an efficient method for TDOA-based source and sensor localization by estimating the reference-distances from TDOA measurements when the numbers of sources and sensors are at least six and eight. The validity of our method is demonstrated by performing synthetic experiments and comparisons with the efficient method proposed by Crocco et al. [24] and the CRLB.

6. ACKNOWLEDGEMENT

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