SINR LOSS OF THE DOMINANT MODE REJECTION BEAMFORMER

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ABSTRACT

The signal to interference and noise ratio (SINR) characterizes a beamformer's ability to attenuate unwanted signals. SINR loss quantifies how close the performance of a beamformer designed with sample statistics is to the performance of a beamformer designed with ensemble statistics. This paper conjectures that the SINR loss of the dominant mode rejection (DMR) beamformer is beta-distributed and provides substantial numerical evidence supporting this claim. Building on results from random matrix theory, the paper proposes a simple perturbative model for the sample covariance eigenvectors, which are known to control DMR performance [1]. SINR loss for the proposed model agrees with the beta distribution conjecture.

***Index Terms*—** adaptive arrays, beamforming, random matrix theory, sample covariance matrix

1. INTRODUCTION

Adaptive beamformers (ABFs) facilitate the detection of quiet sources using weight vectors that place nulls in the directions of loud interfering signals. Typically these weight vectors are designed using a priori knowledge or measurements of the second-order statistics of the array data. The standard minimum variance distortionless response (MVDR) ABF has the weight vector $\mathbf{w}$ [2]:

$$\mathbf{w} = (\mathbf{v}_s^{\mathrm{H}}\mathbf{\Sigma}^{-1}\mathbf{v}_s)^{-1}\mathbf{\Sigma}^{-1}\mathbf{v}_s, \tag{1}$$

where $\mathbf{\Sigma}$ is the ensemble covariance matrix (ECM) of the narrowband data and $\mathbf{v}_s$ is the planewave replica vector for the steering direction. Since the ECM is usually unavailable, it is replaced by the sample covariance matrix (SCM) estimated from snapshots of data. In many applications, only a few snapshots are available to estimate the SCM. Researchers have designed a number of beamformers to operate in snapshot-starved environments. This paper focuses on the dominant mode rejection (DMR) beamformer proposed by Abraham and Owsley [3]. The DMR ABF replaces the ECM with a structured estimate derived from the eigenvectors and eigenvalues of the SCM. The structured covariance requires fewer snapshots to estimate since it models only the D loudest interferers.

The output signal-to-interference and noise ratio (SINR) quantifies a beamformer's ability to attenuate unwanted signals. Since adaptive beamformers are implemented using sample statistics (as opposed to ensemble statistics), SINR loss is often used as a performance metric. SINR loss is defined as the ratio of the SINR for an adaptive beamformer designed using L snapshots to the SINR for a beamformer designed using ensemble statistics. In a seminal paper Reed et al. showed that the SINR loss of the MVDR beamformer is beta-distributed [4]. The beta distribution is a two-parameter distribution:

$$\beta(a,b) = \left(\frac{(a+b-1)!}{(a-1)!(b-1)!}\right)\rho^{a-1}(1-\rho)^{b-1} \tag{2}$$

Reed et al. showed that the beta distribution parameters are functions of the number of sensors (N) and the number of snapshots (L): $a = L-N+2$ and $b = N-1$. The mean of the beta distribution is $a/(a+b)$. Using this, Reed et al. conclude that the number of snapshots must be greater than twice the number of sensors to achieve a mean SINR loss greater than or equal to 0.5.

In previous work no one has derived the distribution of SINR loss for the DMR ABF. Some analysis has been done for a related class of eigenspace (ES) ABFs. Kirsteins and Tufts [5, 6] and Gierull [7] found that the SINR loss for a reduced-rank ES beamformer is beta-distributed with parameters $a = L - D + 1$ and $b = D$. It follows that $L > 2D - 1$ snapshots are required for the ES ABF to attain a mean SINR loss of greater than or equal to 0.5. There is a key difference between the ES and DMR beamformers. The ES ABF projects the data into the subspace spanned by the eigenvectors associated with the D largest eigenvalues and then processes the data in the reduced-dimension subspace using a standard MVDR ABF. The DMR ABF uses the eigendecomposition to form a structured estimate of the covariance matrix, but unlike the ES ABF, it operates on the received data in the full N-dimensional space.

The goal of this paper is to investigate the SINR loss of the DMR ABF. We build on new results from random matrix theory (RMT) that predict the behavior of the eigenvalues and eigenvectors of sample covariance matrices for large arrays and low numbers of snapshots [8, 9, 10]. Mestre used results from RMT to analyze the SINR performance of the diagonally-loaded MVDR beamformer [11]. In previous work we showed that the accuracy of the sample eigenvectors controls DMR performance [1]. Using RMT predictions of generalized cosine between ensemble and sample eigenvectors, we derived a model for the notch depth of the DMR ABF for a single interferer [12, 13]. Notch depth (ND) is defined as the absolute value squared of the beampattern in the interferer direction.

In 2013 we developed approximate expressions for the distributions of notch depth and SINR loss of the DMR beamformer in the case of a single interferer [14]. The expressions are based on approximating SINR loss ρ as follows:

$$\rho = \frac{\mathrm{SINR}}{\mathrm{SINR_{ens}}} \approx \frac{1}{N \cdot \mathrm{INR} \cdot \mathrm{ND} + 1}. \tag{3}$$

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The approximation assumes that the interferer is outside the mainlobe of the steering direction, which implies that the white noise gain is approximately equal to the number of sensors N. Eq. 3 indicates that ρ is a function of ND and the parameters N and INR (interference-to-noise ratio). Our RMT model [12] predicts the mean notch depth. As a part of the analysis in [14], we showed that the mean and standard deviation of ND are approximately equal for a wide range of INRs and snapshots. The exponential distribution is an example of a distribution whose mean and standard deviation are equal. A Kolmogorov-Smirnov (KS) test [15] indicates that the exponential model fits the ND of the single-interferer simulation data when $L \geq 32$ and INR ≥ 30 dB. Assuming that ND is exponentially distributed with a known mean, it is possible to derive the cumulative distribution function for ρ using (3) and differentiate to obtain the probability distribution function (PDF). The result is the PDF f_ρ:

$$f_\rho(\rho) \approx \frac{Le^L}{\rho^2} e^{-L/\rho}. \qquad (4)$$

The approximate distributions of ND and ρ derived in [14] represent an initial attempt to characterize the convergence rate of the DMR beamformer. While the results are reasonable, they have some limitations. First, the PDF of notch depth is not well modeled by an exponential when the number of snapshots is very small. Second, the exponential distribution is based on an empirical fit of the data, making it difficult to extend these results to the multiple interferer case. Given that the SINR loss of other ABFs can be modeled with beta distributions (with different parameters), it is worth considering whether the SINR loss of the DMR ABF fits a beta distribution.

The rest of the paper is organized as follows. Sections 2 and 3 investigate whether a beta distribution can be used to model SINR loss for the DMR ABF for single interferers and multiple interferers, respectively. Section 4 introduces a RMT-based perturbation model for the sample eigenvectors that accurately models the observed SINR loss. Section 5 concludes the paper.

2. SINR LOSS FOR A SINGLE INTERFERER

This section investigates the SINR loss of the DMR ABF for the case of a single interferer in noise. The DMR weight vector has the same form as the MVDR weight vector in Eq. 1 with $\mathbf{\Sigma}$ replaced by the structured estimate $\mathbf{S}_{\text{DMR}}$:

$$\mathbf{S}_{\text{DMR}} = \underbrace{\sum_{n=1}^{D} g_n \mathbf{e}_n \mathbf{e}_n^{\text{H}}}_{\text{largest e-vals}} + \sum_{n=D+1}^{N} s_w^2 \mathbf{e}_n \mathbf{e}_n^{\text{H}}, \qquad (5)$$

where $\mathbf{e}_n$ and g_n are the eigenvectors and eigenvalues of the SCM. DMR assumes that the first D eigenvectors are associated with the strong interferers that the ABF needs to null. In this paper we assume that the rank of the interference subspace is known. In practice D is estimated using standard algorithms, e.g. [16]. s_w^2 is an estimate of the noise power obtained by averaging the remaining eigenvalues (those not associated with the interferers). Most DMR implementations include mainlobe protection to prevent a strong source near the mainlobe from suppressing the desired signal [17, 18]. For the simulations below we ensure that the interferers are outside the mainlobe. Since the goal is to characterize the DMR ABF's ability to null loud interference, the simulations do not include a desired source. This is equivalent to assuming that the source is perfectly matched to the steering direction and passes the ABF with unity gain.

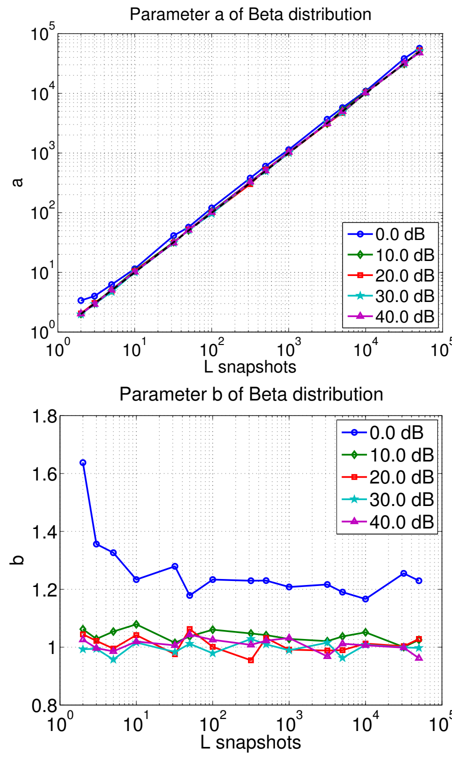

Fig. 1. ML estimates of the parameters of the beta distribution for simulated SINR loss data. The environment consists of one interferer located near the peak sidelobe of the conventional beamformer (CBF) and spatially white noise. Parameter estimates are based on 3000 Monte Carlo trials for each of the 5 values of INR (0 dB to 40 dB) and 15 values of snapshots (2 to 50,000).

Consider the case of a single interferer in spatially white noise. The array has $N = 50$ elements with half-wavelength spacing. The interferer amplitude is modeled as a circular complex Gaussian random variable. The INR and the number of snapshots used to estimate the SCM are parameters of the simulation. A set of 3,000 Monte Carlo trials is generated for each set of parameters. The output SINR of the DMR ABF is computed for each of the trials. The simulated SINR values are used to fit the parameters of a beta distribution. Fig. 1 shows the maximum likelihood (ML) estimates of the a and b parameters of the beta distribution. The plots show that for INRs above 0 dB, the a parameter is approximately equal the number of snapshots (L) and the b parameter is approximately equal to 1. A KS test (5% significance) confirms that a beta distribution with $a = L$ and $b = 1$ is an excellent fit for the simulation data for INR ≥ 0 dB.

If the SINR loss is beta-distributed, it is straightforward to use the relationship between ND and SINR to show that ND is a scaled beta-prime random variable:

$$f_{ND} \sim \beta' = \frac{N \cdot \text{INR} \cdot L}{(1 + N \cdot \text{INR} \cdot \text{ND})^{(L+1)}}. \qquad (6)$$

The histogram in Fig. 2 shows that the beta-prime distribution fits the simulated ND data well for low numbers of snapshots, which was not the case for the exponential distribution considered in our

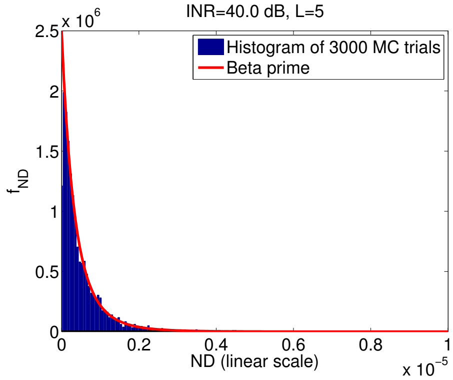

Fig. 2. Beta-prime distribution overlaid on the notch depth histogram for the single interferer case described in Fig. 1. The INR is 40 dB and the DMR ABF was computed using $L = 5$ snapshots.

previous work [14]. The mean and standard deviation of the beta-prime distribution for ND are:

$$\mathcal{E}(\text{ND}) = \frac{1}{N \cdot \text{INR} \cdot (L-1)} \tag{7}$$

$$std(\text{ND}) = \frac{1}{N \cdot \text{INR} \cdot (L-1)} \cdot \frac{\sqrt{L}}{\sqrt{L-2}}. \tag{8}$$

Note that for large L, the mean and standard deviation converge to the same value, which was the observation that lead us to initially consider the exponential distribution.

The mean of the beta distribution for the single interferer case is $L/(L+1)$, which closely tracks the mean of the Monte Carlo simulations. The overall conclusion is that the beta distribution with parameters $a = L$ and $b = 1$ is an excellent fit for the SINR loss of the DMR ABF in the case of a single interferer.

3. SINR LOSS FOR MULTIPLE INTERFERERS

Given that the SINR loss for the DMR ABF is beta-distributed for single interferers, it is reasonable to investigate whether the beta distribution is valid for the case of multiple interferers. This section describes the set of Monte Carlo simulations designed to address this question. The array has 50 elements with half-wavelength spacing and is steered to broadside. The simulation environment includes spatially white noise plus D interferers located outside of the mainlobe of the steering direction; D is one of the simulation parameters. The interferer directions are chosen to guarantee a rank-D interference subspace. INRs vary from 40 dB for the first interferer down to 11 dB for the 30th interferer, in increments of 1 dB. The interferers are independent, with amplitudes modeled by independent complex circular Gaussian random variables. The analysis below uses 3000 Monte Carlo trials for each set of parameters. The simulations include 15 different snapshot values between 2 and 50,000.

The first step is to fit the parameters of beta distribution using the SINR loss from the Monte Carlo simulations. Fig. 3 shows the ML estimates of the beta distribution parameters derived for six different values of D. The plots demonstrate that a is approximately equal to $L - D + 1$ and that b is approximately equal to D. Assuming a and b are equal to these integer values, we use a KS test to analyze how well the measured ρ values fit the beta distribution. The test confirms that SINR loss is beta-distributed for the multiple interferer simulation described above.

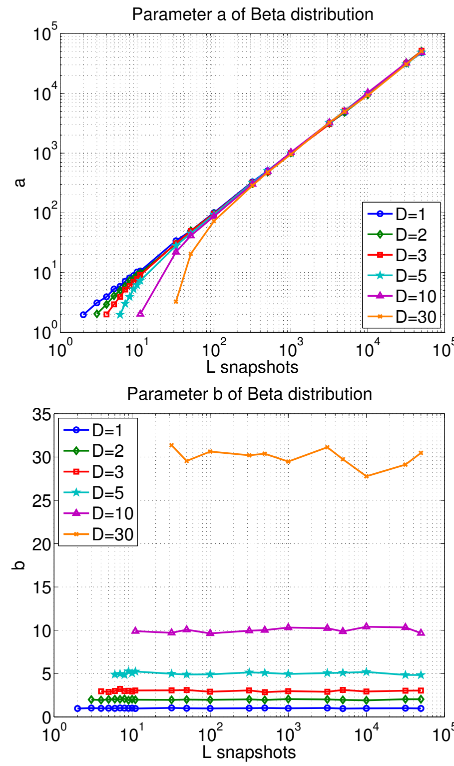

Fig. 3. ML estimates of the beta distribution parameters for the multi-interferer cases. The environment contains D interferers located outside the mainlobe of the steering direction with INRs ranging from 11 to 40 dB. Estimates are based on 3000 Monte Carlo trials for 15 different snapshot levels (2 to 50,000).

The beta distribution is an excellent fit to the SINR loss data for a wide range of parameter values. Fig. 4 shows histograms for the 5-interferer case for three different numbers of snapshots: $L = 6, 10, 100$. The predicted beta distribution (dashed lines) matches the simulation data very well. Note that $L = 6$ is the minimum number of snapshots that could be used to attenuate 5 interferers while maintaining a unity gain constraint in the look direction.

Using the properties of the beta PDF, the mean of SINR loss for the multiple interferer case is equal to $(L - D + 1)/(L + 1)$. Fig. 5 compares the predicted mean to the simulation mean for six different numbers of interferers. The agreement between predictions and simulations is remarkable. Based on the mean prediction we conclude that $L > 2D - 1$ snapshots are required to achieve an expected SINR loss of greater than or equal to 0.5. This is identical to the results for the ES ABFs referenced in the introduction [5, 6, 7]. It is also consistent with other analyses of reduced-rank methods [19, 20], which indicate that the number of snapshots required is proportional to the number of interferers, rather than the number of sensors.

4. EIGENVECTOR PERTURBATION MODEL

In previous work we showed that the sample eigenvectors control DMR performance [1]. RMT predicts that the sample eigenvectors have a phase transition [8, 9, 10]. When the INR is above the threshold, the sample eigenvector is a biased estimate of the true eigenvec-

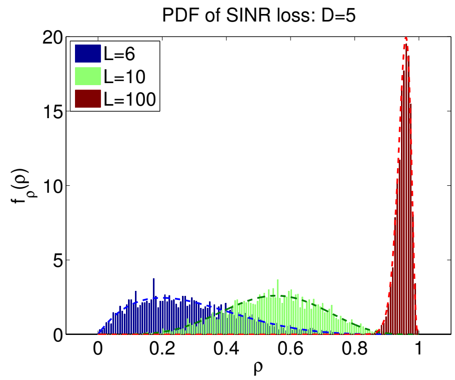

Fig. 4. Comparison of SINR loss histograms with the beta PDF for the 5-interferer case. Results are shown for $L = 6, 10, 100$. The bar graphs show the histograms for 3000 Monte Carlo simulations and the dashed lines show the predicted beta PDF.

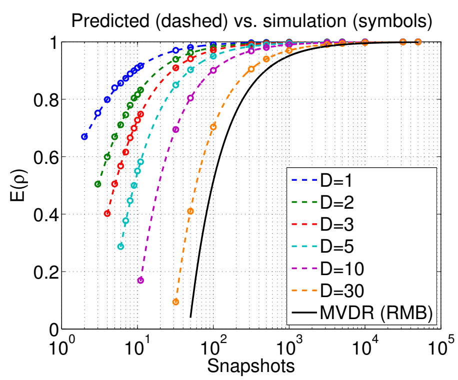

Fig. 5. Mean SINR loss versus snapshots. The dashed lines are the beta distribution predictions and the circles are the simulation results. Results are shown for different numbers of interferers: $D = 1, 2, 3, 5, 10, 30$. The black line shows the MVDR prediction [4].

tor. The sample eigenvector lies on a cone around the true eigenvector, as illustrated in Fig. 6. RMT predicts the radius of the cone in the limit as both N and L approach infinity. As INR and/or number of snapshots increases, the bias, i.e., the radius of the cone, decreases. For a single interferer, Paul [8] shows that the perturbation from the true eigenvector is uniformly distributed on the unit sphere in $N-1$ dimensions. Fig. 6 illustrates this for a three-dimensional example. The blue circles indicate the uniform distribution of sample eigenvectors around the cone.

Paul's results suggest a simple fluctuation model for the sample eigenvector. Since the eigenvectors are a complete basis, the sample eigenvector can be written as a weighted sum of the true eigenvectors, i.e.,

$$\mathbf{e}_1 = \eta \boldsymbol{\xi}_1 + \sum_{i=2}^{N} \delta_i \boldsymbol{\xi}_i. \tag{9}$$

The δ_i's are IID random variables that guarantee the perturbation is uniformly distributed in directions orthogonal to the true direction. η is a function of the perturbations that ensures $\mathbf{e}_1$ maintains its unit norm. The variance of the δ_i's is chosen to make $\mathcal{E}\left\{|\eta|^2\right\} = \cos^2_{RMT}$, where $\cos^2_{RMT}$ is RMT prediction of the squared radius of the cone. Prior research does not describe the distribution of the δ_i perturbations. Our investigation indicates that the perturbations are complex Gaussian-distributed for $L \geq N$, assuming a single interferer with complex Gaussian amplitude in Gaussian noise. This implies $1 - |\eta|^2$ has a χ^2 distribution. For smaller numbers of snapshots, our analysis shows that a gamma distribution provides a good fit for $1-|\eta|^2$. χ^2 is a special case of the gamma distribution. Using the gamma distribution, we can generate realizations of sample eigenvectors from this simple fluctuation model. Given independent realizations of the sample eigenvectors, we can compute the associated DMR weight vectors and the SINR loss. Fig. 7 shows the histogram of SINR loss obtained using this eigenvector fluctuation model alongside the beta distribution predictions. Results are shown for two snapshot values: $L = 3, 10$. The agreement is excellent, indicating that the proposed fluctuation model is an accurate representation of the underlying statistics.

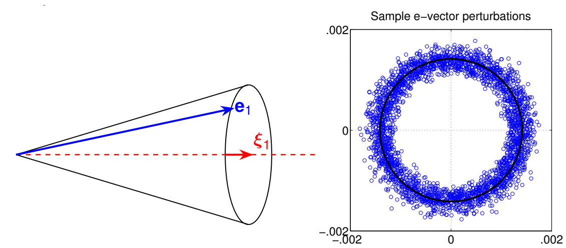

Fig. 6. Relationship of the sample eigenvector $\mathbf{e}_1$ to the ensemble eigenvector $\boldsymbol{\xi}_1$ as predicted by random matrix theory. RMT predicts the generalized cosine squared between $\mathbf{e}_1$ and $\boldsymbol{\xi}_1$, which is the radius of the cone. The plot on the right shows how the sample eigenvectors are distributed around the cone, as predicted by Paul [8].

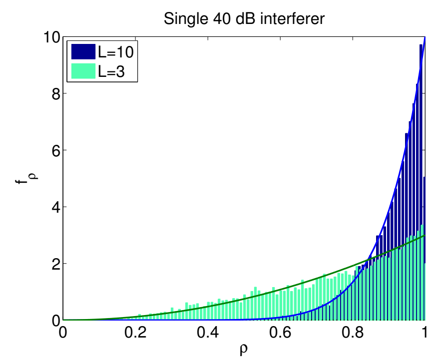

Fig. 7. Comparison of the beta distribution (solid lines) to simulations using the eigenvector perturbation model (histogram) for a single 40 dB interferer.

5. SUMMARY

Numerical simulations support the conjecture that SINR loss for the DMR ABF is beta-distributed. The parameters of the beta distribution are a function of the number of snapshots and the number of interferers: $a = L - D + 1$ and $b = D$, leading to a mean SINR loss of $(L - D + 1)/(L + 1)$. The proposed sample eigenvector fluctuation model accurately predicts DMR performance, and lays the foundation for future theoretical analysis of SINR loss.

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