# Bayesian Compressive Sensing for DOA Estimation using the Difference Coarray

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***Abstract*—In this paper, we utilize Bayesian Compressive Sensing (BCS) for direction-of-arrival (DOA) estimation based on the coarray. This enables estimation of more sources than the number of physical antennas. We adopt the covariance vectorization technique to construct the received signal vectors of coarrays for both fully and partially augmentable arrays. We then apply the single measurement vector BCS (SMV-BCS) for DOA estimation. Supporting simulation results for both sparse linear arrays and circular arrays demonstrate the effectiveness of the proposed approach in terms of high resolution and estimation accuracy compared to the MUSIC and sparse signal reconstruction based methods.**

***Index Terms*—Bayesian compressive sensing, coarray, covariance vectorization, DOA estimation, single vector measurement**

## I. Introduction

Estimating the direction-of-arrival (DOA) using antenna arrays has been an important topic in signal processing with diverse applications, such as radar, satellite navigation and telecommunication to list a few [1]. There has been extensive research on high-resolution DOA estimation techniques, among which the ones evolving around Capon's methods and MUSIC algorithms are commonly used [2]. Another kind of effective DOA estimation techniques based on sparse signal reconstruction (SSR) has emerged in recent years, including the $l_1$-SVD method proposed in [3]. In the case of a single measurement vector (SMV), $l_1$ optimization approach is considered attractive to sparse signal recovery due to its guaranteed recovery accuracy. A major issue encountered in $l_1$ optimization, however, is that reliable recovery is guaranteed only when the restricted isometry property is satisfied [4]. It is worth noting that, for all aforementioned DOA estimation methods, the number of estimated sources cannot exceed the number of physical antennas.

When the number of estimated sources is larger than the number of physical sensors, high-resolution DOA estimation can be accomplished based on two approaches, neither requires increasing the number of physical antennas: (1) Different spatial lags of the covariance matrix of the sparse array are used to form an augmented Toeplitz matrix, which is equivalent to the true covariance matrix of an equivalent filled uniform array [5]–[7]; (2) The covariance matrix of the sparse array is vectorized to emulate observations at the corresponding difference coarray [8]–[10], which is defined as the set of points at which the spatial covariance function can be sampled with the physical array [11], [12]. The former technique requires positive definite Toeplitz completion for partially augmentable arrays [13], which is difficult to implement, especially when there are multiple holes corresponding to missing autocorrelation lags in the coarray points. In the second approach, the sources are replaced by their powers, casting them as mutually coherent. Spatial smoothing must then be applied to decorrelate signals and restore the full rank of the resulting covariance matrix before high-resolution DOA estimation can be performed [8], [14]. Spatial smoothing, however, requires availability of a set of contiguous coarray points without any holes which limits its applicability to partially augmentable arrays.

In order to better utilize the coarray aperture and increase the number of degrees of freedom without the requirement of contiguous spatial lags, an SSR method for DOA estimation has been adopted based on the second approach of covariance matrix vectorization [15], [16]. There are inevitably spurious peaks in the sensing spectrum for SSR based methods due to the coherency of ill-conditioned measurement dictionary. Thus, a more reliable estimation approach is required, especially for partially augmentable arrays. To this end, we utilize the Bayesian Compressive Sensing (BCS) [17] which formulates the problem from a probabilistic perspective and solve it with the relevance vector machine (RVM) concept [18]. The sparse solution is obtained by assuming a Laplace prior for the sources of interest. It has been shown that the BCS-based spectrum sensing method is an effective and robust DOA estimation technique [19]–[21].

A SMV-BCS method based on the covariance vectorization technique is adopted in this paper for both fully and partially augmentable arrays. An attractive approach is to combine multiple measurement vector BCS (MMV-BCS) with the eigenvalue decomposition of the augmented covariance matrix. However, the MMV-BCS is more sensitive to the coherency of the measurement dictionary. Although awaiting further analysis, simulations have shown that the MMV-BCS based on covariance augmentation does not perform as well as the SMV-BCS based on covariance vectorization for fully augmentable arrays especially for sources close to the end-fire.

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The remainder of this paper is organized as follows. DOA estimation based on the difference coarray is formulated in Section II. The SMV-BCS algorithm is introduced in Section III. Section IV provides the supporting simulation results, while Section V contains the concluding remarks.

## II. DOA Estimation Based On Difference Coarray

Assuming the positions of the array elements form the set,

$$\mathcal{S} = \{(x_i, y_i), i = 1, \cdots, M\}, \tag{1}$$

where $x_i = n_i d_0, y_i = m_i d_0$ with $d_0$ being the unit inter-element spacing and $n_i, m_i$ being integer numbers. The corresponding difference coarray has positions,

$$\mathcal{S}_d = \{(x_i - x_j, y_i - y_j), i, j = 1, \cdots, M\}, \tag{2}$$

i.e., the difference coarray is the set of pairwise differences of the array element positions and the received signal correlation can be calculated at all lags comprising the difference coarray [12]. Minimum Redundancy Arrays (MRAs) and Minimum Hole Arrays (MHAs) are the common classes of sparse arrays [11], [22]. MRAs are those configurations of $M$ elements that satisfy minimum (R | H = 0; $M$ = constant), where R and H denote the number of redundancies and holes in the coarray, respectively. MRAs are also referred to as fully augmentable arrays in [23]. MHAs are sparse arrays of $M$ elements that satisfy minimum (H | R = 0; $M$ = constant), which belong to the class of partially augmentable arrays.

Consider $K$ narrowband far-field uncorrelated sources $s_k(t), k = 1, \cdots, K$, impinging on an array of $M$ omnidirectional sensors from directions $\theta_k, k = 1, \cdots, K$ in elevation and $\phi_k, k = 1, \cdots, K$ in azimuth. The array output can be expressed as,

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{e}(t), t = 1, \cdots, \tilde{T}, \tag{3}$$

where $\mathbf{y}(t) = [y_1(t), \cdots, y_M(t)]^T$, $\mathbf{s}(t) = [s_1(t), \cdots, s_K(t)]^T$ and $\mathbf{e}(t) = [e_1(t), \cdots, e_M(t)]^T$ is the noise vector. The matrix $\mathbf{A} = [\mathbf{a}(\mathbf{u}_1), \cdots, \mathbf{a}(\mathbf{u}_K)]$ is the array manifold and $\mathbf{a}(\mathbf{u}_k)$ is the steering vector of the $k$th source, which is defined as,

$$\mathbf{a}(\mathbf{u}_k) = [1, e^{jk_0(u_2^x x_2 + u_2^y y_2)}, \cdots, e^{jk_0(u_M^x x_M + u_M^y y_M)}]^T, \tag{4}$$

where $k_0 = 2\pi/\lambda$ is the wavenumber and $\mathbf{u}_k = [u_k^x, u_k^y]^T = [\cos\theta_k \cos\phi_k, \cos\theta_k \sin\phi_k]^T$.

The correlation matrix $\mathbf{R}$ of the received signal is given by,

$$\mathbf{R} = \sum_{t=1}^{\tilde{T}} \mathbf{y}(t)\mathbf{y}^H(t) = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_0^2\mathbf{I}, \tag{5}$$

where $\mathbf{R}_s$ represents the source correlation matrix, which is diagonal with the source powers $\sigma_1^2, \cdots, \sigma_K^2$ populating its main diagonal, $\mathbf{I}$ is the identity matrix of corresponding rank, $\sigma_0^2$ is the noise variance and the superscript 'H' denotes conjugate transpose. The $ij$th element of $\mathbf{R}$ is,

$$(\mathbf{R})_{ij} = \sum_{k=1}^{K} \sigma_k^2 e^{jk_0(u_k^x(x_i - x_j) + u_k^y(y_i - y_j))} + \sigma_0^2\delta(i - j), \tag{6}$$

where $\delta(i-j)$ is the Kronecker Delta function. It is clear that $(\mathbf{R})_{ij}$ can be treated as the data received by the coarray element position $(x_i - x_j, y_i - y_j)$.

## III. SMV-BCS Based On Covariance Vectorization

We elaborate on the SMV-BCS method based on covariance vectorization for DOA estimation in the case where the sources are more than the physical antennas.

### A. Covariance Vectorization

Vectorizing $\mathbf{R}$, we obtain

$$\tilde{\mathbf{y}} = \text{vec}(\mathbf{R}) = \tilde{\mathbf{A}}\mathbf{b} + \sigma_0^2\tilde{\mathbf{i}}, \tag{7}$$

where $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\mathbf{u}_1), \cdots, \tilde{\mathbf{a}}(\mathbf{u}_K)]$ with $\tilde{\mathbf{a}}(\mathbf{u}_k) = \mathbf{a}(\mathbf{u}_k) \otimes \mathbf{a}^*(\mathbf{u}_k)$. Here $\otimes$ denotes the Kronecker product and '*' is the conjugate operation. The two vectors $\tilde{\mathbf{i}} = \text{vec}(\mathbf{I})$ and $\mathbf{b} = [\sigma_1^2, \cdots, \sigma_K^2]^T$. The vector $\tilde{\mathbf{y}}$ can be viewed as a single snapshot received by a much larger virtual array, whose element positions are given by the difference coarray. Utilizing the coarray measurement vector $\tilde{\mathbf{y}}$ for DOA estimation permits handling of a greater number of sources than the number of physical antennas. The equivalent source signal $\mathbf{b}$ consists of the powers of the estimated sources and the noise becomes a deterministic vector. Therefore, the rank of the covariance matrix of $\tilde{\mathbf{y}}$ is one and subspace-based DOA estimation techniques, such as MUSIC, would fail. It should be noted that if the sparse array is fully augmentable, spatial smoothing can be utilized to restore the rank of the covariance matrix.

### B. SMV-BCS

Let $\hat{\theta} = \{\hat{\theta}_1, \cdots, \hat{\theta}_{N_e}\}$ and $\hat{\phi} = \{\hat{\phi}_1, \cdots, \hat{\phi}_{N_a}\}$ be two fixed sampling grid sets covering all possible DOAs, where $N_e$ and $N_a$ are the number of grid points in elevation and azimuth, respectively. Let $\tilde{\Phi} = [\tilde{\mathbf{a}}(\hat{\mathbf{u}}_1), \cdots, \tilde{\mathbf{a}}(\hat{\mathbf{u}}_{N_e N_a}), \tilde{\mathbf{i}}]$. The observation model in Eq. (7) can be rewritten as,

$$\tilde{\mathbf{y}} = \tilde{\Phi}\mathbf{x} + \mathbf{e}, \tag{8}$$

where $\mathbf{x}$ represents the source signals, with the entry corresponding to the angles $[\theta_k, \phi_k]$ equals $\sigma_k^2$ if $\theta_k \in \{\hat{\theta}_1, \cdots, \hat{\theta}_{N_e}\}$ and $\phi_k \in \{\hat{\phi}_1, \cdots, \hat{\phi}_{N_a}\}, k = 1, \cdots, K$. The signal $\mathbf{x}$ is sparse since $N_e N_a \gg K$. Here, $\mathbf{e}$ denotes the estimation noise which, without loss of generality, is circularly symmetric Gaussian distributed with probability density function,

$$p(\mathbf{e}|\alpha_0) = \mathcal{CN}(\mathbf{e}|\mathbf{0}, \alpha_0^{-1}\mathbf{I}). \tag{9}$$

The complex Normal distribution $\mathcal{CN}(\mathbf{u}|\mu, \Sigma)$ is defined as,

$$\mathcal{CN}(\mathbf{u}|\mu, \Sigma) = \frac{1}{\pi^{N_e N_a}|\Sigma|}\exp\{-(\mathbf{u} - \mu)^H\Sigma^{-1}(\mathbf{u} - \mu)\}. \tag{10}$$

Therefore, we have

$$p(\tilde{\mathbf{y}}|\mathbf{x}, \alpha_0) = \mathcal{CN}(\tilde{\mathbf{y}}|\tilde{\Phi}\mathbf{x}, \alpha_0^{-1}\mathbf{I}). \tag{11}$$

A two-stage hierarchical prior is adopted for $\mathbf{x}$ to introduce a Laplace prior for both real part $\mathcal{R}(\mathbf{x})$ and imaginary part $\mathcal{I}(\mathbf{x})$.

First, we define a zero mean complex Gaussian prior,

$$p(\mathbf{x}|\alpha) = \mathcal{CN}(\mathbf{x}|\mathbf{0}, \Lambda), \tag{12}$$

with $\Lambda = \text{diag}(\alpha)$ and $\alpha = [\alpha_1, \cdots, \alpha_{N_e N_a}]^T \in \mathbb{R}_+^{N_e N_a}$. Further, a Gamma hyperprior is considered over $\alpha$,

$$p(\alpha|\rho) = \Pi_{n=1}^{N_e N_a} \Gamma(\alpha_n|1, \rho), \tag{13}$$

with $\rho \in \mathbb{R}_+$ being fixed a priori. Similarly, a Gamma prior is introduced on $\alpha_0$,

$$p(\alpha_0|c, d) = \Gamma(\alpha_0, c, d) = \alpha_0^{c-1} e^{-\alpha_0/d} d^{-c} \Gamma(c)^{-1}, \tag{14}$$

with $c, d \in \mathbb{R}_+$ being fixed a priori and $\Gamma(c)$ being the Gamma function evaluated at $c$. To make the Gamma prior non-informative, $c, d \to 0$ are adopted [19].

By combining the stages of the hierarchical model, the joint distribution is obtained as

$$p(\mathbf{x}, \tilde{\mathbf{y}}, \alpha_0, \alpha) = p(\tilde{\mathbf{y}}|\mathbf{x}, \alpha_0) p(\mathbf{x}|\alpha) p(\alpha_0) p(\alpha). \tag{15}$$

A type-II ML approach is exploited to perform the Bayesian inference since the posterior distribution $p(\mathbf{x}, \alpha_0, \alpha|\tilde{\mathbf{y}})$ cannot be expressed explicitly. First, it is easy to show that the posterior for $\mathbf{x}$ is a complex Gaussian distribution, [19]

$$p(\mathbf{x}|\mathbf{y}, \alpha_0, \alpha) = \frac{p(\mathbf{y}|\mathbf{x}, \alpha_0) p(\mathbf{x}|\alpha)}{p(\mathbf{y}|\alpha_0, \alpha)} = \mathcal{CN}(\mathbf{x}|\mu, \Sigma), \tag{16}$$

with mean and covariance,

$$\mu = \alpha_0 \Sigma \tilde{\Phi}^H \tilde{\mathbf{y}}, \ \Sigma = (\Lambda^{-1} + \alpha_0 \tilde{\Phi}^H \tilde{\Phi})^{-1}. \tag{17}$$

Then, the hyperparameters $\alpha_0$ and $\alpha$ are estimated by the maxima of the posterior $p(\alpha_0, \alpha|\tilde{\mathbf{y}})$, or equivalently, the maxima of the joint distribution $p(\tilde{\mathbf{y}}, \alpha_0, \alpha) \propto p(\alpha_0, \alpha|\tilde{\mathbf{y}})$, which can be expressed as

$$p(\tilde{\mathbf{y}}, \alpha_0, \alpha) = p(\tilde{\mathbf{y}}|\alpha_0, \alpha) p(\alpha_0) p(\alpha). \tag{18}$$

We can readily show that $p(\tilde{\mathbf{y}}|\alpha_0, \alpha)$ is the convolution of two Gaussian distributions, i.e.,

$$p(\tilde{\mathbf{y}}|\alpha_0, \alpha) = \int p(\tilde{\mathbf{y}}|\mathbf{x}, \alpha_0) p(\mathbf{x}|\alpha) \mathrm{d}\mathbf{x} = \mathcal{CN}(\tilde{\mathbf{y}}|\mathbf{0}, \mathbf{C}), \tag{19}$$

with

$$\mathbf{C} = \alpha_0^{-1} \mathbf{I} + \tilde{\Phi} \Lambda \tilde{\Phi}^H. \tag{20}$$

To alleviate the computational complexity of Eq.(17), we use the Woodbury matrix identity to obtain

$$\Sigma = \Lambda - \Lambda \tilde{\Phi}^H \mathbf{C}^{-1} \tilde{\Phi} \Lambda. \tag{21}$$

The Maximum Likelihood (ML) function is the logarithm of the joint PDF, $\mathcal{L}(\alpha_0, \alpha) = \text{log} p(\tilde{\mathbf{y}}, \alpha_0, \alpha)$. An expectation maximization (EM) algorithm is implemented to maximize the ML function. Hence, the update of $\alpha_n$ is given as

$$\alpha_n^{\text{new}} = \frac{\sqrt{1 + 4\rho(|\mu_n|^2 + \Sigma_{nn})} - 1}{2\rho}, \tag{22}$$

where $\mu_n$ and $\Sigma_{nn}$ denote the $n_{th}$ entry of the mean vector $\mu$ and the diagonal of the covariance matrix $\Sigma$ respectively.

TABLE I: DOA estimation MSE of MUSIC, SSR and BCS for both MRA and MHA.

| Arrays | MUSIC(°) | SSR(°) | SMV-BCS(°) |
|---|---|---|---|
| MRA | 2.35 | 1.07 | 0.012 |
| MHA | - | 4.78 | 0.035 |

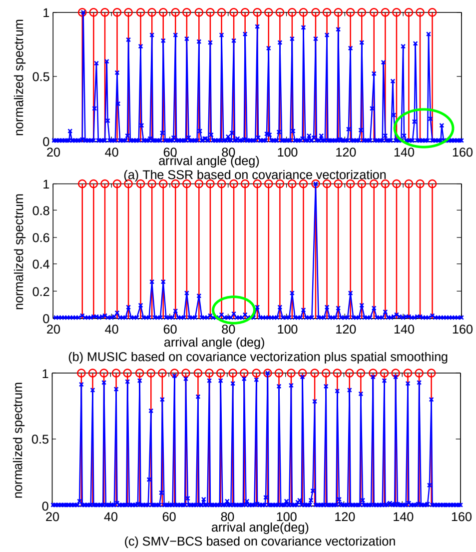


Fig. 1: Sensing spectra of three DOA estimation algorithms for fully augmentable arrays.

Similarly, for $\alpha_0$ we have

$$\alpha_0^{\text{new}} = \frac{M_a + c - 1}{\|\tilde{\mathbf{y}} - \tilde{\Phi}\mu\|_2^2 + \alpha_0^{-1} \sum_{n=1}^{N_e N_a} \gamma_n + d}, \tag{23}$$

with $\gamma_n = 1 - \alpha_n^{-1} \Sigma_{nn}$ and $c, d$ are defined in Eq.(14). Here, $M_a$ is the number of virtual antennas in the coarray. Most entries of $\mu$ and $\Sigma$ converge to very small values, which implies that the posterior for these $x_n$ becomes strongly peaked at zero. As a result, these $x_n$ are zeros and, hence, sparsity is realized. The source power in the direction $[\hat{\theta}_{n_1}, \hat{\phi}_{n_2}]$ with $n = N_a * (n_1 - 1) + n_2$ is estimated by $\sigma_n^2 = |\mu(n)|$.

## IV. Simulation Results

### A. Sparse Linear Array

We use two sparse linear arrays with 10 antennas with configurations $[0, 1, 3, 6, 13, 20, 27, 31, 35, 36]\lambda/2$ for MRA and $[0, 1, 6, 10, 23, 26, 34, 41, 53, 55]\lambda/2$ for MHA respectively [11], [24]. We set the BCS parameters as $\rho = c = d = 1e^{-4}$ and initialize $\alpha_0 = 100/\text{Var}(\tilde{\mathbf{y}})$, $\alpha = |\tilde{\Phi}^H \tilde{\mathbf{y}}|$. The number of time snapshots is 2000 for covariance matrix estimation. The performance comparison of the different techniques reported in this section also holds when using higher and lower number of snapshots. The MRA is fully augmentable with the coarray being a 73-antenna uniform linear array (ULA). Therefore, in addition to the SSR and SMV-BCS, the MUSIC equipped with spatial smoothing can also be utilized for DOA

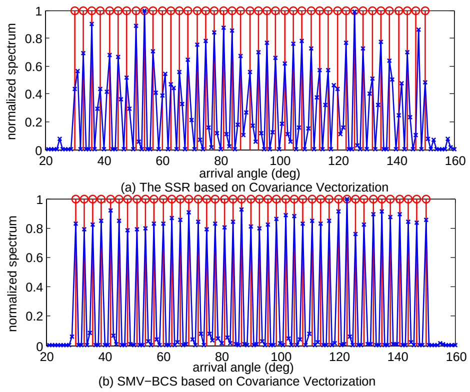


Fig. 2: Sensing spectra of two DOA estimation algorithms for partially augmentable arrays.

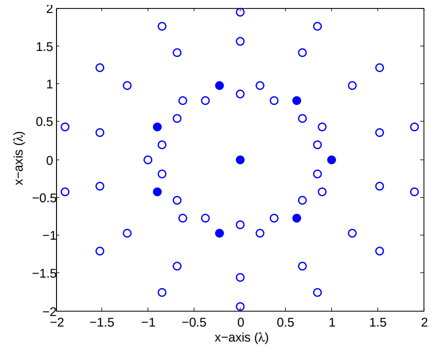


Fig. 3: The 8-antenna circular array and its difference coarray.

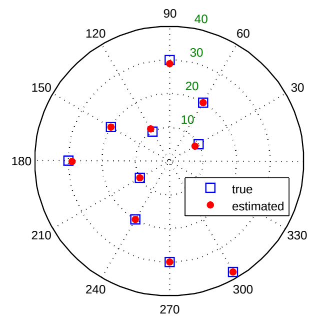


Fig. 4: 2-D DOA estimation: the square and circle indicate the true and estimated directions respectively.

estimation. Consider 31 sources uniformly distributed within the range $[30°, 150°]$ with an angular interval $4°$ and signal-to-noise ratio (SNR) 20dB. The sensing spectra of the three methods are shown in Fig. 1. We can observe that there are spurious and biased peaks in the spectrum of the SSR method. Some spectral lines are too weak to identify for the MUSIC pseudo-spectrum. The SMV-BCS based on the covariance vectorization exhibits the best performance, and clearly shows 31 peaks with almost the same power level.

In order to examine the estimation accuracy, we utilize the mean squared error (MSE) as the metric defined as,

$$\mathrm{MSE} = \frac{1}{K}\sum_{k=1}^{K}(\theta_k - \tilde{\theta}_k)^2, \tag{24}$$

where $\tilde{\theta}_k$ denotes the estimated angle of the $k$th source. The MSE values of the three considered methods, MUSIC, the SSR and the SMV-BCS, are listed in the first row of Table I with 200 Monte-Carlo runs. It is evident that the MUSIC approach exhibits the worst estimation performance, whereas the proposed SMV-BCS approach based on covariance vectorization demonstrates much higher estimation accuracy than the other two methods.

The coarray for the 10-antenna MHA is a 91-antenna linear array with 10 holes, thus only the SSR and the SMV-BCS can be used for DOA estimation. Consider 41 sources uniformly distributed within the range $[30°, 150°]$ with an angular interval $3°$ and 20dB SNR. The sensed spectra of the two methods are shown in Fig. 2. Again, the proposed method is superior to the SSR and enjoys high resolution DOA estimation for this case of closely spaced sources. Similar to the MRA, we also utilize 200 Monte-Carlo runs to calculate the estimation MSE, which is provided in the second row of Table I. It is clear that the proposed method presents high estimation accuracy for partially augmentable arrays as well. Note that we do not present the DOA estimation performance based on the physical array with its limited degrees of freedom, as it cannot deal with more sources than the number of antennas.

### B. Circular Arrays

A circular array with a single antenna in the center and 7 antennas uniformly distributed along the circumference, as indicated by filled dots in Fig. 3, is typically employed in GPS. The corresponding coarray is a four-circle concentric array, also shown in Fig. 3. Suppose there are 10 jammers arriving from $[10°, 20°, 30°, 10°, 20°, 30°, 10°, 20°, 30°, 38°]$ in elevation and uniformly distributed in azimuth sector $[30°, 300°]$ with an angular interval $30°$ and 20dB interference to noise ratio. The two-dimensional (2-D) DOA estimation in both elevation and azimuth is shown in Fig. 4, where the radial direction denotes the elevation angle and the circumference direction is the azimuth. We can observe that the estimated 2-D angles coincide with the true angles which further validates the effectiveness of the proposed approach.

## V. Conclusion

In this paper, we utilized a probabilistic Bayesian inference method for DOA estimation based on the difference coarray. This allows DOA estimation of more sources than the sensors. We combined the SMV-BCS approach with covariance vectorization for both fully and partially augmentable arrays. Simulation results showed that the proposed method can overcome the shortcomings of the SSR method and the MUSIC, such as spurious peaks and inaccurate power estimation. However, this performance superiority has to be weighted against slow convergence of BCS for high dimension dictionary matrix.

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