MULTIPLE SOURCE LOCALIZATION WITH MOVING CO-PRIME ARRAYS

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ABSTRACT

In this paper, we explore the use of temporal signal coherence to increase the degrees of freedom in a non-uniform under-sampled linear moving array. In particular, we develop bounds on the minimum Temporal-Coherence-Period (TCP) required to synthetically fill-in missing inter-element sensors in a moving array system. In addition, we show how an existing technique that exploits properties of the sensor co-array can be used to overcome positive-definite deficiency in the spatial covariance matrix estimate derived from the synthetic sensor array. This type of synthetic aperture processing facilitates the localization of a number of far-field sources on the order of the number of half-wavelength sensor spacings over the array aperture, rather than on the number of sensors present in the sparse array.

Index Terms— Co-Prime Linear Array, Synthetic Aperture Methods, Rank Enhanced Spatial Smoothing

1. INTRODUCTION

Synthetic Aperture (SA) methods are well known techniques for exploiting platform motion for sensor array systems. Synthetic aperture methods use temporal signal coherence to achieve improved array processing performance across a synthesized array aperture. Traditional SA techniques require temporal coherence that is proportional to the time required for the array to travel the distance of the entire physical aperture. These methods have been used to improve angular resolution and signal gain over that provided by the physical array [1–4].

While typically SA methods are used with spatially uniformly sampled arrays, the class of physical arrays studied in this paper are non-uniformly spatially under-sampled arrays known as co-prime sensor arrays. A co-prime array is a sparse array formed by nesting two uniform linear arrays (ULA's) with inter-element spacings $M\frac{\lambda}{2}$ and $N\frac{\lambda}{2}$, where M and N are co-prime integers.

Co-prime arrays are of particular interest because this geometry provides the required spatial covariance measurements required to identify more sources than the number of sensors present in the array [5]. In addition, they are sparse arrays that can be designed to span large apertures with relatively few sensor elements, by simply nesting two different ULA's. The ease of forming a co-prime array can be contrasted with the Minimally Redundant Line Array (MRA), which for large apertures is computationally expensive to determine where to place the sensor elements [6, 7]. However, the spatial covariances afforded by a co-prime array only account for half of the array aperture. To mediate this shortcoming, co-prime sensor arrays have been recently studied by the authors in the synthetic aperture framework [8]. It was shown by using SA methods, the achievable resolution of the co-prime array can be increased from the typical half-aperture resolution limit to the full extent of the physical aperture.

The focus of this paper is to develop minimum temporal coherence criteria needed to maximize the degrees of freedom of a SA co-prime sensor array system. In addition, we show how the Rank Enhanced Spatial Smoothing algorithm, proposed by [9], can be used to overcome positive-definite deficiency in the Direct Augmentation Algorithm for covariance matrix estimation.

2. SYNTHETIC APERTURE ARRAY MODEL

Consider a co-prime array with a physical aperture L meters composed of $Q = M + 2N - 1$ sensors. This geometry is the result of nesting two ULA's, one with inter-element spacing of $M\frac{\lambda}{2}$ array with $2N$ sensors and the other ULA with $N\frac{\lambda}{2}$ element spacing and M sensors, where $N < M$. Assume the array moves with a constant velocity v along a staight-line course and that K static radiating sources with common frequency Ω_0 and bearings $\theta_1, \cdots, \theta_K$ are in the far-field of the array.

The Doppler shifted plane-wave signal received at the array from the l-th source is given by,

$$\mathbf{s}_l(t) = \alpha_l \exp(j\Omega_0 t)\exp(-j\Omega_0 \frac{vt\sin(\theta_l)}{c})\mathbf{b}(\theta_l) \tag{1}$$

where $\alpha_l \sim \mathcal{CN}(0, \sigma_l^2)$, and $\mathbf{b}(\theta_l) = [1, \exp(-j\Omega_0 \frac{d_1 \sin(\theta_l)}{c}), \cdots, \exp(-j\Omega_0 \frac{d_{Q-1}\sin(\theta_l)}{c})]^\top$ is the array manifold vector. The sensor locations relative to the left most sensor are given by $d_1, \cdots, d_{Q-1}$. The snapshot collected at the array at time

t from a K-source field is given by,

$$\mathbf{x}(t) = \sum_{l=1}^{K} \mathbf{s}_l(t) + \mathbf{w}(t), \;\; \mathbf{w}(t) \sim \mathcal{CN}(0, \sigma_w^2 \mathbf{I}_Q) \quad (2)$$

Array motion is used for synthetically inserting sensor elements into the co-prime array. The increase of the dimension of the array manifold vector leads to an increase in the degrees of freedom of the array [9]. To insert sensors at integral multiples of half-wavelengths, let τ be the time for the array to travel a distance of $\frac{\lambda}{2}$ meters. The synthetic array snapshot is then formed by appropriately interleaving the physical array snapshots from times t, $t+\tau, \cdots$, and $t+n\tau$. For a more in-depth development of the synthesis process see [8].

Spatial processing across the synthetic array snapshot requires temporal coherence over the synthesis time interval. The coherence time interval is a function of the number of physical array snapshots used to form the synthetic array.

3. MINIMUM TEMPORAL-COHERENCE-PERIOD ARRAY SYNTHESIS

Our objective is to minimize the Temporal-Coherence-Period (TCP) such that the degrees of freedom for a moving co-prime array are maximized over the array aperture. The degrees of freedom of a line array are related to the number of spatial covariances that can be observed from the array snapshot. For a given array snapshot the co-array tells us if and how many times a spatial lag has been measured. For co-prime arrays, Vaidyanathan and Pal have shown the co-array has a hole-free contiguous region of approximately half of the array aperture [9] [10]. This hole-free region is directly related to the degrees of freedom present in the array and is the primary reason the co-array plays a central role in SA methods [11]. For co-prime arrays we have $\mathcal{O}(MN)$ degrees of freedom from only $M + 2N - 1$ sensors. Ramirez Jr. *et al.* have developed an upper bound on the TCP required to produce a filled co-array without synthesizing every missing sensor over the array aperture in a moving co-prime sensor array system [8]. This however, only provides a sufficient condition for filling the co-array. The necessary and sufficient condition for minimizing TCP while maximizing the degrees of freedom over the array aperture of a moving co-prime array, is developed here, and can be formalized as,

Lemma 3.1 *The minimum Temporal-Coherence-Period for a (N, M) $\lambda/2$ co-prime array with $M+2N-1$ sensor elements and $N < M$ to have a hole-free co-array across the array aperture is given by $TCP = \eta_0\tau$ where,*

$$\eta_0 = \begin{cases} \frac{N}{2} & \text{if } N \text{ is even} \\ \frac{N-1}{2} & \text{if } N \text{ is odd} \end{cases} \quad (3)$$

and τ is the time required for the array to travel a distance of $\lambda/2$ meters.

To justify Lemma 3.1 we need to show that for the above value of η_0 the corresponding co-array is hole-free and that for any $\gamma < \eta_0$ the co-array will contain at least one hole. Figure 1 shows a diagram of the general setting for the co-prime array. The major concepts in proving this Lemma are presented in what follows.

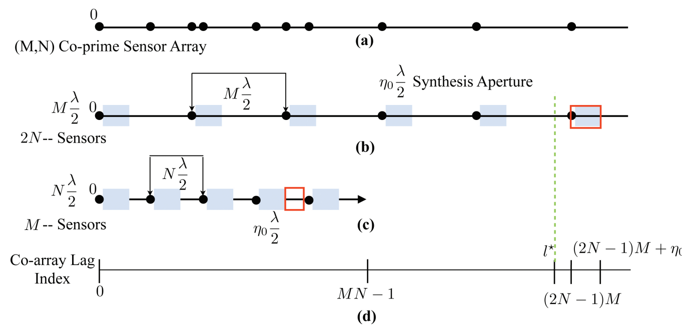

Fig. 1. (a) Co-prime array, (b)-(c) Co-prime array components with synthetic aperture window (shaded regions). (d) Co-array Index Partition

The co-array for a thinned regular array is defined as the autocorrelation of the array element weights

$$c(l) = \sum_{m=0}^{P-|l|-1} h_m h_{m+|l|}^* \quad (4)$$

where $h_m \in \{0, 1\}$ and P is the number elements in the full aperture. The vector $h = \left[h_0, h_1, \cdots, h_{P-1}\right]^\top$ encodes if a sensor is missing ,0, or present ,1, in the aperture.

To show that the co-array is hole-free for η_0 we consider the following partition on the spatial lag indices, l:

Case 1: $0 \leq l \leq MN - 1$, In this case, the co-array is hole-free since the physical array is co-prime.

Case 2: $MN \leq l \leq (2N-1)M - 1$, We observe that for any value l within this range the autocorrelation of the sensor weights contains the contiguous set of active elements in the last $\eta_0 + 1$ section of the array aperture (red box in Figure 1 (b)). It can be shown that this section of aperture is larger than any hole in the $N\frac{\lambda}{2}$ subarray component of the co-prime array (red box in Figure 1 (c)). With this property in mind we have the following lower bound on $c(l)$ for l within this range,

$$\begin{aligned} c(l) &= \sum_{m=0}^{P-1-|l|} h_m h_{m+|l|} \quad (5) \\ &> \underbrace{\left[h_{P-1-|l|-\eta_0} \;\; \cdots \;\; h_{P-1-|l|}\right]}_{\text{At least one non-zero element}} \underbrace{\begin{bmatrix} h_{(2N-1)M} \\ \vdots \\ h_{(2N-1)M+\eta_0} \end{bmatrix}}_{\text{1's Vector}} \\ &\geq 1 \end{aligned}$$

For spatial lags in the range $MN \leq l \leq (2N-1)M - 1$, we can see that $c(l) > 0$ and is therefore hole-free.

Case 3: $(2N-1)M \leq l \leq (2N-1)M + \eta_0$, In this range both h_0 and h_l are non-zero and a lower bound on the co-array value given by, $c(l) > h_0 h_l = 1$.

We can therefore conclude that for η_0 the co-array is filled over the entire aperture of the synthetic array.

Next, to show that for any $\gamma < \eta_0$ the co-array will contain at least one hole, let $\gamma = \eta_0 - 1$ and consider the following cases partitioned by the magnitude of the co-prime factor N:

Case 1: $N = 2$ or 3, In this situation, $\eta_0 = 1$ and subsequently $\gamma = 0$. Here no sensor synthesis is performed and therefore the holes in the co-array are inherited from the physical co-prime array.

Case 2: $N \geq 4$, In this situation, $c(l^\star) = 0$ for,

$$l^\star = \begin{cases} P-1-N+1 & \text{if } N \text{ is even} \\ P-1-(N-1)-1 & \text{if } N \text{ is odd} \end{cases} \tag{6}$$

with $P - 1 = (2N-1)M + \gamma$. The co-array expression for $c(l^\star)$ can be partitioned into,

$$\begin{aligned} c(l^\star) &= \sum_{m=0}^{P-1-|l^\star|} h_m h_{m+|l^\star|} \\ &= \begin{bmatrix} h_0 & \cdots & h_g \end{bmatrix} \overbrace{\begin{bmatrix} h_{l^\star} \\ \vdots \\ h_{(2N-1)M-1} \end{bmatrix}}^{\text{0's Vector}} \\ &+ \underbrace{\begin{bmatrix} h_{g+1} & \cdots & h_{P-1-l^\star} \end{bmatrix}}_{\text{0's Vector}} \begin{bmatrix} h_{(2N-1)M} \\ \vdots \\ h_{(2N-1)M+\gamma} \end{bmatrix} = 0 \end{aligned} \tag{7}$$

Here the first term becomes zero since h_l for $l^* \leq l \leq (2N-1)M-1$ represent inactive sensor elements. To verify this it can be shown that $(2N-2)M + \gamma < l^*$.

The second term becomes zero since h_l for $g+1 \leq l \leq P-1-l^\star$ represents inactive sensor elements. To verify this it can be shown that $\gamma < g+1$ and that $P-1-l^\star < N$ where $g = (2N-1)M - 1 - l^\star$.

We can therefore conclude that $c(l^\star) = 0$ when $\gamma < \eta_0$ and $\gamma = \eta_0 - 1$, the co-array will have at least one hole. For subsequent smaller values of γ the hole found at $l^\star$ will persist and the number of holes in the co-array will increase approaching the total number of holes found in the physical co-prime array.

This Lemma suggests that the minimum TCP is a function of the co-prime factor N and independent of the larger co-prime factor M. From this perspective, we can see that the minimum TCP is proportional to the time required for the array to travel $\frac{N}{2}\frac{\lambda}{2}$ (m) of the physical aperture.

The significance of the minimum TCP is that it describes the minimum distance required for a co-prime sensor array needs to travel for SA methods to produce a filled co-array. The hole-free co-array over the array aperture provides a full set of spatial covariances required to maximize the degrees of freedom required for signal subspace based source localization. The advantage of the work presented here is that via SA methods we are able to maximize the degrees of freedom over a given aperture with the minimum total synthesis time.

4. SPATIAL COVARIANCE AUGMENTATION

SA methods applied over the minimum TCP produce a full set of spatial covariances over the aperture of the non-uniformly sampled array, but does not lead directly to a positive-definite covariance matrix. Using the Direct Augmentation Algorithm (DAA), the co-array and the available spatial covariances provide an estimate of the l-th lag of the spatial covariance given by,

$$\hat{R}_x(l,t) = \frac{1}{c(l)} \sum_{\substack{(m_1,m_2)\in\mathcal{V}(l) \\ m_1 \leq m_2}} (\mathbf{x}(t))_{m_1} (\mathbf{x}(t))^*_{m_2} \tag{8}$$

where $\mathcal{V}(l)$ is the set of array snapshot element index pairs whose difference is l. By collecting the samples into a Toeplitz matrix, we can obtain an estimate of the augmented spatial-covariance matrix. However, estimating the covariance matrix in this manner does not guarantee positive-definiteness and may lead to negative eigenvalues in the spatial covariance matrix estimate. Abramovich *et al.* has proposed methods for Toeplitz Positive-Definite covariance matrix completion from the DAA spatial covariance matrix estimate [12]. However, these methods lead to unstable localization results and are computationally expensive.

Vaidyanathan and Pal have proposed a Rank Enhanced Spatial Smoothing (RESS) algorithm that exploits the hole-free co-array property of the co-prime array [5, 9]. In this work, they have shown how to develop a positive-definite spatial covariance matrix for source localization from the co-prime array snapshot vector. The RESS algorithm can be applied to other array geometries that have hole-free co-arrays. Shakeri *et al.* have applied the RESS algorithm to Sparse Ruler Array designs for the localization of more sources than sensors [13].

The RESS algorithm for the minimum TCP synthetic array can be described in the following major stages, a detailed mathematical description can be found in [9]:

Stage 1: Co-array

The co-array is hole-free for spatial lags in the range, $0 < l < (2N-1)M + \eta_0$. This corresponds to an underlying ULA with $J = (2N-1)M + \eta_0 + 1$ sensor elements.

Stage 2: Direct Data Covariance (DDC)

Since the co-array for the synthetic array is hole-free, spatial covariances for lags $0 < l < (2N-1)M + \eta_0$ are present in the DDC matrix, the outer product of the synthetic array snapshot $\mathbf{R}_{\mathbf{x}'(t)} = \mathbf{x}'(t)\mathbf{x}'^H(t)$.

Stage 3: Vectorize DDC

$$\mathbf{y} = vec(\mathbf{R}_{\mathbf{x}'(t)}) = \mathbf{B}\mathbf{p} + \sigma_n^2 \hat{\mathbf{e}} \tag{9}$$

Where $\mathbf{B}$ is a $Q'^2 \times K$ complex matrix and each column of $\mathbf{B}$ is the Kronecker product between the array manifold vectors of the sparse array and their conjugates and $\mathbf{p} = [\sigma_1^2, \cdots, \sigma_K^2]^\top$, $\hat{\mathbf{e}} = [\mathbf{e}_1^\top, \mathbf{e}_2^\top, \cdots, \mathbf{e}_{Q'}^\top]$, with $\mathbf{e}_i$ a vector of all zeros except for the i-th element.

Stage 4: $(2J-1)$-ULA

Each column of $\mathbf{B}$ has $2J-1$ distinct values. These distinct values can be used to form the spatial covariances from a uniform linear array with $2J-1$ sensors. This ULA is developed by extracting the distinct rows of $\mathbf{B}$, this results in a new matrix $\mathbf{B}_1$ with dimensions $(2J-1) \times K$. The columns of $\mathbf{B}_1$ form the array manifold vectors for a uniform line array with $2J-1$ sensors with a phase center at the J-th sensor.

Stage 5: J-ULA Subarray The $(2J-1)$-ULA can subdivide into overlapping J-ULA subarrays. The J-ULA subarray snapshot can be written as, $\mathbf{y}_{1,i} = \mathbf{B}_{1i}\mathbf{p} + \sigma_n^2\mathbf{e}'_i$ where $\mathbf{B}_{1i}$ is the $J \times K$ array manifold vector matrix for the i-th subarray and $\mathbf{e}'_i$ is a zeros vector except for a 1 in the i-th position. For each subarray we can compute the $J \times J$ Rank-1 spatial covariance matrix as, $\mathbf{R}_{1i} = \mathbf{y}_{1,i}\mathbf{y}_{1,i}^H$.

Stage 6: Rank Enhancement

$$\mathbf{R}_{s\mathbf{x}'(t)} = \frac{1}{J}\sum_{1=1}^{J}\mathbf{R}_{1i} \tag{10}$$

Theorem 1 in [5] proves that $\mathbf{R}_{s\mathbf{x}'(t)}$ has the same form as the positive-definite spatial covariance matrix of a ULA with J sensors. This matrix provides access to the degrees of freedom on the order of the synthetic array aperture.

5. NUMERICAL SIMULATIONS

To examine the above synthesis algorithm we consider a co-prime physical array with $N = 3$ and $M = 5$ and aperture of $L = 25\frac{\lambda}{2}$ (m). The synthetic array is formed by using temporal coherence period of TCP $= \eta_0\tau$ where $\eta_0 = 1$ and τ is the time required to travel a distance of $\frac{\lambda}{2}$ (m). The synthetic array has a total aperture of $L' = 26\frac{\lambda}{2}$ (m) and consists of 10 physical sensors and 8 virtual sensors. The SNR of the signal field was chosen to be unity and is made up of 18 far-field sources uniformly distributed between $-70°$ and $70°$, relative to array broadside. The wave propagation speed, array velocity and signal frequency Ω_0 were chosen to be $1.3\frac{m}{s}$, $1500\frac{m}{s}$, and 1500Hz, respectively.

A total of 200 snapshots are utilized to estimate the DDC matrix $\mathbf{R}_{\mathbf{x}'(t)}$. For each snapshot the minimum TCP is used to form the synthetic array snapshot. The RESS algorithm is then applied to produce a positive definite spatial covariance matrix for source localization via the MUSIC algorithm. Figure 2 and 3 shows the results of the synthesis algorithm and source localization performance. Here we see that after synthesis with the minimum TCP, the co-array for the synthetic array is hole-free and we are able to localize all 18 far-field sources.

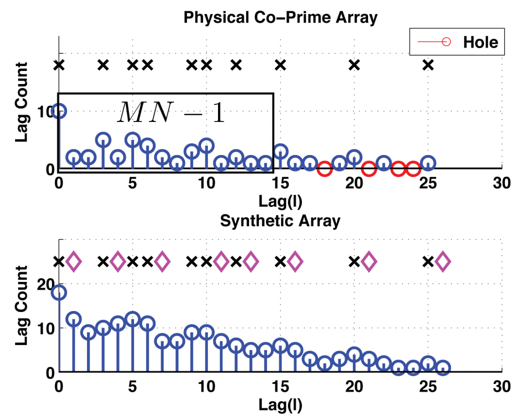

Fig. 2. Array geometry and co-array for $(N = 3, M = 5)$ co-prime physical and synthetic array. $\times$ and $\diamond$ mark the physical and virtual sensor, respectively.

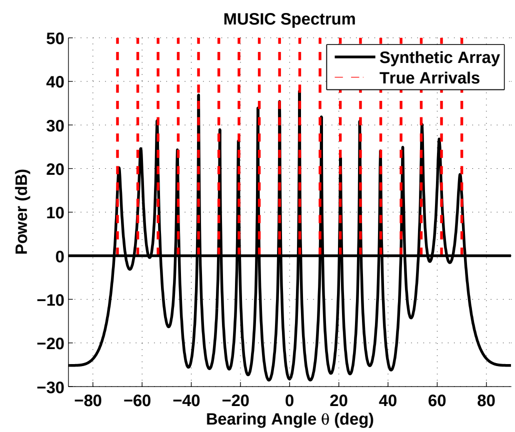

Fig. 3. Synthetic Array MUSIC Specturm

6. CONCLUSIONS

We have found the minimum temporal coherence period needed to fill the co-array of a moving co-prime array. This factor is proportional to only a fraction of the total aperture. By using the RESS algorithm we are able to transform the DDC estimate of the spatial covariance matrix into a positive-definite matrix that encodes the angles of arrivals of sources in the wave-field. This covariance estimate has been used to identify more sources than physical sensors in the moving co-prime array. Future work will examine the rank of the smoothed covariance matrix and develop methods to achieve the localization performance of the J-ULA.

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