OPTIMAL DESIGN OF DIRECTIVITY PATTERNS FOR ENDFIRE LINEAR MICROPHONE ARRAYS

Liheng Zhao[1], *Jacob Benesty*[1], *and Jingdong Chen*[2]

[1]INRS-EMT, University of Quebec
800 de la Gauchetiere Ouest, Suite 6900
Montreal, QC H5A 1K6, Canada

[2]Northwestern Polytechnical University
127 Youyi West Road
Xi'an, Shaanxi 710072, China

ABSTRACT

Directivity pattern or beampattern is an important performance measure in all fixed beamformers. Given a microphone array, how to design the beamforming filter so that the resulting directivity pattern is close to the desired one is a critical issue. In this paper, we study the design of such patterns for endfire uniform linear microphone arrays. By considering the frequency-independent Chebyshev pattern as the desired one, we derive an optimal beamforming filter based on the minimization of the mean-squared error (MSE) under the distortionless constraint. It is shown that the proposed beamformer design can generate beampatterns that are very close to the desired ones and, the larger is the number of microphones, the better is the designed beampattern.

***Index Terms*—** Linear microphone arrays, differential microphone arrays (DMAs), endfire arrays, directivity pattern, beampattern design, white noise gain, directivity factor.

1. INTRODUCTION

In real-world environments, speech quality and intelligibility are adversely affected by noise and reverberation. Therefore, the speech enhancement technology, which aims at combating these problems, is essential in many applications such as hands-free telecommunication and hearing aids. Microphone arrays, which are very promising for speech enhancement, have been widely studied for several decades [1], [2], [3]. Among them, differential microphone arrays (DMAs) have received an increasing research attention recently. DMAs have the nice property that their beampatterns are almost frequency independent [4], [5], [6], [7], [8]. In this paper, we basically show how to design patterns that resemble the DMA ones. Our objective is to describe an optimal approach so that the designed beampattern is as close as possible to the desired one. For this purpose, we first define the MSE criterion between the endfire array beampattern and the desired directivity pattern. Then, we formulate the design into an optimization problem that comprises the minimization of the MSE criterion subject to the distortionless constraint. Thanks to the modified Bessel function, which naturally appears in the formulation, we find the optimal solution to the described problem. Simulation results show that the designed beampattern is very close to the desired directivity pattern and this design gets better as the number of microphones increases.

2. SIGNAL MODEL, PROBLEM FORMULATION, AND DEFINITIONS

We consider a source signal (plane wave), in the farfield, that propagates in an anechoic acoustic environment at the speed of sound, i.e., $c = 340$ m/s, and impinges on a uniform linear sensor array consisting of M omnidirectional microphones, where the distance between two successive sensors is equal to δ. The direction of the source signal to the array is parameterized by the azimuth angle θ. In this scenario, the steering vector (of length M) is given by

$$\mathbf{d}(\omega,\theta) = \begin{bmatrix} 1 & e^{-\jmath\omega\tau_0\cos\theta} & \cdots & e^{-\jmath(M-1)\omega\tau_0\cos\theta} \end{bmatrix}^T, \quad (1)$$

where the superscript ^{T} is the transpose operator, $\jmath = \sqrt{-1}$ is the imaginary unit, $\omega = 2\pi f$ is the angular frequency, $f > 0$ is the temporal frequency, and $\tau_0 = \delta/c$ is the delay between two successive sensors at the angle $\theta = 0$. The acoustic wavelength is $\lambda = c/f$.

In order to avoid spatial aliasing [1], which has the negative effect of creating grating lobes (i.e., copies of the main lobe, which usually points toward the desired signal), it is necessary that the interelement spacing is less than $\lambda/2$, i.e.,

$$\omega\tau_0 < \pi. \quad (2)$$

The condition (2) easily holds for small values of δ and at low frequencies but not at high frequencies.

We consider designing directivity patterns identical to the ones obtained with DMAs [4], [5], [6], [7], [8], where the main lobe is at the angle $\theta = 0$ (endfire direction). For that, a complex weight, $H_m^*(\omega)$, $m = 1, 2, \ldots, M$, is applied at the output of each microphone, where the superscript * denotes complex conjugation. The weighted outputs are then summed together to form the beamformer output. Putting all the gains together in a vector of length M, we get

$$\mathbf{h}(\omega) = \begin{bmatrix} H_1(\omega) & H_2(\omega) & \cdots & H_M(\omega) \end{bmatrix}^T. \quad (3)$$

It is assumed that the desired signal propagates from the endfire direction, so that the corresponding steering vector is $\mathbf{d}(\omega, 0)$. In our context, the distortionless constraint is desired, i.e.,

$$\mathbf{h}^H(\omega)\,\mathbf{d}(\omega,0) = 1, \quad (4)$$

where the superscript ^{H} is the conjugate-transpose operator. Then, the objective is to design the filter, $\mathbf{h}(\omega)$, in such a way that the beampattern of the array is as close as possible to a desired directivity pattern.

With the filter, $\mathbf{h}(\omega)$, and the source at the endfire direction, the array gain in signal-to-noise ratio (SNR) is defined as [8]

$$\mathcal{G}[\mathbf{h}(\omega)] = \frac{\left|\mathbf{h}^H(\omega)\,\mathbf{d}(\omega,0)\right|^2}{\mathbf{h}^H(\omega)\,\mathbf{\Gamma}_{\mathbf{v}}(\omega)\,\mathbf{h}(\omega)}, \quad (5)$$

where $\mathbf{\Gamma_v}(\omega)$ is the pseudo-coherence matrix of the noise signal vector.

The most convenient way to evaluate the sensitivity of the array to some of its imperfections is via the so-called white noise gain (WNG), which is defined by taking $\mathbf{\Gamma_v}(\omega) = \mathbf{I}_M$ in (5), where $\mathbf{I}_M$ is the $M \times M$ identity matrix, i.e.,

$$\mathcal{W}[\mathbf{h}(\omega)] = \frac{\left|\mathbf{h}^H(\omega)\mathbf{d}(\omega,0)\right|^2}{\mathbf{h}^H(\omega)\mathbf{h}(\omega)}. \quad (6)$$

The maximum WNG is given by [8]

$$\mathcal{W}_{\max}[\mathbf{h}(\omega)] = M. \quad (7)$$

Another important measure, which quantifies how the microphone array performs in the presence of reverberation, is the directivity factor (DF). Considering the spherically isotropic (diffuse) noise field, the DF is defined as

$$\mathcal{D}[\mathbf{h}(\omega)] = \frac{\left|\mathbf{h}^H(\omega)\mathbf{d}(\omega,0)\right|^2}{\mathbf{h}^H(\omega)\mathbf{\Gamma}_{\mathrm{d}}(\omega)\mathbf{h}(\omega)}, \quad (8)$$

where the elements of the $M \times M$ matrix $\mathbf{\Gamma}_{\mathrm{d}}(\omega)$ are

$$[\mathbf{\Gamma}_{\mathrm{d}}(\omega)]_{ij} = \frac{\sin[\omega(j-i)\tau_0]}{\omega(j-i)\tau_0} = \mathrm{sinc}[\omega(j-i)\tau_0]. \quad (9)$$

The maximum DF is given by [8]

$$\mathcal{D}_{\max}[\mathbf{h}(\omega)] = \mathbf{d}^H(\omega,0)\mathbf{\Gamma}_{\mathrm{d}}^{-1}(\omega)\mathbf{d}(\omega,0), \quad (10)$$

and it can be shown that [9]

$$\lim_{\delta\to 0}\mathcal{D}_{\max}[\mathbf{h}(\omega)] = M^2. \quad (11)$$

3. BEAMPATTERNS

The beampattern or directivity pattern describes the sensitivity of the beamformer to a plane wave (source signal) impinging on the array from the direction θ. For a uniform linear array, it is mathematically defined as

$$\begin{aligned}\mathcal{B}_M[\mathbf{h}(\omega),\theta] &= \mathbf{d}^H(\omega,\theta)\mathbf{h}(\omega) \quad (12)\\ &= \sum_{m=1}^{M} H_m(\omega)e^{\jmath(m-1)\omega\tau_0\cos\theta}.\end{aligned}$$

Recall that $\mathbf{h}(\omega)$ is designed so that the array looks in the direction $\theta = 0$. For a fixed $\mathbf{h}(\omega)$, it is obvious that

$$\mathcal{B}_M[\mathbf{h}(\omega),-\theta] = \mathcal{B}_M[\mathbf{h}(\omega),\theta] \quad (13)$$

and

$$\mathcal{B}_M[\mathbf{h}(\omega),\theta+2\pi] = \mathcal{B}_M[\mathbf{h}(\omega),\theta]. \quad (14)$$

Therefore, the complex function $\mathcal{B}_M[\mathbf{h}(\omega),\theta]$ is even and periodic. As a result, the study of $\mathcal{B}_M[\mathbf{h}(\omega),\theta]$ is limited to $\theta \in [0,\pi]$.

Let $\mathcal{B}(\theta)$ be a real even periodic function with period 2π and such that $\int_0^\pi |\mathcal{B}(\theta)|\,d\theta$ exists, then $\mathcal{B}(\theta)$ can be written in terms of its Fourier cosine series:

$$\mathcal{B}(\theta) = \sum_{n=0}^{\infty} b_n\cos(n\theta), \quad (15)$$

where

$$b_0 = \frac{1}{\pi}\int_0^\pi \mathcal{B}(\theta)\,d\theta, \quad (16)$$

$$b_i = \frac{2}{\pi}\int_0^\pi \mathcal{B}(\theta)\cos(i\theta)\,d\theta,\ i \geq 1. \quad (17)$$

Now, if we limit this series to the order N, $\mathcal{B}(\theta)$ can be approximated by

$$\mathcal{B}_N(\theta) = \sum_{n=0}^{N} b_n\cos(n\theta), \quad (18)$$

which is a trigonometric polynomial of order N. The function $\mathcal{B}_N(\theta)$ is, in fact, a very general definition of a frequency-independent directivity pattern of order N. It is very much related to the directivity pattern of an Nth-order DMA [4], [8]:

$$\mathcal{B}'_N(\theta) = \sum_{n=0}^{N} a_n\cos^n\theta, \quad (19)$$

and any DMA pattern can be designed with $\mathcal{B}_N(\theta)$. Indeed, we know from the usual trigonometric identities that

$$\cos^n\theta = \sum_i c(n,i)\cos[(n-2i)\theta], \quad (20)$$

where $c(n,i)$ are some binomial coefficients. Substituting (20) into (19), we deduce that any DMA pattern can be written as a general pattern, $\mathcal{B}_N(\theta)$. It is well known that

$$\cos(n\theta) = T_n(\cos\theta), \quad (21)$$

where $T_n(\cdot)$ is the nth Chebyshev polynomial of the first kind [10], which have the recurrence relation:

$$T_{n+1}(\cos\theta) = 2\cos\theta \times T_n(\cos\theta) - T_{n-1}(\cos\theta), \quad (22)$$

with

$$T_0(\cos\theta) = 1,\ T_1(\cos\theta) = \cos\theta.$$

Thus, $\cos(n\theta)$ can be expressed as a sum of powers of $\cos\theta$. Consequently, any general pattern can be written as a DMA pattern. We can then conclude that $\mathcal{B}_N(\theta)$ and $\mathcal{B}'_N(\theta)$ are strictly equivalent.

The relations between the coefficients $b_n,\ n = 0,1,\ldots,N$ of $\mathcal{B}_N(\theta)$ and the coefficients $a_n,\ n = 0,1,\ldots,N$ of $\mathcal{B}'_N(\theta)$ for the first three orders are as follows:

- $N = 1$: $b_0 = a_0$, $b_1 = a_1$;
- $N = 2$: $b_0 = a_0 + \frac{a_2}{2}$, $b_1 = a_1$, $b_2 = \frac{a_2}{2}$; and
- $N = 3$: $b_0 = a_0 + \frac{a_2}{2}$, $b_1 = a_1 + \frac{3a_3}{4}$, $b_2 = \frac{a_2}{2}$, $b_3 = \frac{a_3}{4}$.

For convenience, we can also express (18) as

$$\mathcal{B}_N(\theta) = \mathbf{t}^T(\theta)\mathbf{b}, \quad (23)$$

where

$$\mathbf{t}(\theta) = \begin{bmatrix} 1 & \cos\theta & \cdots & \cos(N\theta) \end{bmatrix}^T \quad (24)$$

and

$$\mathbf{b} = \begin{bmatrix} b_0 & b_1 & \cdots & b_N \end{bmatrix}^T \quad (25)$$

are two vectors of length $N + 1$.

In the rest, we need to make sure that at $\theta = 0$, we have

$$\mathcal{B}_N(0) = \mathcal{B}'_N(0) = 1. \tag{26}$$

Therefore, we will always choose

$$a_0 = 1 - \sum_{n=1}^{N} a_n, \tag{27}$$

$$b_0 = 1 - \sum_{n=1}^{N} b_n. \tag{28}$$

4. MEAN-SQUARED ERROR CRITERION

Considering the frequency-independent Chebyshev pattern, $\mathcal{B}_N(\theta)$, as the desired directivity pattern, the objective becomes to find a proper filter, $\mathbf{h}(\omega)$, so that the array beampattern, $\mathcal{B}_M[\mathbf{h}(\omega), \theta]$, is as close as possible to $\mathcal{B}_N(\theta)$. From now on, it is assumed that θ is a real random variable, which is uniformly distributed in the interval $[0, \pi]$. We define the MSE criterion between the array beampattern and the desired directivity pattern as

$$\begin{aligned} \text{MSE}[\mathbf{h}(\omega)] &= E\left\{|\mathcal{B}_M[\mathbf{h}(\omega), \theta] - \mathcal{B}_N(\theta)|^2\right\} \\ &= E\left[\left|\mathbf{d}^H(\omega, \theta)\mathbf{h}(\omega) - \mathbf{t}^T(\theta)\mathbf{b}\right|^2\right] \\ &= \mathbf{h}^H(\omega)\mathbf{\Phi}_{\mathbf{d}}(\omega)\mathbf{h}(\omega) - \mathbf{h}^H(\omega)\mathbf{\Phi}_{\mathbf{dt}}(\omega)\mathbf{b} \\ &\quad - \mathbf{b}^T\mathbf{\Phi}_{\mathbf{dt}}^H(\omega)\mathbf{h}(\omega) + \mathbf{b}^T\mathbf{\Phi}_{\mathbf{t}}\mathbf{b}, \end{aligned} \tag{29}$$

where $E\{\cdot\}$ denotes mathematical expectation, and

$$\mathbf{\Phi}_{\mathbf{d}}(\omega) = E\left[\mathbf{d}(\omega, \theta)\mathbf{d}^H(\omega, \theta)\right], \tag{30}$$

$$\mathbf{\Phi}_{\mathbf{dt}}(\omega) = E\left[\mathbf{d}(\omega, \theta)\mathbf{t}^T(\theta)\right], \tag{31}$$

$$\mathbf{\Phi}_{\mathbf{t}} = E\left[\mathbf{t}(\theta)\mathbf{t}^T(\theta)\right]. \tag{32}$$

The (i, j)th element of the $M \times M$ matrix $\mathbf{\Phi}_{\mathbf{d}}(\omega)$ can be computed as

$$\begin{aligned} [\mathbf{\Phi}_{\mathbf{d}}(\omega)]_{ij} &= E\left[e^{\jmath\omega(j-i)\tau_0\cos\theta}\right] \\ &= \frac{1}{\pi}\int_0^{\pi} e^{\jmath\omega(j-i)\tau_0\cos\theta} d\theta \\ &= I_0[\jmath\omega(j-i)\tau_0], \end{aligned} \tag{33}$$

where

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\, d\theta \tag{34}$$

is the integral representation of the modified Bessel function of the first kind [10].

In the same way, we can compute the (i, j)th element of the $M \times (N+1)$ matrix $\mathbf{\Phi}_{\mathbf{dt}}(\omega)$ as follows:

$$\begin{aligned} [\mathbf{\Phi}_{\mathbf{dt}}(\omega)]_{ij} &= E\left\{e^{-\jmath\omega(i-1)\tau_0\cos\theta}\cos[(j-1)\theta]\right\} \\ &= \frac{1}{\pi}\int_0^{\pi} e^{-\jmath\omega(i-1)\tau_0\cos\theta}\cos[(j-1)\theta]\, d\theta \\ &= I_{j-1}[-\jmath\omega(i-1)\tau_0]. \end{aligned} \tag{35}$$

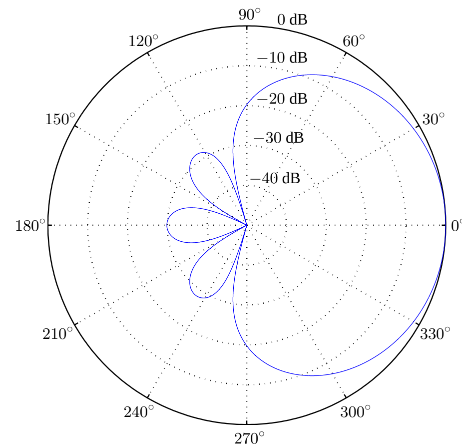

Fig. 1. Desired directivity pattern.

5. OPTIMAL DESIGN

To find the optimal filter, in the MSE sense, for the desired directivity pattern, it is important to minimize the MSE criterion given in (29) subject to the distortionless constraint given in (4), i.e.,

$$\min_{\mathbf{h}(\omega)} \text{MSE}[\mathbf{h}(\omega)] \quad \text{subject to} \quad \mathbf{h}^H(\omega)\mathbf{d}(\omega, 0) = 1. \tag{36}$$

We easily find that the optimal solution is

$$\begin{aligned} &\mathbf{h}_{\text{o}}(\omega) \\ &= \mathbf{h}_{\text{u}}(\omega) + \frac{1 - \mathbf{d}^H(\omega, 0)\mathbf{h}_{\text{u}}(\omega)}{\mathbf{d}^H(\omega, 0)\mathbf{\Phi}_{\mathbf{d}}^{-1}(\omega)\mathbf{d}(\omega, 0)}\mathbf{\Phi}_{\mathbf{d}}^{-1}(\omega)\mathbf{d}(\omega, 0), \end{aligned} \tag{37}$$

where

$$\mathbf{h}_{\text{u}}(\omega) = \mathbf{\Phi}_{\mathbf{d}}^{-1}(\omega)\mathbf{\Phi}_{\mathbf{dt}}(\omega)\mathbf{b} \tag{38}$$

is the unconstrained filter obtained by minimizing $\text{MSE}[\mathbf{h}(\omega)]$.

6. SIMULATIONS

In this section, we carry out simulations to evaluate the performance of the proposed approach. We consider a linear array with $\delta = 2.5$ cm. In the rest, we would like to design the following frequency-independent Chebyshev pattern:

$$\mathcal{B}_2(\theta) = 0.3095 + 0.484\cos\theta + 0.2065\cos(2\theta). \tag{39}$$

This is equivalent to the second-order supercardioid pattern [6]:

$$\mathcal{B}'_2(\theta) = 0.103 + 0.484\cos\theta + 0.413\cos^2\theta. \tag{40}$$

Figure 1 illustrates the desired directivity pattern [eq. (39)]. It should be noticed that similar simulation results can be obtained for other patterns and orders.

In order to avoid numerical problems with the inversion of $\mathbf{\Phi}_{\mathbf{d}}(\omega)$, we replace $\mathbf{\Phi}_{\mathbf{d}}^{-1}(\omega)$ in (37) and (38) by $[\mathbf{\Phi}_{\mathbf{d}}(\omega) + \epsilon\mathbf{I}_M]^{-1}$, where $\epsilon = 10^{-12}$.

First, we derive the optimal design by setting the number of microphones to $M = 3$. The derived beampattern and SNR gains are plotted in Figs. 2 and 3. We can see that the beampattern is similar to the desired directivity pattern in that the desired signal at the endfire direction is perfectly preserved while the signals from the other directions are attenuated. We can also see that the DF is higher than 7 dB while the WNG is negative at low frequencies. Then, we increase the number of microphones to $M = 5$ and investigate its effect on the performance of the optimal design. The beampattern and SNR gains are presented in Figs. 4 and 5. Comparing the patterns in Figs. 1, 2, and 4, we observe that, as the number of microphones increases, the designed beampattern gets closer to the desired directivity pattern as it can be expected. Comparing the SNR gains in Figs. 3 and 5, we can see that increasing the number of microphones slightly improves the DF and makes it frequency independent but degrades the WNG.

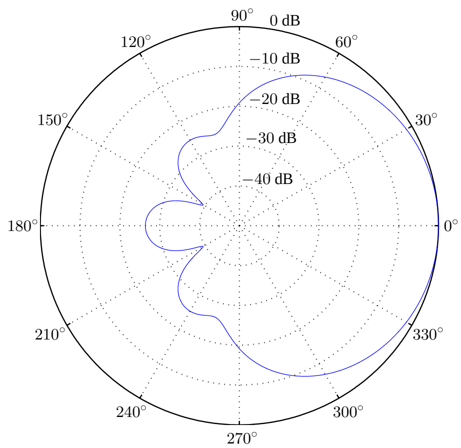

Fig. 2. Designed beampattern. $M = 3$, $\delta = 2.5$ cm, and $f = 1$ kHz.

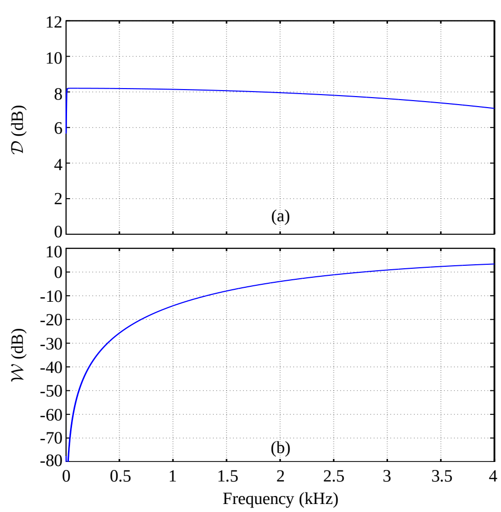

Fig. 3. SNR gains with the designed beampattern: (a) DF and (b) WNG. $M = 3$ and $\delta = 2.5$ cm.

7. CONCLUSIONS

In this paper, we focused on the design of beampatterns that resemble the DMA directivity patterns, which are almost frequency independent. We proposed an optimal design in the MSE sense and it

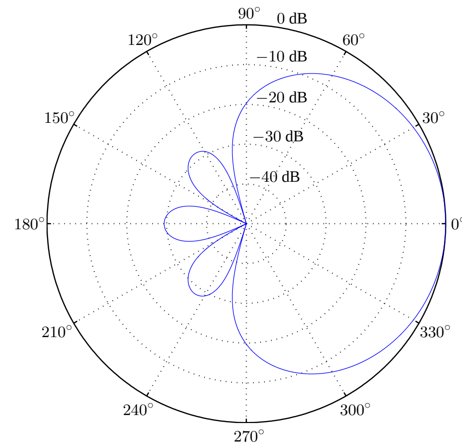

Fig. 4. Designed beampattern. $M = 5$, $\delta = 2.5$ cm, and $f = 1$ kHz.

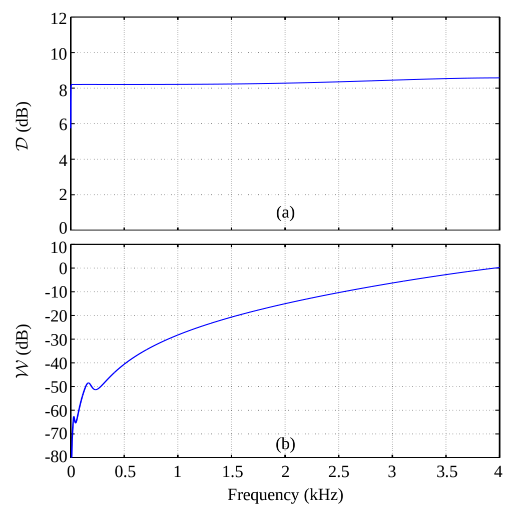

Fig. 5. SNR gains with the designed beampattern: (a) DF and (b) WNG. $M = 5$ and $\delta = 2.5$ cm.

was shown that we can approach the desired directivity patterns with very high precision. Our next step is to extend these ideas and make the proposed approach more robust to white noise amplification by including, for example, some ideas from [11].

8. RELATION TO PRIOR WORK

Microphone arrays can be applied to speech enhancement in noisy and reverberant environments [1], [2], [3]. In the design of fixed beamformers, the beampattern is a very important performance criterion. Among the many types of microphone arrays, DMAs are designed in such a way that their beampatterns obey some desired directivity patterns. In [4], [5], [6], [7], DMAs are constructed by using simple delays and band-pass filters. In [8], [12], DMAs are designed by passing the microphone outputs through a filter with some fundamental constraints on nulls. In this paper, we propose another way to design DMA patterns, which is based on the MSE criterion.

9. REFERENCES

[1] G. W. Elko and J. Meyer, "Microphone arrays," in *Springer Handbook of Speech Processing*, J. Benesty, M. M. Sondhi, and Y. Huang, Eds., Berlin, Germany: Springer-Verlag, 2008, Chapter 50, pp. 1021–1041.

[2] M. Brandstein and D. B. Ward, Eds., *Microphone Arrays: Signal Processing Techniques and Applications*. Berlin, Germany: Springer-Verlag, 2001.

[3] J. Benesty, J. Chen, and Y. Huang, *Microphone Array Signal Processing*. Berlin, Germany: Springer-Verlag, 2008.

[4] G. W. Elko, "Superdirectional microphone arrays," in *Acoustic Signal Processing for Telecommunication*, S. L. Gay and J. Benesty, Eds., Boston, MA: Kluwer Academic Publishers, 2000, Chapter 10, pp. 181–237.

[5] M. Buck, "Aspects of first-order differential microphone arrays in the presence of sensor imperfections," *European Trans. Telecommunications*, vol. 13, pp. 115–122, Mar.-Apr. 2002.

[6] E. De Sena, H. Hacihabiboğlu, and Z. Cvetković, "On the design and implementation of higher-order differential microphones," *IEEE Trans. Audio, Speech, Language Process.*, vol. 20, pp. 162–174, Jan. 2012.

[7] T. D. Abhayapala and A. Gupta, "Higher order differential-integral microphone arrays," *J. Acoust. Soc. Am.*, vol. 127, pp. EL227–EL233, May 2010.

[8] J. Benesty and J. Chen, *Study and Design of Differential Microphone Arrays*. Berlin, Germany: Springer-Verlag, 2012.

[9] A. I. Uzkov, "An approach to the problem of optimum directive antenna design," *Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS*, vol. LIII, no. 1, pp. 35–38, 1946.

[10] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover, 1970.

[11] E. Mabande, A. Schad, and W. Kellermann, "Design of robust superdirective beamformers as a convex optimization problem," in *Proc. IEEE ICASSP*, 2009, pp. 77–80.

[12] J. Chen and J. Benesty, "A general approach to the design and implementation of linear differential microphone arrays," in *Proc. APSIPA ASC*, 2013.