ANGULAR RESOLUTION LIMIT OF TWO CLOSELY-SPACED POINT SOURCES BASED ON INFORMATION THEORETIC CRITERIA

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ABSTRACT

The statistical angular resolution limit (RL) of two closely-spaced point sources in array processing is analyzed based on the framework of hypothesis test. For the first time, the general case where neither the source signals nor the parameters of interest are known under both the null hypothesis and the alternative is considered. Using the theory of misspecified model, the asymptotic distribution of the twice log-generalized-likelihood-ratio statistic under the alternative hypothesis is derived. It is shown to be non-central chi-squared distributed with degrees of freedom equal to the difference in dimensionality of the parameters under both hypotheses, and non-centrality parameter closely related to the Kullback-Leibler divergence of the probability density functions under both hypotheses. The information-theoretic-criteria-based model order selection rules are adopted and a novel detection-based RL is derived. The Cramér-Rao bound (CRB)-based RL is also derived for comparison. It is observed that the detection-based RL and the CRB-based RL are fundamentally different in such general case. Numerical simulations verify the theoretical results.

***Index Terms*—** resolution limit, misspecified model, array processing, hypothesis test, generalized likelihood ratio test, Cramér-Rao bound, information-theoretic-criteria

1. INTRODUCTION

The resolvability of two closely-spaced sources is a fundamental problem in a variety of applications and has been widely studied in the literature. The resolution limit (RL), which is generally defined as the minimum distance (w.r.t. the parameter of interest) beyond which accurate resolution of the signals is highly probable [1, 2], is commonly used to characterize the resolvablity of closely spaced signals. Basically, there are three definitions of resolution. The first one concerns a specific algorithm (e.g., MUSIC) and the relevant criteria are focused on approximately testing the presence of two peaks. We refer the interested readers to [3] for detailed definitions, motivations and comparisons.

The second one is based on estimation accuracy and the RL is defined in the sense of reliability of the parameter estimates [4, 5]. Hence, most works tend to adopt a Cramér-Rao bound (CRB) based criterion [2, 4–6]. One of the most commonly used definition of the RL is $\delta = \gamma\sqrt{\mathrm{CRB}(\delta)}$, where δ is the parameters distance, and the user-selected γ is related to the probability that the parameter estimates of two sources are disjoint [4] ($\gamma = 1$ in [2] and $\gamma = 2$ in [4]). Two source signals are defined to be resolvable if the distance between the two sources (w.r.t. the parameter of interest) is greater than γ times standard deviation of the distance estimation [2, 4].

The last one is based on detection theory and uses the hypothesis test formulation [1, 7–9]. In this framework, *resolvablity* refers to the ability to distinguish whether the measurements are generated by one point source (hypothesis $\mathcal{H}_0$) or two (hypothesis $\mathcal{H}_1$). The generalized likelihood ratio test (GLRT) is the most commonly used method to deal with this problem [1, 7, 8, 10–13], except for [9, 14] employing a Bayesian approach. However, most previous works based on GLRT make some ideal assumptions on the parameters of interest, e.g., the true parameters of the two radar targets are assumed to be known in [10], the frequency of the single real sinusoid under $\mathcal{H}_0$ is assumed to be known in [1] in the spectral analysis context, and the parameter of the single source under $\mathcal{H}_0$ is assumed to be the *center* parameter and known in [11, 12] in array processing context. To the best of our knowledge, no results are available for the general case where all the parameters are unknown.

In addition, the comparison between the detection-based RL and the estimation-based RL has not yet been sufficiently studied. In particular, it is revealed in [8] that a strong relation exists between the RL based on the CRB criterion and the hypothesis test formulation, i.e., based on hypothesis testing, the relation $\delta = \gamma\sqrt{\mathrm{CRB}(\delta)}$ can also be obtained and γ is analytically determined by the pre-specified probability of detection and probability of false alarm. Whereas in an earlier work [15], it is claimed that one can detect the presence of closely-spaced signals at a signal-to-noise ratio (SNR) lower than the SNR required for reliable estimation of the signals' parameters.

The aim of this work is to fill the gap in the framework of hypothesis test and revisit the relation between the detection-based RL and the estimation-based RL shown in [8]. In this paper, the angular RL of two closely-spaced sources in array processing is considered based on the framework of hypothesis test, and the general case where neither the source signals nor the direction of arrivals (DOA) are known under both the null hypothesis and the alternative is addressed. The rest of this paper is organized as follows. In section 2, we introduce the signal model and formulate a binary hypothesis test. In section 3, we derive the detection-based RL by adopting the information-theoretic-criterion (ITC) to determine the number of sources. In section 4, we compare the ITC-based RL with the estimation-based RL. To this end, the closed-form expression of the RL based on Smith's criterion [2] is derived. We present numerical examples in section 5 and finally draw the conclusions in section 6.

Notations: Throughout this paper, matrices are denoted by bold capital letters, and vectors by bold lowercase letters. $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and conjugate transpose, respectively. $\|\cdot\|$ denotes the Euclidean norm. $\otimes$ denotes the Kronecker product. $\boldsymbol{I}_L$

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stands for the $L \times L$ identity matrix. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all real numbers and complex numbers, respectively. $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real part and imaginary part of the argument, respectively. $\chi_L'^2(\lambda)$ is a noncentral chi-squared distribution with L degrees of freedom and non-centrality λ. $Q_{\chi_L'^2(\lambda)}$ is the right tail probability for a noncentral chi-squared probability density function (PDF).

2. PROBLEM FORMULATION

Consider a linear symmetrical sensor array consisting of N elements with known positions given by the vector $\boldsymbol{d} = [d_1 \; d_2 \; \cdots \; d_N]^T$, $\boldsymbol{d} \in \mathbb{R}^{N\times 1}$ and the origin of coordinate being the center of the array. Let $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{C}^{L\times 1}$ denote two far-field and narrow-band source signals in L snapshots. $\boldsymbol{v}(\omega_k) = [e^{j\omega_k d_1} \; e^{j\omega_k d_2} \cdots e^{j\omega_k d_N}]^T, k = 1, 2$ denote the two linearly-independent array steering vectors, where $\omega_k = \frac{2\pi}{\nu}\cos\theta_k$ is the parameter of interest, with θ_k denoting the DOA relative to the baseline of the array and ν the wavelength. The array outputs are corrupted by spatially and temporally white circularly symmetric complex Gaussian noises with mean zero and variance σ^2. After collecting the array outputs from L snapshots into a vector $\boldsymbol{z}$, $\boldsymbol{z} \in \mathbb{C}^{NL\times 1}$, the data model is given by

$$\boldsymbol{z} = \boldsymbol{V}_1\boldsymbol{s}_1 + \boldsymbol{V}_2\boldsymbol{s}_2 + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{V}_1 = \boldsymbol{I}_L \otimes \boldsymbol{v}(\omega_1)$ and $\boldsymbol{V}_2 = \boldsymbol{I}_L \otimes \boldsymbol{v}(\omega_2)$, $\boldsymbol{V}_1, \boldsymbol{V}_2 \in \mathbb{C}^{NL\times L}$.

We assume that ω_1, ω_2, $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are deterministic but unknown. The noise variance σ^2 is assumed to be known. An extension to the case of unknown σ^2 has been considered in [16] but is beyond the scope of this paper. To simplify the derivations, we define $K_r = \sum_{i=1}^N d_i^{2r}, r = 0, 1, 2$. Without loss of generality, we assume that $\omega_1 < \omega_2$ and the parameters distance of the two sources are defined by $\delta \triangleq \omega_2 - \omega_1$.

Let the hypothesis $\mathcal{H}_0$ embody the case where the two sources are unresolvable and are taken as a single source $\boldsymbol{s}_0$ with parameter ω_0, and the hypothesis $\mathcal{H}_1$ represent the case where the two distinct sources are resolvable. A binary hypothesis test formulation of the resolution problem is given by

$$\begin{cases} \mathcal{H}_0 : \boldsymbol{z} = \boldsymbol{V}_0\boldsymbol{s}_0 + \boldsymbol{w} \\ \mathcal{H}_1 : \boldsymbol{z} = \boldsymbol{V}_1\boldsymbol{s}_1 + \boldsymbol{V}_2\boldsymbol{s}_2 + \boldsymbol{w} \end{cases} \tag{2}$$

where $\boldsymbol{V}_0 = \boldsymbol{I}_L \otimes \boldsymbol{v}(\omega_0)$. Both $\boldsymbol{s}_0$ and ω_0 are assumed to be unknown, in contrast to previous works [1, 7, 11–13].

3. RESOLUTION LIMIT BASED ON ITC

The GLRT is the most commonly used tool for the problem (2). Unfortunately, the hypothesis test (2) is nonstandard [17] and the Wilks' theorem does not apply [18]. The asymptotic chi-squared distribution of the GLRT statistic under the null hypothesis no more holds and, in fact, is asymptotically distributed as the maximum of a χ^2 random field [19]. Nevertheless, in section 3.1, we shall show that based on the theory of misspecified model [18, 20–22], we can still obtain the asymptotic distribution of the GLRT statistic under $\mathcal{H}_1$ for the resolution problem. However, as the distribution of the GLRT statistic under the null hypothesis is rather complicated, it is hard to choose the detection threshold given a pre-specified probability of false alarm. In section 3.2, we resort to the ITC to determine the detection threshold and derive an ITC-based RL. The ITC has been used in [23] to discuss the resolution of complex exponentials in the spectral analysis context, using an unconditional model. However, only a heuristic resolution threshold is considered therein and the result is limited to equi-powered signals.

3.1. Asymptotic distribution of the GLRT statistic under $\mathcal{H}_1$

Let $\breve{\boldsymbol{z}} = \left[\Re\{\boldsymbol{z}\}^T \; \Im\{\boldsymbol{z}\}^T\right]^T$ and $\breve{\boldsymbol{w}} = \left[\Re\{\boldsymbol{w}\}^T \; \Im\{\boldsymbol{w}\}^T\right]^T$. We rewrite the binary hypothesis test in the following "realified" form

$$\begin{cases} \mathcal{H}_0 : \breve{\boldsymbol{z}} = \breve{\boldsymbol{H}}_0\breve{\boldsymbol{x}}_0 + \breve{\boldsymbol{w}} \\ \mathcal{H}_1 : \breve{\boldsymbol{z}} = \breve{\boldsymbol{H}}_1\breve{\boldsymbol{x}}_1 + \breve{\boldsymbol{w}} \end{cases} \tag{3}$$

where $\breve{\boldsymbol{x}}_1 = \left[\Re\{\boldsymbol{s}_1\}^T \; \Im\{\boldsymbol{s}_1\}^T \; \Re\{\boldsymbol{s}_2\}^T \; \Im\{\boldsymbol{s}_2\}^T\right]^T$ and $\breve{\boldsymbol{x}}_0 = \left[\Re\{\boldsymbol{s}_0\}^T \; \Im\{\boldsymbol{s}_0\}^T\right]^T$. Let $\boldsymbol{\vartheta}_1 = [\breve{\boldsymbol{x}}_1^T \; \omega_1 \; \omega_2]^T$ and $\boldsymbol{\vartheta}_0 = [\breve{\boldsymbol{x}}_0^T \; \omega_0]^T$ denote the unknown parameters under $\mathcal{H}_1$ and $\mathcal{H}_0$, respectively. When hypothesis $\mathcal{H}_1$ is true, the signal model under hypothesis $\mathcal{H}_0$ can be viewed as a misspecified model [21, 24]. In this case, the maximum-likelihood (ML) estimator under a misspecified model is called the quasi-maximum-likelihood (QML) estimator. It is shown in [20, 21] that the QML estimate converges to a limit that minimizes the Kullback-Leibler (KL) divergence [25] between the true PDF $p_1(\breve{\boldsymbol{z}}|\boldsymbol{\vartheta}_1)$ and the misspecified PDF $p_0(\breve{\boldsymbol{z}}|\boldsymbol{\vartheta}_0)$, i.e.,

$$\begin{aligned} \boldsymbol{\vartheta}_0^* &= \arg\min_{\boldsymbol{\vartheta}_0} D(p_1||p_0) = \arg\min_{\boldsymbol{\vartheta}_0} \|\breve{\boldsymbol{H}}_1\breve{\boldsymbol{x}}_1 - \breve{\boldsymbol{H}}_0\breve{\boldsymbol{x}}_0\|^2/\sigma^2 \\ &= \arg\min_{\boldsymbol{\vartheta}_0} \|\boldsymbol{V}_1\boldsymbol{s}_1 + \boldsymbol{V}_2\boldsymbol{s}_2 - \boldsymbol{V}_0\boldsymbol{s}_0\|^2/\sigma^2 \end{aligned} \tag{4}$$

In order to obtain an analytical expression for the ITC-based RL, we exploit the fact that δ is small and approximate $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ using a first-order Taylor expansion around ω_0, then we have

$$D(p_1||\,p_0) \approx \|\boldsymbol{V}_0(\boldsymbol{s}_1 + \boldsymbol{s}_2 - \boldsymbol{s}_0) + \dot{\boldsymbol{V}}_0(\delta_2\boldsymbol{s}_2 - \delta_1\boldsymbol{s}_1)\|^2/\sigma^2 \tag{5}$$

where $\delta_1 = \omega_0 - \omega_1, \delta_2 = \omega_2 - \omega_0$ and $\dot{\boldsymbol{V}}_0 = \frac{\partial \boldsymbol{V}_0}{\partial \omega_0}$. It is easy to obtain that

$$\breve{\boldsymbol{x}}_0^* = \left[\Re\{\boldsymbol{s}_0^*\}^T \; \Im\{\boldsymbol{s}_0^*\}^T\right]^T, \quad \boldsymbol{s}_0^* = \boldsymbol{s}_1 + \boldsymbol{s}_2 \tag{6}$$

$$\omega_0^* = \frac{\omega_1 + \omega_2}{2} + \frac{\|\boldsymbol{s}_2\|^2 - \|\boldsymbol{s}_1\|^2}{2(\|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2 + 2\Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\})}\delta \tag{7}$$

$$D_{\min} \approx \frac{\|\dot{\boldsymbol{V}}_0\boldsymbol{r}_0^*\|^2}{\sigma^2} = \frac{\|\boldsymbol{s}_1\|^2\|\boldsymbol{s}_2\|^2 - \Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}^2}{\|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2 + 2\Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}}\frac{K_1\delta^2}{\sigma^2} \tag{8}$$

where $D_{\min} = \min_{\boldsymbol{\vartheta}_0} D(p_1||\,p_0)$, $\boldsymbol{r}_0^* = \delta_2^*\boldsymbol{s}_2 - \delta_1^*\boldsymbol{s}_1$, $\delta_2^* = \omega_2 - \omega_0^*$ and $\delta_1^* = \omega_0^* - \omega_1$. The QML estimate of $\boldsymbol{\vartheta}_0$ is given by

$$\hat{\boldsymbol{\vartheta}}_0 = \arg\min_{\boldsymbol{\vartheta}_0} \frac{2}{\sigma^2}\|\breve{\boldsymbol{H}}_1\breve{\boldsymbol{x}}_1 - \breve{\boldsymbol{H}}_0\breve{\boldsymbol{x}}_0 + \breve{\boldsymbol{w}}\|^2 \tag{9}$$

It is also proved that the QML estimate is asymptotically Gaussian for large data records [20] or high SNR [21]. Based on the implicit function theorem [26, Theorem 3.5.1] and following [21, 27], we can obtain

$$\hat{\boldsymbol{\vartheta}}_0 - \boldsymbol{\vartheta}_0^* \approx -\boldsymbol{\Phi}_0^{-1}\boldsymbol{\Psi}_0\breve{\boldsymbol{w}} \tag{10}$$

in the asymptotic sense (large data records or high SNR), where

$$\boldsymbol{\Psi}_0 = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{\vartheta}_0, \breve{\boldsymbol{w}})}{\partial \breve{\boldsymbol{w}}}\right|_{(\boldsymbol{\vartheta}_0^*, 0)}, \quad \boldsymbol{\Phi}_0 = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{\vartheta}_0, \breve{\boldsymbol{w}})}{\partial \boldsymbol{\vartheta}_0}\right|_{(\boldsymbol{\vartheta}_0^*, 0)} \tag{11}$$

$$\boldsymbol{f}(\boldsymbol{\vartheta}_0, \breve{\boldsymbol{w}}) = \frac{2}{\sigma^2}\left(\frac{\partial(\breve{\boldsymbol{H}}_0\breve{\boldsymbol{x}}_0)}{\partial \boldsymbol{\vartheta}_0}\right)^T\left(\breve{\boldsymbol{H}}_1\breve{\boldsymbol{x}}_1 - \breve{\boldsymbol{H}}_0\breve{\boldsymbol{x}}_0 + \breve{\boldsymbol{w}}\right) \tag{12}$$

Let $\hat{\boldsymbol{\vartheta}}_1$ denote the ML estimate of $\boldsymbol{\vartheta}_1$. Based on the implicit function theorem, we can also have [27]

$$\hat{\boldsymbol{\vartheta}}_1 - \boldsymbol{\vartheta}_1 \approx -\boldsymbol{\Phi}_1^{-1}\boldsymbol{\Psi}_1\breve{\boldsymbol{w}} \tag{13}$$

asymptotically, where

$$\boldsymbol{\Psi}_1 = \frac{2}{\sigma^2}\left(\frac{\partial(\breve{\boldsymbol{H}}_1\breve{\boldsymbol{x}}_1)}{\partial\boldsymbol{\vartheta}_1}\right)^T, \qquad \boldsymbol{\Phi}_1 = -\frac{\boldsymbol{\Psi}_1\boldsymbol{\Psi}_1^T}{2}\sigma^2 \tag{14}$$

The Taylor expansion of $\ln p_0(\boldsymbol{z}|\boldsymbol{\vartheta}_0^*)$ and $\ln p_1(\boldsymbol{z}|\boldsymbol{\vartheta}_1)$ around $\hat{\boldsymbol{\vartheta}}_0$ and $\hat{\boldsymbol{\vartheta}}_1$ lead to [18,22,28,29]

$$\ln p_0(\boldsymbol{z}|\boldsymbol{\vartheta}_0^*) \approx \ln p_0(\boldsymbol{z}|\hat{\boldsymbol{\vartheta}}_0) + (\hat{\boldsymbol{\vartheta}}_0 - \boldsymbol{\vartheta}_0^*)^T\boldsymbol{\Phi}_0(\hat{\boldsymbol{\vartheta}}_0 - \boldsymbol{\vartheta}_0^*)/2 \tag{15}$$

$$\ln p_1(\boldsymbol{z}|\boldsymbol{\vartheta}_1) \approx \ln p_1(\boldsymbol{z}|\hat{\boldsymbol{\vartheta}}_1) + (\hat{\boldsymbol{\vartheta}}_1 - \boldsymbol{\vartheta}_1)^T\boldsymbol{\Phi}_1(\hat{\boldsymbol{\vartheta}}_1 - \boldsymbol{\vartheta}_1)/2 \tag{16}$$

Define $\boldsymbol{P}_0 = -\frac{\sigma^2}{2}\boldsymbol{\Psi}_0^T\boldsymbol{\Phi}_0^{-1}\boldsymbol{\Psi}_0$ and $\boldsymbol{P}_1 = \boldsymbol{\Psi}_1^T(\boldsymbol{\Psi}_1\boldsymbol{\Psi}_1^T)^{-1}\boldsymbol{\Psi}_1$, then

$$\ln p_0(\boldsymbol{z}|\boldsymbol{\vartheta}_0^*) \approx \ln p_0(\boldsymbol{z}|\hat{\boldsymbol{\vartheta}}_0) - \breve{\boldsymbol{w}}^T\boldsymbol{P}_0\breve{\boldsymbol{w}}/\sigma^2 \tag{17}$$

$$\ln p_1(\boldsymbol{z}|\boldsymbol{\vartheta}_1) \approx \ln p_1(\boldsymbol{z}|\hat{\boldsymbol{\vartheta}}_1) - \breve{\boldsymbol{w}}^T\boldsymbol{P}_1\breve{\boldsymbol{w}}/\sigma^2 \tag{18}$$

After some mathematical manipulations, we have

$$\boldsymbol{\Psi}_0 = \frac{2}{\sigma^2}\begin{bmatrix}\Re\{\boldsymbol{V}_0^*\} & -\Im\{\boldsymbol{V}_0^*\} & \Re\{\dot{\boldsymbol{V}}_0^*\boldsymbol{s}_0^*\}\\ \Im\{\boldsymbol{V}_0^*\} & \Re\{\boldsymbol{V}_0^*\} & \Im\{\dot{\boldsymbol{V}}_0^*\boldsymbol{s}_0^*\}\end{bmatrix}^T \tag{19}$$

$$\boldsymbol{\Phi}_0 \approx -\frac{2}{\sigma^2}\begin{bmatrix}K_0\boldsymbol{I}_L & \boldsymbol{0} & -K_1\Re\{\boldsymbol{r}_0^*\}\\ \boldsymbol{0} & K_0\boldsymbol{I}_L & -K_1\Im\{\boldsymbol{r}_0^*\}\\ -K_1\Re\{\boldsymbol{r}_0^*\}^T & -K_1\Im\{\boldsymbol{r}_0^*\}^T & K_1\|\boldsymbol{s}_0^*\|^2\end{bmatrix} \tag{20}$$

$$\boldsymbol{\Psi}_1 = 2/\sigma^2 \times$$
$$\begin{bmatrix}\Re\{\boldsymbol{V}_1\} & -\Im\{\boldsymbol{V}_1\} & \Re\{\boldsymbol{V}_2\} & -\Im\{\boldsymbol{V}_2\} & \Re\{\dot{\boldsymbol{V}}_1\boldsymbol{s}_1\} & \Re\{\dot{\boldsymbol{V}}_2\boldsymbol{s}_2\}\\ \Im\{\boldsymbol{V}_1\} & \Re\{\boldsymbol{V}_1\} & \Im\{\boldsymbol{V}_2\} & \Re\{\boldsymbol{V}_2\} & \Im\{\dot{\boldsymbol{V}}_1\boldsymbol{s}_1\} & \Im\{\dot{\boldsymbol{V}}_2\boldsymbol{s}_2\}\end{bmatrix}^T \tag{21}$$

where $\boldsymbol{V}_0^* = \boldsymbol{I}_L \otimes \boldsymbol{v}(\omega_0^*)$, $\dot{\boldsymbol{V}}_0^* = \frac{\partial \boldsymbol{V}_0^*}{\partial \omega_0^*}$, $\dot{\boldsymbol{V}}_1 = \frac{\partial \boldsymbol{V}_1}{\partial \omega_1}$ and $\dot{\boldsymbol{V}}_2 = \frac{\partial \boldsymbol{V}_0}{\partial \omega_2}$. We have proved in [16] that

$$\boldsymbol{P}_0 \overset{F}{\approx} \boldsymbol{P}_0' = \boldsymbol{\Psi}_0^T(\boldsymbol{\Psi}_0\boldsymbol{\Psi}_0^T)^{-1}\boldsymbol{\Psi}_0, \ \text{ if } \|\boldsymbol{V}_0\boldsymbol{s}_0^*\|^2 \gg \|\dot{\boldsymbol{V}}_0\boldsymbol{r}_0^*\|^2 \tag{22}$$

where "$\overset{F}{\approx}$" denotes "approximately equivalent in the sense of Frobenius norm". We note that the condition $\|\boldsymbol{V}_0\boldsymbol{s}_0^*\|^2 \gg \|\dot{\boldsymbol{V}}_0\boldsymbol{r}_0^*\|^2$ acts similarly to the *weak* signal condition for the asymptotic performance of GLRT [30, pp. 205], as we try to detect the component $\|\dot{\boldsymbol{V}}_0\boldsymbol{r}_0^*\|^2$ in the presence of $\|\boldsymbol{V}_0\boldsymbol{s}_0^*\|^2$ and noise. It is the principal case that we are interested in for the resolution problem. An equivalent form of the condition will be

$$\delta \ll \sqrt{\frac{K_0}{K_1}}\frac{\|\boldsymbol{s}_1 + \boldsymbol{s}_2\|^2}{\sqrt{\|\boldsymbol{s}_1\|^2\|\boldsymbol{s}_2\|^2 - \Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}^2}} \tag{23}$$

Let

$$\boldsymbol{\xi} = \sqrt{\frac{2}{\sigma^2}}\begin{bmatrix}\Re\{\dot{\boldsymbol{V}}_0\boldsymbol{r}_0^*\}\\ \Im\{\dot{\boldsymbol{V}}_0\boldsymbol{r}_0^*\}\end{bmatrix}, \ \bar{\boldsymbol{w}} = \sqrt{\frac{2}{\sigma^2}}\breve{\boldsymbol{w}}, \ \boldsymbol{P} = \boldsymbol{P}_1 - \boldsymbol{P}_0' \tag{24}$$

then from (17) and (18) we have

$$2\ln L_G(\boldsymbol{z}) \approx \boldsymbol{\xi}^T\boldsymbol{\xi} + 2\boldsymbol{\xi}^T\bar{\boldsymbol{w}} + \bar{\boldsymbol{w}}^T\boldsymbol{P}\bar{\boldsymbol{w}} \tag{25}$$

where $2\ln L_G(\boldsymbol{z})$ is the twice log-generalized-likelihood-ratio statistic. In [16], we also prove that $\boldsymbol{P}$ is approximately a symmetric, idempotent matrix with rank $2L+1$ and $\boldsymbol{P}\boldsymbol{\xi} \approx \boldsymbol{\xi}$ as δ is small, then

$$2\ln L_G(\boldsymbol{z}) \approx (\boldsymbol{\xi} + \bar{\boldsymbol{w}})^T\boldsymbol{P}(\boldsymbol{\xi} + \bar{\boldsymbol{w}}) \tag{26}$$

As $E\{\bar{\boldsymbol{w}}\bar{\boldsymbol{w}}^T\} = \boldsymbol{I}_{2NL}$, we have [31, Theorem 5.11]

$$2\ln L_G(\boldsymbol{z}) \overset{a}{\sim} \chi_{2L+1}'^2(\lambda), \quad \text{under } \mathcal{H}_1 \tag{27}$$

where

$$\lambda = \frac{2K_1}{\sigma^2}\frac{\|\boldsymbol{s}_1\|^2\|\boldsymbol{s}_2\|^2 - \Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}^2}{\|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2 + 2\Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}}\delta^2 \approx 2D_{\text{min}} \tag{28}$$

It is crucial to emphasize that $2L+1$ is exactly the difference in dimensionality of the parameters under both hypotheses.

3.2. RL based on ITC

The commonly used model order selection rules based on ITC for the problem considered have a common form, i.e., [32]

$$k = \arg\min_k \ -2\ln p_k(\boldsymbol{z}, \hat{\boldsymbol{\vartheta}}^k) + k(2L+1)C(k, NL) \tag{29}$$

where k is the model order, NL is the data length, $C(k, NL) = 2$ for the Akaike information criterion (AIC) [33] and $C(k, NL) = \ln NL$ for the minimum description length (MDL) [34], $p_k(\boldsymbol{z}, \hat{\boldsymbol{\vartheta}}^k)$ is the likelihood function, $\hat{\boldsymbol{\vartheta}}^k$ is the ML estimate of the unknown parameters and $k(2L+1)$ is the total number of model parameters. It is easy to show that for the two point sources resolution problem, we have

$$2\ln L_G(\boldsymbol{z}) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta_{\text{ITC}} \tag{30}$$

where $\eta_{\text{ITC}} = (2L+1)C(k, NL)$ is the threshold determined by ITC. The probability of detection (also referred as the resolution success rate [9,14]) P_s is then given by

$$P_s = Q_{\chi_{2L}'^2(\lambda)}(\eta_{\text{ITC}}) \tag{31}$$

For a pre-specified resolution success rate P_s, the ITC-based RL δ_{I} can be given by

$$\delta_{\text{I}} = \sqrt{\frac{\lambda(P_s)\sigma^2}{2K_1}\frac{\|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2 + 2\Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}}{\|\boldsymbol{s}_1\|^2\|\boldsymbol{s}_2\|^2 - \Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}^2}} \tag{32}$$

from (28), where $\lambda(P_s)$ is obtained by solving (31). We can see that when $\frac{\|\boldsymbol{s}_1+\boldsymbol{s}_2\|^2}{\sigma^2} \gg \frac{\lambda(P_s)}{2K_0}$, the RL δ_{I} satisfies the condition (23).

4. COMPARISON WITH CRB-BASED RL

In previous works, the CRB-based RLs based on $\delta = \gamma\sqrt{\text{CRB}(\delta)}$ are limited to non-closed-form solutions [2] or simplified context (e.g., known source signals [35,36] or one known DOA [37]). To the best of our knowledge, there is no general closed-form expression for the CRB-based RL in the case of both unknown source signals and DOA's. To make a comparison with the CRB-based RL $\delta_{\text{C}} = \sqrt{\text{CRB}(\delta_{\text{C}})}$ according to [2], we obtain the CRB on δ via the *complexified* approach [2] and derive an approximate closed-form expression of δ_{C} based on a high-order Taylor expansion (see [16] for more details, here we simply present the results):

$$\delta_{\text{C}} \approx \left(\frac{2\sigma^2}{K_2 - K_1^2/K_0}\frac{\|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2 - 2\Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}}{\|\boldsymbol{s}_1\|^2\|\boldsymbol{s}_2\|^2 - \Re\{\boldsymbol{s}_1^H\boldsymbol{s}_2\}^2}\right)^{1/4} \tag{33}$$

We can see that the CRB-based RL (33) and the detection-based RL (32) are fundamentally different and no explicit relation exists between them. It is worth noting that [8] requires the binary hypothesis test to be a standard one which can be formulated as testing whether $\delta = 0$ or not, such that the twice log-generalized-likelihood-ratio statistic under $\mathcal{H}_1$ follows a $\chi_1'^2(\lambda')$ distribution where $\lambda' = \delta^2\text{CRB}^{-1}(\delta)$ (i.e., $\delta = \sqrt{\lambda'}\sqrt{\text{CRB}(\delta)}$). In comparison, the equivalent test for (2) is [38]

$$\begin{cases}\mathcal{H}_0: & \delta = 0 \ \text{ or } \ \boldsymbol{s}_1 = \boldsymbol{0} \ \text{ or } \ \boldsymbol{s}_2 = \boldsymbol{0}\\ \mathcal{H}_1: & \text{otherwise}\end{cases} \tag{34}$$

which is nonstandard due to loss of identifiability under the null hypothesis [17] [31, pp. 291].

Remark 1: Define $\mathrm{SNR}_1 = \|\boldsymbol{s}_1\|^2/\sigma^2$, $\mathrm{SNR}_2 = \|\boldsymbol{s}_2\|^2/\sigma^2$ and the correlation factor $\rho = \frac{\Re\{\boldsymbol{s}_1^H \boldsymbol{s}_2\}}{\|\boldsymbol{s}_1\|\|\boldsymbol{s}_2\|}$. For two equi-powered, orthogonal source signals ($\rho = 0$ and $\mathrm{SNR}_1 = \mathrm{SNR}_2$), we can show that $\delta_{\mathrm{I}} \propto \mathrm{SNR}_{\mathrm{T}}^{-1/2}$ while $\delta_{\mathrm{C}} \propto \mathrm{SNR}_{\mathrm{T}}^{-1/4}$, where $\mathrm{SNR}_{\mathrm{T}} = \mathrm{SNR}_1 + \mathrm{SNR}_2$ is the total SNR. Such a result is consistent with [15], although the eigenvalue-based nonparametric detection method is considered in [15] rather than the GLRT and ITC.

Remark 2: For equi-powered sources, $\delta_{\mathrm{I}} \propto (1-\rho)^{-1/2}$ while $\delta_{\mathrm{C}} \propto (1+\rho)^{-1/4}$.

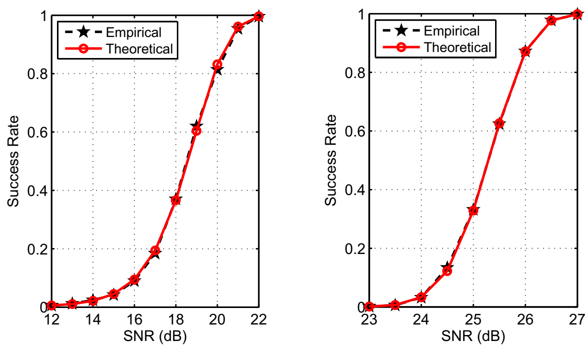
Remark 3: The derivations of both δ_{I} and δ_{C} require that $|\rho|$ is not so close to 1 that only keeping a low-order term in Taylor expansion is already accurate enough. The case of $|\rho| \to 1$ is rather complicate and is beyond the scope of this paper. We also note that in the case of $\boldsymbol{s}_1 = \boldsymbol{s}_2$ ($\rho = 1$), (33) is still meaningful and is reduced to $\delta_{\mathrm{C}}' \approx \left(\frac{2}{K_2 - K_1^2/K_0}\frac{\sigma^2}{\|\boldsymbol{s}_1\|^2}\right)^{1/4} = \left(\frac{2}{K_2 - K_1^2/K_0}\frac{\sigma^2}{\|\boldsymbol{s}_2\|^2}\right)^{1/4}$. It coincides with the result in [2] obtained via symbolic algebra packages, in the case of uniform linear array.

Remark 4: When both $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are known and $\boldsymbol{s}_0 = \boldsymbol{s}_1 + \boldsymbol{s}_2$ for the hypothesis test (2), we can treat either of ω_1 and ω_2 as a nuisance parameter and test whether $\delta = 0$ or not. In this case, the conclusions in [8] do apply and the detection-based RL and CRB-based RL share the similar form. The CRB-based RL for such case has been given in [35].

5. NUMERICAL EXAMPLES

In this section, numerical simulations are performed to verify the theoretical results. We consider a uniform linear array with $N = 16$ sensors and half-wavelength inter-element spacing. The number of snapshots is $L = 20$.

Fig. 1 shows both theoretical and numerical resolution success rate versus $\mathrm{SNR}_{\mathrm{T}}$ for ITC-based detector (30). 500 Monte Carlo simulations are conducted for each $\mathrm{SNR}_{\mathrm{T}}$. The two source signals are equi-powered, with correlation factor $\rho = -0.5$ and DOA's given by $60°$ and $61°$, respectively. With comparison, we can see that numerical simulations match well with the asymptotic theoretical analysis.



(a) AIC-based detector (b) MDL-based detector

Fig. 1. Resolution success rate versus $\mathrm{SNR}_{\mathrm{T}}$ for ITC-based detector

Fig. 2 shows both closed-form and the exact CRB-based RL (i.e., the solution of $\delta_{\mathrm{C}} = \sqrt{\mathrm{CRB}(\delta_{\mathrm{C}})}$) via numerical evaluation, normalized by the Rayleigh limit $\delta_R = \frac{2\pi}{N} \cdot \frac{2}{\nu}$ [39, pp. 48], versus $\mathrm{SNR}_{\mathrm{T}}$ for different ρ. It can be seen that (33) is a good approximation to the exact CRB-based RL. We also notice that in the low SNR range, the closed-form RL for $\rho = -0.8$ deviates from the exact solution. This is because the CRB-based RL becomes close to the Rayleigh limit such that the Taylor polynomial approximation fails.

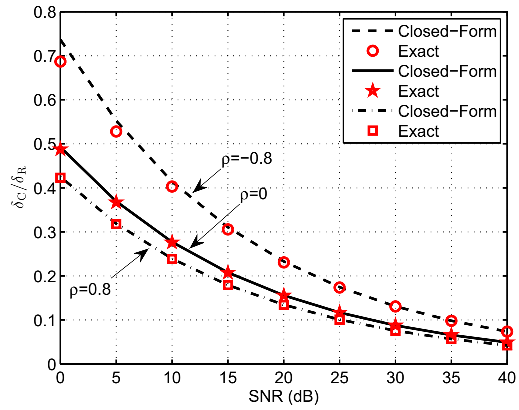


Fig. 2. Normalized CRB-based RL versus $\mathrm{SNR}_{\mathrm{T}}$

Fig. 3 compares the AIC-based RL, MDL-based RL according to (32) and the CRB-based RL according to (33) for the case of two orthogonal, equi-powered source signals. All of them are normalized by the Rayleigh limit. The pre-specified resolution success rate for ITC-based RL is set to $P_s = 0.7$. One can observe that to reduce δ_{I} and δ_{C} by a factor of 10, the total SNR must be increased by 20 dB and 40 dB, respectively.

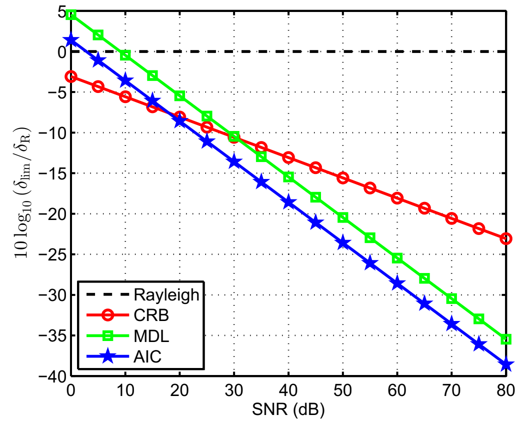


Fig. 3. Comparison of the ITC-based RL and the CRB-based RL

6. CONCLUSIONS AND FUTURE WORK

In this paper, the statistical angular RL of two closely-spaced point sources in array processing based on hypothesis test is considered for the general case where neither the source signals nor the parameters of interest are known under both the null hypothesis and the alternative. The asymptotic distribution of the twice log-generalized-likelihood-ratio statistic under the alternative hypothesis is derived based on the theory of misspecified model. The ITC-based model order selection rules are adopted and the relevant RLs are obtained. To make a comparison, the closed-form expression of the CRB-based RL is also derived. It is shown that in such general case, the detection-based RL and the CRB-based RL are fundamentally different. We also show that for two orthogonal, equi-powered sources, the CRB-based RL is more pessimistic, which accords with earlier observations [15, 23].

An alternative approach based on approximate GLRT has been addressed by the authors and a similar form of detection-based RL is also obtained. Due to space limitations, we shall present it in a journal paper in preparation [16].

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