ESTIMATION OF ARMA STATE PROCESSES BY PARTICLE FILTERING

Iñigo Urteaga and Petar M. Djurić

Department of Electrical & Computer Engineering
Stony Brook University, Stony Brook, NY 11794 USA
{inigo.urteaga, petar.djuric}@stonybrook.edu

ABSTRACT

There are many practical signal processing settings where a state-space model consists of a state described by an ARMA process that is observed via non-linear functions of the state. In this paper, we propose a particle filtering method for sequentially estimating the ARMA process in the presence of unknown parameters. In the considered problem, we have static and dynamic unknowns, and we show how to handle the static parameters so that the estimation of the state process does not degrade with time. We propose a new particle filter that approximates the posterior of all the unknowns by a Gaussian distribution, in combination with a Monte Carlo approach to the Rao-Blackwellization of the static parameters. We demonstrate the performance of the proposed method by extensive computer simulations.

Index Terms— ARMA processes, particle filtering, Rao-Blackwellization, state-space estimation.

1. INTRODUCTION

In many signal processing applications, including speech processing, communications and finance, state-space models are often used [1]. In this paper, we study models where the state is represented by an autoregressive moving average (ARMA) process and the observations are non-linear functions of the state. ARMA processes have two sets of parameters, autoregressive (AR) and moving average (MA) parameters. Modeling data as ARMA processes is suitable for stationary time series. In these models, a sample of the process depends on its past samples as well as on a current and past unobserved random perturbations. An ARMA process is often referred to as ARMA(p, q), where p is the order of the AR part and q the order of the MA part.

Inference of hidden-states in state-space models is a widely studied problem. When the model is linear and the driving noises are Gaussian, the optimal solution is provided by the Kalman filter [2]. When the models deviate from assumptions of linearity and Gaussianity, the processing of the data under such models requires alternative solutions. Sequential Monte Carlo methods, also known as particle filters (PFs), constitute a possible approach. The PFs already have a nice track record of applications in diverse disciplines [3, 4, 5]. Motivated by applications such as representation of asset returns in financial econometrics [6, 7], we are interested in inference of hidden ARMA time series observed through non-linear models.

The sequential estimation of states modeled as ARMA processes is difficult because the MA parameters of the model require non-linear estimation. If the observations are also non-linear functions of the states, one has to look for approaches that can handle such difficulties. Particle filtering methods have the capacity to overcome these challenges.

A problem in using particle filtering methods is that the ARMA model has static unknown parameters and therefore, one has to take special care in treating them. The application of PFs to AR states with non-linear observations has already been studied [8, 9, 10], where inference has been performed using different methods with both known and unknown parameters. When dealing with unknown AR parameters, the parameter estimation has been carried out by generating particles of all the states and parameters [8, 11] or by using Rao-Blackwellization to integrate out the AR parameters [12]. The latter approach is possible when the noise in the state equation is Gaussian. More specifically, after Rao-Blackwellization, one does not have to generate particles of the AR parameters and the variance of the state noise (which is also assumed unknown) and can directly draw particles of the state from a t-distribution. Consequently, the performance of the PF is improved because the generated particles come from a space that has a considerably reduced dimension.

In the literature, there has not been much work on applying particle filtering to ARMA processes in the state equation. In [13], estimation of both the state and time-varying ARMA parameters was proposed. On the contrary, the ARMA parameters are static in our problem, which is usually an inconvenience in particle filtering. We note that for these processes Rao-Blackwellization of the static parameters cannot be applied analytically. In this paper, we show how we can achieve this approximately. We propose an approach that models the posterior of the state and the unknown parameters as a joint Gaussian distribution. This allows us to implement a form of

This work has been supported by NSF under Award CCF-1320626.

Rao-Blackwellization that yields state estimates with reduced variance.

The paper is organized as follows. First, in Section 2 we state the problem. In Section 3, we provide the details of the proposed solution. In the following section, we show results of extensive computer simulations where we demonstrate the feasibility of the proposed method. We conclude the paper with Section 5.

2. PROBLEM FORMULATION

The mathematical description of the state-space model is as follows:

$$x_t = \sum_{i=1}^{p} a_i x_{t-i} + \sum_{i=1}^{q} b_i u_{t-i} + u_t, \qquad \text{state eq. (1)}$$

$$y_t = h(x_t, v_t), \qquad \text{observation eq. (2)}$$

The state is modeled by a generic ARMA(p, q) process, with AR parameters a_i and MA parameters b_i, and the observation is a non-linear function $h(x_t, v_t)$ of the state. The symbols u_t and v_t represent zero-mean white Gaussian processes independent of each other. The model orders p and q as well as the variances of the noise processes are assumed known.

Given the observations $y_{1:t} \equiv \{y_1, y_2, \cdots, y_t\}$, we want to estimate sequentially the posterior distribution of x_t, $f(x_t|y_{1:t})$. We note that the parameters a_i and b_i are of secondary importance. Clearly, we cannot estimate $f(x_t|y_{1:t})$ without the joint posterior of the parameters, and in the next section we show how this posterior is obtained.

3. THE PROPOSED METHOD

Particle filtering [14] is a well known approach for inference in non-linear/non-Gaussian state-space models that approximates the posterior density of the states given all the available observations by

$$f(x_t|y_{1:t}) \approx \sum_{m=1}^{M} w_t^{(m)} \delta(x_t - x_t^{(m)}), \qquad (3)$$

where $x_t^{(m)}$ are particles drawn from a proposal distribution and $w_t^{(m)}$ are the weights associated to the particles.

The method proceeds sequentially, i.e., $f(x_t|y_{1:t})$ is obtained from $f(x_{t-1}|y_{1:t-1})$ according to

$$\begin{aligned} f(x_t|y_{1:t}) &\propto f(y_t|x_t) \int f(x_t|x_{0:t-1}) f(x_{0:t-1}|y_{1:t-1}) \mathrm{d}x_{0:t-1} \\ &\approx f(y_t|x_t) \sum_{m=1}^{M} w_{t-1}^{(m)} f(x_t|x_{0:t-1}^{(m)}). \end{aligned} \qquad (4)$$

In the considered problem with ARMA state processes, the predictive density of the state given all the previous states, $f(x_{t+1}|x_{0:t})$, can only be derived analytically when dealing with known ARMA parameters. However, in the presence of unknown ARMA parameters, new alternatives must be explored. We propose to perform Rao-Blackwellization of the ARMA(p, q) parameters to derive the corresponding predictive density

$$f(x_{t+1}|x_{0:t}) = \int_{\boldsymbol{\theta}} f(x_{t+1}|\boldsymbol{\theta}, x_{0:t}) f(\boldsymbol{\theta}|x_{0:t}) \mathrm{d}\boldsymbol{\theta}, \qquad (5)$$

where $\boldsymbol{\theta} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top = \begin{pmatrix}a_1 & \cdots & a_p & b_1 & \cdots & b_q\end{pmatrix}^\top$. The exact solution to the previous expression is analytically intractable [15, 16], due to (a) the non-standard form of the posterior of the parameters, $f(\boldsymbol{\theta}|x_{0:t})$; and (b) the resulting complex integration. To overcome these difficulties, we propose a two-step approximation of the above integral:

1. Approximation of the joint posterior of the state and the parameters as a multivariate Gaussian, i.e.,

$$f(x_t, \boldsymbol{\theta}_t) \approx \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t),$$

   where we note that the subscript t in $\boldsymbol{\theta}_t$ indicates approximation at time instant t and not that the parameter is time-varying.

2. Monte Carlo integration of equation (5). We obtain a set of J samples for $\boldsymbol{\theta}_t$, $\boldsymbol{\theta}_t^{(j)}$, and assign them appropriate weights $\rho_t^{(j)}$, which allows numerical computation of the predictive density as

$$f(x_{t+1}|x_{0:t}) \approx \sum_{j=1}^{J} \rho_t^{(j)} f(x_{t+1}|\boldsymbol{\theta}_t^{(j)}, x_{0:t}).$$

Using the previously described concepts, we introduce a new PF designed for ARMA hidden-state estimation by numerical Rao-Blackwellization.

At time instant t, consider the random measure

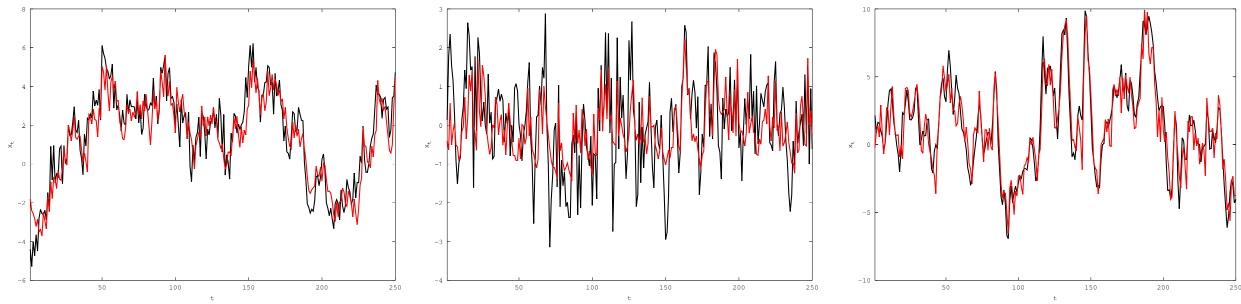
$$\chi_t = \left\{ x_t^{(m,j)}, \boldsymbol{\theta}_t^{(m,j)}, w_t^{(m,j)} \right\},$$

where $m = 1, \cdots, M$ and $j = 1, \cdots, J$. Note that the notation (m, j) means that for a given particle m, we have J children (one per sample of the static parameters obtained through the Monte Carlo integration of (5)), for a total of MJ particles.

Upon reception of a new observation at time instant $t + 1$, the algorithm proceeds as follows:

1. Approximate the joint state-parameter posterior distribution with a multivariate Gaussian:

$$f(x_t, \boldsymbol{\theta}_t|x_{0:t-1}) \approx \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t),$$



(a) AR(2) (b) MA(2) (c) ARMA(1,1)

Fig. 1: Estimation (red) of the hidden-state (black)

where

$$\boldsymbol{\mu}_t = \sum_{i=1}^{M\cdot J} \boldsymbol{\nu}_t \cdot w_t^{(m,j)},$$

$$\boldsymbol{\Sigma}_t = \sum_{i=1}^{M\cdot J} (\boldsymbol{\nu}_t - \boldsymbol{\mu}_t)(\boldsymbol{\nu}_t - \boldsymbol{\mu}_t)^\top \cdot w_t^{(m,j)}$$

with $\boldsymbol{\nu}_t$ being defined as $\boldsymbol{\nu}_t = \begin{pmatrix} x_t^{(m,j)} & \boldsymbol{\theta}_t^{(m,j)} \end{pmatrix}^\top$.

2. Downsample from MJ to M particles, obtaining a set of resampled particles $\overline{x}_t^{(m)}, m = 1, \cdots, M$.

3. Draw J parameter samples for each particle $\overline{x}_t^{(m)}$ from the conditional Gaussian:

$$\boldsymbol{\theta}_{t+1}^{(m,j)} \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{\theta}_t|\overline{x}_t^{(m)}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}_t|\overline{x}_t^{(m)}}) \qquad j = 1, \cdots, J.$$

4. Propagate the particles by drawing from the numerically Rao-Blackwellized mixture distribution:

$$x_{t+1}^{(m,j)} \sim f(x_{t+1}|x_{0:t}^{(m,j)}) \approx \frac{1}{J}\sum_{j=1}^{J} f(x_{t+1}|\boldsymbol{\theta}_{t+1}^{(m,j)}, \overline{x}_t^{(m)}).$$

5. Compute the unnormalized weights for the drawn particles according to

$$\widetilde{w}_{t+1}^{(m,j)} = f(y_{t+1}|x_{t+1}^{(m,j)}),$$

and normalize them to obtain a new random measure $\chi_{t+1} = \left\{x_{t+1}^{(m,j)}, \boldsymbol{\theta}_{t+1}^{(m,j)}, w_{t+1}^{(m,j)}\right\}$.

4. SIMULATION RESULTS

We evaluated the proposed method on the following stochastic volatility model:

$$x_t = \sum_{i=1}^{p} a_i x_{t-i} + \sum_{i=1}^{q} b_i u_{t-i} + u_t, \qquad (6)$$

$$y_t = e^{(x_t/2)} v_t, \qquad (7)$$

where the log-volatility x_t is an ARMA(p, q) process with unknown parameters $\boldsymbol{\theta} = \begin{pmatrix} a_1 & \cdots & a_p & b_1 & \cdots & b_q \end{pmatrix}^\top$ and the driving noises are independent and identically distributed standard Gaussian variables.

The proposed PF successfully estimates the log-volatility x_t for different ARMA(p, q) state models. Figure 1 shows the tracking results for a specific run of the following processes, each evolving for 250 time units:

- AR(2): $x_t = 0.49x_{t-1} + 0.49x_{t-2} + u_t$,
- MA(2): $x_t = u_t + 0.49u_{t-1} + 0.49u_{t-2}$,
- ARMA(1,1): $x_t = 0.8x_{t-1} + u_t + u_{t-1}$.

In order to show the estimation accuracy of the proposed method, the following simulation results were obtained by averaging over 1000 realizations, using $M = 1000$ and $J = 50$ particles, unless otherwise indicated. First, we evaluated the AR(p) case because then one can obtain the closed-form solution to the integral in (5). The tracking accuracy of the proposed method is compared to three other PFs: (a) the benchmark PF that assumes knowledge of the static AR parameters (referred to as `PF Known Param`), (b) the PF where the unknown AR parameters are analytically integrated (`PF Analytical RB`), and (c) the PF where the unknown AR parameters and the state are jointly estimated without any Rao-Blackwellization (`PF Param Estimation`).

Table 1: AR(p) performance

PF type	AR(1) MSE	AR(2) MSE
`PF Known Parameters`	0.84950	1.1262
`PF Analytical RB`	0.86583	1.1520
`Proposed PF Method`	0.86807	1.3658
`PF Param Estimation`	0.87332	1.6689

The results are shown in Table 1 where the entries are the Mean Square Errors (MSEs) of the estimates of the hidden-state variable x_t. The results reveal the validity of the pro-

Table 2: ARMA(1,1) performance

PF Type	ARMA(1,1) state estimation error (MSE)		
	$a_1 = 0.75, b_1 = 0.6$	$a_1 = 0.5, b_1 = 0.5$	$a_1 = 0.2, b_1 = 0.75$
`PF Known Parameters`	1.5418	1.1852	1.1251
`Proposed PF Method (J=100)`	1.5453	1.1999	1.1350
`Proposed PF Method (J=50)`	1.5484	1.2001	1.1353
`Proposed PF Method (J=20)`	1.5553	1.2004	1.1353
`PF Param Estimation`	1.5801	2.2108	1.1401

Table 3: ARMA Vs AR performance

PF Type	State estimation error (MSE)		
	$a_1 = 0.75, b_1 = 0.6$	$a_1 = 0.5, b_1 = 0.5$	$a_1 = 0.2, b_1 = 0.75$
`PF Known Parameters`	1.5332	1.1824	1.1278
`Proposed PF Method (J=50)`	1.5423	1.1933	1.1373
`Analytically RB AR(2)`	1.5475	1.2071	1.1512
`Analytically RB AR(3)`	1.5497	1.2161	1.1582
`Analytically RB AR(4)`	1.5568	1.2253	1.1651
`Analytically RB AR(5)`	1.5676	1.2331	1.1755

posed method because it outperforms the PF that jointly estimates all the parameters and has accuracy comparable to that of `PF Analytical RB`.

We extended the simulation study to the general ARMA case. Once again, the proposed method provides an accurate tracking of the hidden state, as shown in Fig. 2 for a realization of an ARMA(1,1) with parameters $a_1 = 0.8, b_1 = 1$.

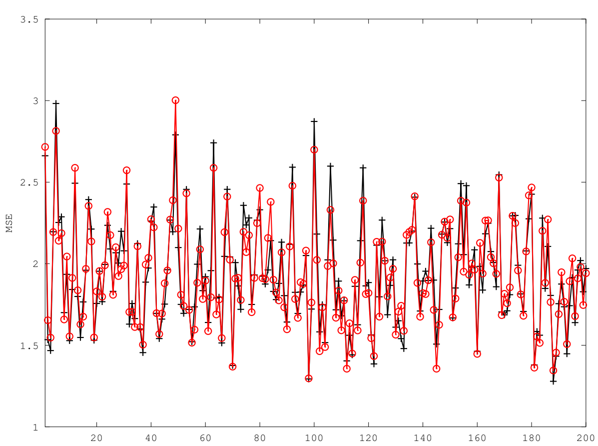


Fig. 2: ARMA(1,1) estimation (red) of the state (black)

In Table 2, we present the MSEs of the estimates of x_t for three ARMA models. They show that the proposed method is consistently more accurate than `PF Param Estimation`. We note that we did not implement `PF Analytical RB` because there is no closed-form solution to equation (5). We observe that the accuracy of the proposed method is reasonably close to the ideal case (i.e., `PF Known Parameters`). As expected, the more particles we use for approximating the integral in (5) (larger J), the more accurate our estimates become.

Finally, we provide more insight into the advantage of our approach by emphasizing the importance of considering ARMA processes in the state equation instead of approximating them by higher order AR processes. We reiterate that for the latter we can apply analytical Rao-Blackwellizion of the parameters. The results are shown in Table 3. They suggest that the proposed method outperforms the method based on AR process approximation and analytical Rao Blackwellization. Interestingly, the results also show that the performance of the AR-based method deteriorates with the increase of the order of the approximating AR process.

5. CONCLUSIONS

In this paper we propose a new particle filter that tracks a hidden ARMA state in the presence of non-linear observations. The method is based on a Monte Carlo form of Rao-Blackwellization of the static ARMA parameters and an approximation to the posterior of all the unknowns by a multivariate Gaussian distribution. We conducted extensive simulation studies on the stochastic volatility model. All the results show the validity of the proposed method. Future work includes detailed study of the method where the state is modeled with higher dimensional ARMA processes.

6. REFERENCES

[1] R. H. Shumway and D. S. Stoffer, *Time Series Analysis and Its Applications: With R Examples (Springer Texts in Statistics)*. Springer, 2010.

[2] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Prentice-Hall, 1979.

[3] J. S. Liu and R. Chen, "Sequential Monte Carlo methods for dynamic systems," *Journal of the American Statistical Association*, vol. 93, pp. 1032–1044, 1998.

[4] P. M. Djurić and M. F. Bugallo, *Particle Filtering*. John Wiley & Sons, Inc., 2010, ch. Chapter 5 in "Adaptive Signal Prcoessing: Next Generation Solutions", pp. 271–331.

[5] A. Doucet and A. M. Johansen, "A tutorial on particle filtering and smoothing: Fifteen years later," The Institute of Statistical Mathematics Department of Statistics, University of Warwick, Tech. Rep., December 2008.

[6] S. T. Rachev, J. S. J. Hsu, B. S. Bagasheva, and F. J. Fabozzi, *Bayesian Methods in Finance*. John Wiley & Sons, 2008.

[7] D. E. Johnston, I. Urteaga, and P. M. Djurić, "Replication and optimization of hedge fund risk factor exposures," in *2013 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2013, pp. 8712–8716.

[8] C. M. Carvalho and H. F. Lopes, "Simulation-based sequential analysis of Markov switching stochastic volatility models," *Computational Statistics & Data Analysis*, vol. 51, no. 9, pp. 4526 – 4542, 2007.

[9] H. F. Lopes and R. S. Tsay, "Particle filters and bayesian inference in financial econometrics," *Journal of Forecasting*, vol. 30, no. 1, pp. 168–209, 2011.

[10] A. Doucet and V. B. Tadic, "Parameter estimation in general state-space models using particle methods," *Annals of the Institute of Statistical Mathematics*, vol. 55, pp. 409–422, 2003.

[11] J. Liu and M. West, *Combined Parameter and State Estimation in Simulation Based Filtering*. Springer, 2001, ch. Chapter 10 in "Sequential Monte Carlo Methods in Practice", pp. 197–224.

[12] P. M. Djurić, M. Khan, and D. E. Johnston, "Particle filtering of stochastic volatility modeled with leverage," *IEEE Journal of Selected Topics in Signal Processing*, vol. 6, no. 4, pp. 327–336, 2012.

[13] R. Huang, H. Zheng, and E. E. Kuruoglu, "Time-varying ARMA stable process estimation using sequential Monte Carlo," *Signal, Image and Video Processing*, pp. 1–8, 2011.

[14] P. M. Djurić, J. H. Kotecha, J. Zhang, Y. Huang, T. Ghirmai, M. F. Bugallo, and J. Míguez, "Particle filtering," *IEEE Signal Processing Magazine*, vol. 20, no. 5, pp. 19–38, September 2003.

[15] J. F. Monahan, "Fully Bayesian analysis of ARMA time series models," *Journal of Econometrics*, vol. 21, pp. 307–331, 04 1983.

[16] S. Chib and E. Greenberg, "Bayes inference in regression models with ARMA(p,q) errors," *Journal of Econometrics*, vol. 64, no. 12, pp. 183 – 206, 1994.