PERFECT PERIODIC SEQUENCES FOR IDENTIFICATION OF EVEN MIRROR FOURIER NONLINEAR FILTERS

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ABSTRACT

In this paper we consider the identification of a class of linear-in-the parameters nonlinear filters that has been recently introduced, the so-called even mirror Fourier nonlinear filters. We show that perfect periodic sequences can be derived for these filters. A periodic sequence is perfect for a nonlinear filter if all cross-correlations between two different basis functions, estimated over a period, are zero. By applying perfect periodic sequences as input signals to even mirror Fourier nonlinear filters, it is possible to model unknown nonlinear systems exploiting the cross-correlation method. Then, the most relevant basis functions, i.e., those that guarantee the most compact representation of the nonlinear system according to some information criterion, can be easily estimated. Experimental results on the identification of a real nonlinear system illustrate the effectiveness of the proposed approach.

***Index Terms*—** Nonlinear filters, even mirror Fourier nonlinear filters, perfect periodic sequences, cross-correlation method.

1. INTRODUCTION

In the field of system identification, appropriate deterministic input signals have been proposed in the literature as an alternative to the use of white random signals. Among them, perfect periodic sequences [1], [2] have been studied and proposed as inputs for linear system identification [3]. A periodic sequence is called perfect if all the cross-correlations between two different basis functions of the modeling filter, estimated over a period, are zero. Thus, with perfect periodic sequences (PPSs) as inputs, the linear basis functions $x(n-i)$, with i ranging between 0 and the sequence period N, form an orthogonal set. It has been proved in the literature that PPSs optimize the convergence speed of the NLMS algorithm [4, 5]. Moreover, without output noise, an NLMS algorithm excited by a PPS of period N is able to identify a linear system within N samples [4], [5]. The approach has been recently extended to the identification of multichannel linear systems [6], [7]. Efficient identification algorithms for linear filters, which require just a multiplication, an addition and a subtraction per sample, have been developed for perfect sequences [8] and have been then extended to imperfect periodic sequences [9]. PPSs have been also considered in areas related to signal processing, such as information theory [10], [11], communications [12], [13], [14], [15], and acoustics [4], [16].

In the field of nonlinear signal processing, input signals alternative to the usual Gaussian white noise inputs have been investigated for the identification of nonlinear systems. For example, pseudo-random multilevel sequences have been proposed in [17] to identify Volterra and extended Volterra filters. In [18], a Wiener model has been estimated using a multilevel sequence, generated using a single binary maximal length sequence, and subsequently used to obtain a truncated Volterra model by a change of basis. An efficient algorithm for the identification of linear-in-the-parameters (LIP) nonlinear systems using periodic sequences has been described in [19], [20], as an extension of the method for linear filters proposed in [8], [9].

The class of LIP nonlinear systems has been recently enriched by a novel family of nonlinear filters, called even mirror Fourier nonlinear (EMFN) filters [21], [22]. Their basis functions are even mirror symmetric, as the waveforms defining the discrete cosine transform. It has been shown in [21], [22] that the EMFN basis functions and their linear combinations constitute an algebra on the interval $[-1, 1]$ that satisfies all the requirements of the Stone-Weierstrass theorem [23]. As a consequence, EMFN filters are universal approximators for causal, time invariant, finite-memory, continuous, nonlinear systems, as the well-known Volterra filters. Moreover, the basis functions of the EMFN filters are orthogonal for white uniform input signals in the interval $[-1, 1]$, and thus they can be simply estimated with cross-correlation methods [24]. A white uniform input signal is useful for modeling nonlinear systems since it is broadband and explicitly bounds the range of inputs for which the model is valid. According to the orthogonality property of the EMFN basis functions, it is possible to conjecture the existence of a deterministic quasi-uniform sequence for which the orthogonality condition is guaranteed on a finite period. Indeed, we found that it is possible to develop PPSs for EMFN filters. These sequences allow a simple and effective identification of nonlinear systems by means of the cross-correlation method, avoiding the drawbacks of the stochastic inputs, i.e., the length of the time averages necessary to obtain a reasonable accuracy of the coefficient estimates [24, page 77]. In this paper, we refer specifically to the novel family of EMFN filters and develop perfect periodic sequences for their efficient identification.

The paper is organized as follows. EMFN filters and their properties are summarized in Section 2. PPSs for EMFN filters are introduced in Section 3. Nonlinear systems identification using PPS is discussed in Section 4. Experimental results concerning the construction of PPSs and the identification of a real nonlinear system are presented in Section 5. Concluding remarks are given in Section 6.

The following notation is used throughout the paper. Intervals are represented with square brackets, $E[\,]$ indicates expectation, $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^+$ the set of positive natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_1$ is the unit interval $[-1, +1]$, $< x(n) >_L$ indicates time average over L successive samples of $x(n)$.

2. EVEN MIRROR FOURIER NONLINEAR FILTERS

EMFN filters have been recently introduced [21], [22] to approximate the input-output relationship of discrete-time, time-invariant,

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finite-memory, causal, continuous, nonlinear systems given by

$$y(n) = f[x(n), x(n-1), \ldots, x(n-N+1)], \qquad (1)$$

where f is a real continuous function and $x(n)$ belongs to $\mathbb{R}_1$. For 1-dimensional functions,

$$y = f[\xi], \qquad (2)$$

a complete set of even mirror basis functions that can arbitrarily well approximate (2) is given by

$$1, \sin(\frac{1}{2}\pi\xi), \cos(\pi\xi), \sin(\frac{3}{2}\pi\xi), \ldots, \cos(k\pi\xi), \sin(\frac{2k+1}{2}\pi\xi), \ldots \qquad (3)$$

where 1 is the basis function of order 0, $\sin(\frac{2k+1}{2}\pi\xi)$ is a basis function of order $2k+1$, and $\cos(k\pi\xi)$ is of order $2k$, with $k \in \mathbb{N}$.

To develop a set of even mirror basis functions for the N-dimensional case, we write the functions in (3) for $\xi = x(n)$, $\xi = x(n-1), \ldots, \xi = x(n-N+1)$, and multiply in any possible manner the functions of different variable, taking care of avoiding repetitions. The order of each N-dimensional basis function is then defined as the sum of the orders of the constituent 1-dimensional basis functions. It has been shown in [21], [22] that the linear combination of these basis functions form an algebra that satisfies all the requirements of the Stone-Weierstrass theorem and can arbitrarily well approximate (1). Therefore, the EMFN filters are universal approximators, as the well known Volterra filters. Moreover, their basis functions are orthogonal for white uniform input signals in $\mathbb{R}_1$. In this case, we can easily find an unbiased estimate for the coefficients of the EMFN filter approximating (1) using the cross-correlation method [24]. In fact, the generic coefficient g_l of the EMFN basis function $f_l(n)$ is given by

$$g_l = \frac{E[f_l(n)y(n)]}{E[f_l^2(n)]}, \qquad (4)$$

where the expectation can be estimated using time averages.

The EMFN basis functions of order 1, 2, 3 are shown in Table 1, while the basis function of order 0 is equal to 1. An EMFN filter of order K and memory N is the linear combination of

$$\mathcal{N}_T(K, N) = \binom{N+K}{N} \qquad (5)$$

EMFN basis functions and has the same complexity of a Volterra filter with same order and memory. In what follows, $S_f(K, N)$ indicates the set of basis functions of order less than or equal to K and memory N, with cardinality $\mathcal{N}_T(K, N)$. $S_{f,n}(K, N)$ indicates the subset of $S_f(K, N)$ formed by the basis functions that are function of $x(n)$, which can be proved to have cardinality $\mathcal{N}_T(K-1, N)$. $f_l(n)$ indicates the l-th EMFN basis function estimated at time n, with l ranging between 1 and the cardinality of the set $f_l(n)$ belongs to. R_l indicates the number of sine and cosine terms in $f_l(n)$. Thus, for a white uniform input signal in $\mathbb{R}_1$,

$$E[f_l^2(n)] = 2^{-R_l}.$$

The orthogonality property of the basis functions of the EMFN filters is lost when the input signal is not white nor uniformly distributed in $\mathbb{R}_1$, and thus the cross-correlation technique may become not useful for system identification. In general, the main problem of this technique, when stochastic inputs are used, is the length of the time averages necessary to obtain a reasonable estimate of the coefficients [24, page 77]. In fact, the estimate may require a huge number of samples, and thus the cross-correlation approach is often not feasible in practice. To overcome this difficulty, in the next section we introduce PPSs for EMFN filters, i.e., periodic sequences that guarantee the orthogonality of the basis functions on a finite time interval. Using these sequences, it is possible to obtain an exact estimate of the coefficients of the EMFN filter applying again (4), where the expectations are replaced by time averages on one or a few periods of the PPS.

Table 1. Basis functions of even mirror Fourier nonlinear filters

Order 1:

$$\sin[\tfrac{1}{2}\pi x(n)], \ldots, \sin[\tfrac{1}{2}\pi x(n-N+1)].$$

Order 2:

$$\begin{gathered}
\cos[\pi x(n)], \ldots, \cos[\pi x(n-N+1)], \\
\sin[\tfrac{1}{2}\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-1)], \ldots, \\
\sin[\tfrac{1}{2}\pi x(n-N+2)]\sin[\tfrac{1}{2}\pi x(n-N+1)], \\
\sin[\tfrac{1}{2}\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-2)], \ldots, \\
\sin[\tfrac{1}{2}\pi x(n-N+3)]\sin[\tfrac{1}{2}\pi x(n-N+1)], \\
\vdots \\
\sin[\tfrac{1}{2}\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-N+1)].
\end{gathered}$$

Order 3:

$$\begin{gathered}
\sin[\tfrac{3}{2}\pi x(n)], \ldots, \sin[\tfrac{3}{2}\pi x(n-N+1)], \\
\cos[\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-1)], \ldots, \\
\cos[\pi x(x-N+2)]\sin[\tfrac{1}{2}\pi x(n-N+1)], \\
\vdots \\
\cos[\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-N+1)], \\
\sin[\tfrac{1}{2}\pi x(n)]\cos[\pi x(n-1)], \ldots, \\
\sin[\tfrac{1}{2}\pi x(n-N+2)]\cos[\pi x(n-N+1)], \\
\vdots \\
\sin[\tfrac{1}{2}\pi x(n)]\cos[\pi x(n-N+1)], \\
\sin[\tfrac{1}{2}\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-1)]\sin[\tfrac{1}{2}\pi x(n-2)], \ldots \\
\vdots \\
\sin[\tfrac{1}{2}\pi x(n)]\sin[\tfrac{1}{2}\pi x(n-N+2)]\sin[\tfrac{1}{2}\pi x(n-N+1)].
\end{gathered}$$

3. PERFECT PERIODIC SEQUENCES FOR EMFN FILTERS

Let us consider a sequence $x_0, x_1, \ldots, x_{L-1}$ of period L. Such a sequence is perfect for an EMFN filter of order K and memory N if all the cross-correlations between two different basis functions, estimated over a period, are zero, i.e., if

$$< f_l(n) \cdot f_m(n) >_L = 0, \qquad (6)$$

for all $f_l(n) \in S_{f,n}(K, N)$, $f_m(n) \in S_f(K, N)$ with $f_l(n) \neq f_m(n)$. We consider $f_l(n) \in S_{f,n}(K, N)$ instead of $S_f(K, N)$ since, for example, if the condition

$$< \sin[\frac{1}{2}\pi x(n)]\sin[\frac{1}{2}\pi x(n-1)] >_L = 0$$

is verified, then for the periodicity of the sequence it also results

$$< \sin[\frac{1}{2}\pi x(n-j)]\sin[\frac{1}{2}\pi x(n-j-1)] >_L = 0,$$

for all $j = 1, \ldots, N-1$.

Together with the conditions in (6) it is convenient to impose

$$< f_l(n) \cdot f_l(n) >_L = 2^{-R_l}, \qquad (7)$$

for all $f_l(n) \in S_{f,n}(K, N)$ and $f_l(n) \neq 1$, in order to have the same power of a white uniform input signal in $\mathbb{R}_1$ for the basis functions. The condition in (7) allows also to easily form an orthonormal set of basis functions by considering $\phi_l(n) = 2^{R_l} f_l(n)$. It is interesting to note that the system of nonlinear equations defined in (6) and (7) is equivalent to the following simpler system

$$< f_l(n) >_L = 0, \tag{8}$$

for all $f_l(n) \in S_{f,n}(2K, N)$. In fact, the product of two basis functions of order k and h, respectively, can be expanded in the sum of basis functions of maximum order $k + h$. Each basis function in (8) appears in the expansion of at least one of the products in (6). Moreover, imposing (8), both (6) and (7) are satisfied. Indeed, if we expand the products in (6), we find a linear combination of basis functions different from $f_1(n) = 1$, while if we expand the terms in (7) we find a linear combination of basis functions which includes $f_1(n)$. For example, $\sin^2[\frac{k}{2}\pi x(n)] = \{1 + \cos[k\pi x(n)]\}/2$.

For sufficiently large L, the equations in (8) provide an underdetermined system of nonlinear equations in the variables $x_0, x_1, \ldots, x_{L-1}$, that may have infinite solutions. The system in (8) has $Q = \mathcal{N}_T(2K - 1, N)$ equations and, for L ranging between $1.5Q$ and $2Q$, we have always been able to find a solution for it. Any algorithm for the solution of systems of nonlinear equations can be used, in principle, for this purpose. Indeed, we found particularly useful the Newton-Raphson method. This method has been implemented as described in [25, ch. 9.7] with the only modification of reflecting the variables $x_0, x_1, \ldots, x_{L-1}$ in $\mathbb{R}_1$ when they exceeded the range. In our approach, the iterations start from a random distribution of $x_0, x_1, \ldots, x_{L-1}$ in $\mathbb{R}_1$ and the Jacobian matrix is computed analytically. Obviously, employing a numerical method, only an approximate solution for the PPS can be obtained. Nevertheless, the resulting cross-correlations can be made as small as desired, depending on the precision fixed by the stop-condition of the Newton-Raphson method. In our experiments, the algorithm has always been able to converge in few iterations for any selected precision.

The main problem of the system in (8) remains the large number of equations, which increases exponentially with the order K and geometrically with the memory N. In order to reduce the number of equations and variables, it is possible to exploit periodic sequences with specific structures. In the following, a few conditions that allow us to almost halve the number of equations, and thus the number of variables, are given.

1) Symmetry: the PPS is formed with the terms $a_1, a_2, \ldots, a_M$ and the reversed ones $a_M, a_{M-1}, \ldots, a_1$. For any couple of symmetric basis functions, only one equation is considered.

2) Oddness: the PPS is formed with the terms $a_1, a_2, \ldots, a_M$ and the negated ones $-a_1, -a_2, \ldots, -a_M$. All odd basis functions have zero average and can be removed from the system in (8).

3) Oddness-1: the PPS is formed with the terms $a_1, a_2, \ldots, a_M$ and those obtained by alternatively negating one every two terms $a_1, -a_2, a_3, -a_4, \ldots, -a_M$. By construction, all the basis functions that change their sign by alternatively negating one every two samples, as for example $\sin[\frac{1}{2}\pi x(n)] \sin[\frac{1}{2}\pi x(n-1)]$, can be removed by the system in (8).

It is then possible to introduce "oddness-2", "oddness-4", ... conditions by alternatively negating two every four samples, four every eight samples, and so on. It is also possible to exploit two or more conditions together. The reduction in the number of equations comes at the expense of a longer period of the resulting PPS. Nevertheless, the complexity reduction is determinant to solve the system in (8). Indeed, the Newton-Raphson algorithm has memory and processing time requirements that grow with the cube of the number of equations. Therefore, in general, the method is effective for not too large orders K and memory lengths N. A strategy to avoid these difficulties is to resort to simplified models for the EMFN filters, as done for Volterra filters in [26]. As a consequence of the reduction of the involved basis functions, the derivation of PPSs becomes again feasible and effective.

4. NONLINEAR SYSTEM IDENTIFICATION USING PPS

In this Section we deal with the identification of a discrete-time, time-invariant, finite-memory, causal, continuous, nonlinear system. Let us first assume that the input-output relationship of the nonlinear system is expressed as a linear combination of EMFN basis functions up to a given order K and memory of N samples,

$$y(n) = \sum_l g_l f_l(n). \tag{9}$$

We can identify the system using a PPS of period L, designed with the procedure outlined in the previous Section. The estimation of the coefficients g_l is immediate by estimating the cross-correlations between the output of the system and the basis functions over a period mL with $m \in \mathbb{N}^+$. Indeed, it results

$$\widehat{g}_l = \frac{< f_l(n)y(n) >_{mL}}{< f_l^2(n) >_{mL}} \approx g_l, \tag{10}$$

Since the autocorrelations $< f_l^2(n) >_{mL}$ can be pre-computed, the computational cost of (10) is just a multiplication and an addition per basis function and per input sample.

Let us now consider the case where the input-output relationship of the system to be identified is a linear combination of EMFN basis functions with memory N and maximum order greater than K. It is possible to see that, if we use for the identification of this system a PPS for EMFN filters of order K and memory N, there is an error affecting mainly the coefficients of the higher-order basis functions, while, in general, this error has only a marginal influence on the coefficients of the lower-order basis functions. Similarly, in the case where the input-output relationship of the system to be identified is a linear combination of EMFN basis functions with order K and memory greater than N, there is still an error affecting mainly the coefficients of the basis functions associated with the most recent samples $x(n)$, $x(n-1)$, ..., while, in general, this error has only a marginal influence on the coefficients of the basis functions associated with the less recent samples $x(n-N+1)$, $x(n-N+2)$, ... The proofs are omitted here because of the space limitations.

Another advantage offered by PPSs is that, according to the orthogonality of the basis functions on a period, it is possible to rank them with reference to the reduction obtained in the mean square error (MSE). The reduction in the MSE obtained by the l-th basis function is given by

$$\delta \text{MSE}_l = \frac{< f_l(n)y(n) >_{mL}^2}{< f_l^2(n) >_{mL}}. \tag{11}$$

Using (10) and (11), it is possible to minimize any information criterion, in order to obtain a compact representation for the nonlinear system. The most common criteria, exploited in the experiment described in the next Section, are

- the Akaike's information criterion (AIC) [27],
- the Final Prediction Error (FPE) [27],
- the Khundrin's law of iterated logarithm criterion (LILC) [28],
- the Bayesian information criterion (BIC) or Schwarz criterion [29].

5. EXPERIMENTAL RESULTS

We present in this Section some experimental results that illustrate the generation of PPSs, using the proposed procedure, and their potentialities in the identification of real nonlinear systems.

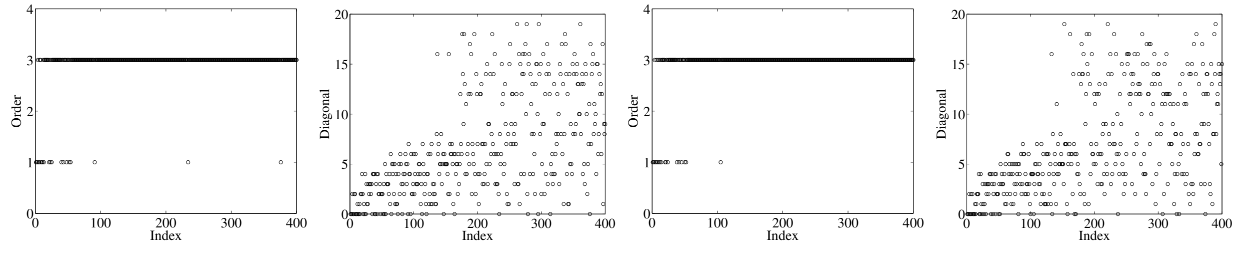

Fig. 1. Order and diagonal number of the first 400 selected basis functions for (a)-(b) the EMFN filter, (c)-(d) the Volterra filter.

Table 2. Results of Newton-Raphson method

Seq.	Q	M	L	Iter.	Max XC
1	3003	4505	4505	12	1.3 E-16
2	2232	3348	6696	12	1.2 E-16
3	1567	2351	4702	9	1.5 E-16
4	1116	1675	6700	10	1.3 E-16
5	593	891	7124	9	1.2 E-16

In the first experiment, we highlight the ability of the Newton-Raphson to solve the system in (8). In particular, we develop PPSs for an order 3, memory 10 EMFN filter (i.e., with 286 coefficients). The algorithm starts from a random uniform distribution of the variables $x_0, \ldots, x_{N-1}$ and optimizes them till the maximum absolute value of the averages of the basis functions on a period is less than 10^{-15}. Table 2 summarizes the results obtained applying the Newton-Raphson method to the full system in (8) (Seq. 1) and to the reduced systems obtained exploiting the sequence oddness (Seq. 2), symmetry (Seq. 3), oddness and oddness-1 (Seq. 4), oddness, oddness-1, and symmetry (Seq. 5). Table 2 provides the number of equations in the system Q, the number of independent variables M, the period L of the sequence, the number of iterations (Iter.) necessary for the Newton-Raphson method to converge, and the maximum cross-correlation (Max XC) between the basis functions of the resulting sequence. We can notice that the Newton-Raphson method converges in at most 12 iterations and that a number of independent variables $M \simeq 1.5Q$ is sufficient to find a solution.

In the second experiment, we consider the identification of a guitar pedal, a Behringer Overdrive TO100. At 8 kHz sampling frequency, the system has a memory length lower than 20 samples. Thus, a PPS for an EMFN of order 3, memory 20, exploiting oddness, oddness-1, and symmetry, and with period of $L = 201412$ samples has been fed to the amplifier input. The corresponding output has been recorded with a notebook. Since similar results have been obtained for all settings of the pedal "drive" control, only the results for one of them are reported here. Table 3 shows the number of terms selected by the AIC (with parameter 4), FPE, LILC, and BIC information criteria, and the corresponding MSE for the EMFN filter, estimated on a period L with the cross-correlation method of (10), and for a Volterra filter, estimated with the method of [32] on the same data. In this experiment, the EMFN filter provides results at least as good as the Volterra filter in the identification of the nonlinear system. The main advantage of our method is the remarkable reduction in the computational complexity. The method of [32] has been chosen for comparison purposes since it is one of the most computationally efficient identification methods for LIP nonlinear systems. Nevertheless, the computational cost of the method of [32] is of order TBS^2 operations, i.e., multiplications and additions, with T the number of samples used for the identification, B the number of candidate basis functions, and S the number of selected basis functions. In contrast, the cross-correlation method has a computational cost of only TB operations. In our experiment, while the execution of the cross-correlation required a processing time of few minutes, the method of [32] requested hours of simulation.

Table 3. Results of identification of Behringer TO100.

Filter	Information Criterion	Selected Bases	MSE
EMFN	AIC(4)	372	$2.01E{-}3$
	FPE	576	$2.00E{-}3$
	LILC	313	$2.01E{-}3$
	BIC	176	$2.02E{-}3$
Volterra	AIC(4)	352	$2.23E{-}3$
	FPE	577	$2.22E{-}3$
	LILC	205	$2.23E{-}3$
	BIC	161	$2.24E{-}3$

Figure 1 shows the order and the diagonal number of the first 400 selected basis functions. By definition the "diagonal number" of a basis function is the maximum time difference between the samples involved in its expression. For example, $\sin[\frac{1}{2}\pi x(n)]\sin[\frac{1}{2}\pi x(n-2)]$ has diagonal number 2. We can see that the system is odd (since only odd terms are selected) and that low diagonal numbers are selected in the first terms. According to Figure 1, if a compact representation is desired, we could model the system with a simplified filter (EMFN or Volterra) with maximum diagonal number 5 or 6. Also from this viewpoint, the EMFN filter provides results at least as good as the Volterra filter.

6. CONCLUSIONS AND OPEN ISSUES

In this paper, we derive periodic sequences which guarantee perfect orthogonality of the EMFN basis functions on a finite period. A methodology to build such PPSs, based on the Newton-Raphson algorithm, is suggested. Using PPSs as input signals, nonlinear systems can be easily and efficiently identified with the cross-correlation approach. Identification of a compact representation of the nonlinear system can be also easily obtained by ranking the basis functions according to their ability to reduce the MSE. Experimental results involving the identification of a real nonlinear system are presented.

Topics not dealt with in this paper, due to space limitations, are the derivation of PPSs for different filter configurations, such as simplified EMFN filters or linear and EMFN filters.

Open issues include the derivation of some theoretical result about the minimal length of a PPS for an EMFN filter of order K and memory N. Reasonably, the minimal length should be around the number of nonlinear equations Q of the system in (8), but with the Newton-Raphson method it has been impossible to find PPSs unless the number of independent variables is larger than Q.

Examples of PPSs can be downloaded from `http://www.units.it/ipl/res_PSeqs.htm`.

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