

# INTRODUCING LEGENDRE NONLINEAR FILTERS

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## ABSTRACT

This paper introduces a novel sub-class of linear-in-the-parameters nonlinear filters, the Legendre nonlinear filters. Their basis functions are polynomials, specifically, products of Legendre polynomial expansions of the input signal samples. Legendre nonlinear filters share many of the properties of the recently introduced classes of Fourier nonlinear filters and even mirror Fourier nonlinear filters, which are based on trigonometric basis functions. In fact, Legendre nonlinear filters are universal approximators for causal, time invariant, finite-memory, continuous, nonlinear systems and their basis functions are mutually orthogonal for white uniform input signals. In adaptive applications, gradient descent algorithms with fast convergence speed and efficient nonlinear system identification algorithms can be devised. Experimental results, showing the potentialities of Legendre nonlinear filters in comparison with other linear-in-the-parameters nonlinear filters, are presented and commented.

**Index Terms**— Nonlinear system identification, linear-in-the-parameters nonlinear filters, universal approximators, orthogonality property.

## 1. INTRODUCTION

Linear-in-the-parameters (LIP) nonlinear filters constitute a broad class of nonlinear models, which includes most of the commonly used finite-memory or infinite-memory nonlinear filters. The class is characterized by the property that the filter output depends linearly on the filter coefficients. Among LIP filters, the most popular are the truncated Volterra filters [1], still actively studied and used in applications [2, 3, 4, 5, 6, 7, 8, 9]. Other elements of the class include particular cases of Volterra filters, as the Hammerstein filters [1, 10, 11, 12, 13], and modified forms of Volterra filters, as memory and generalized memory polynomial filters [14, 15]. Filters based on functional expansions of the input samples, as functional link artificial neural networks (FLANN) [16] and radial basis function networks [17], also belong to the LIP class. A review under a unified framework of finite-memory LIP nonlinear filters can be found in [18]. Infinite-memory LIP nonlinear filters have also been studied [19, 20, 21, 22, 23] and used in applications.

Recently, Fourier Nonlinear (FN) filters [24, 25] and Even Mirror Fourier Nonlinear (EMFN) filters [26, 25] have been added to the finite-memory LIP class. FN and EMFN filters derive from the truncation of a multidimensional Fourier series expansion of a periodic repetition or an even mirror periodic repetition, respectively, of the nonlinear function we want to approximate. They are based on trigonometric function expansions as the FLANN filters. Differently from the FLANN filters, the FN and EMFN basis functions form an

algebra that satisfies all the requirements of the Stone-Weierstrass approximation theorem [27]. Consequently, these filters can arbitrarily well approximate any causal, time invariant, finite-memory, continuous, nonlinear system, as the well known Volterra filters. In contrast to Volterra filters, their basis functions are mutually orthogonal for white uniform input signals in  $[-1, +1]$ . This property, is particularly appealing since it allows the derivation of gradient descent algorithms with fast convergence speed. Moreover, efficient identification algorithms for nonlinear systems can be devised. Between the two filter classes, EMFN filters should be the family of choice since they are able to provide a much more compact representation of nonlinear systems than FN filters [25]. In terms of modelling performance, it has been shown that EMFN filters can be better models than Volterra filters in presence of strong nonlinearities. In contrast, for weak or medium nonlinearities Volterra filters should be preferred to EMFN filters [25].

In this paper, we introduce a novel sub-class of finite-memory LIP nonlinear filters based on orthogonal polynomials. While the approach to introduce the novel filter class can be applied to any family of orthogonal polynomials defined on a finite interval, we specifically focus on the class of Legendre polynomials, which have been already considered for nonlinear filtering [28, 29]. Thus, the novel LIP nonlinear filters are named Legendre Nonlinear (LN) filters. They are based on polynomial basis functions as the Volterra filters, and present properties similar to FN and EMFN filters. Specifically, their basis functions satisfy all the requirements of the Stone-Weierstrass approximation theorem and thus LN filters are universal approximators, as well as the Volterra, FN, and EMFN filters. Their basis functions are orthogonal for white uniform input signals in  $[-1, +1]$  and thus LN filters share all the benefits offered by FN and EMFN filters in terms of convergence speed of gradient descent adaptation algorithms and efficient identification algorithms. Moreover, since LN filters are based on polynomial basis functions including the linear function, their modelling capabilities are similar to those of Volterra filters. Therefore, LN filters can provide more compact models than EMFN filters for weak or medium nonlinearities.

It is worth noting that Legendre polynomials have already been used for nonlinear filtering in Hammerstein models [28, 29], FLANN filters [30, 31, 32], and neural networks [33]. Nevertheless, the approaches of the literature do not make use of cross-terms, i.e., products between different basis functions, which can be very important for modelling nonlinear systems [18]. The corresponding basis functions do not form an algebra, because they are not complete under product. Thus, in contrast to the filters proposed in this paper, those in [28, 29] are not universal approximators for causal, time invariant, finite-memory, continuous, nonlinear systems.

The rest of the paper is organized as follows. Section 2 de-

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rives the LN filters and discusses their properties. Section 3 presents experimental results that illustrate the advantages of the novel LIP class. Concluding remarks follow in Section 4.

Throughout the paper the following notation is used. Sets are represented with curly brackets, intervals with square brackets, while the following convention for brackets:  $\{[(\dots\{[(\dots)]\dots)]\}$  is used elsewhere.  $\text{leg}_i(x)$  indicates a Legendre polynomial of order  $i$ .  $\delta_{ij}$  is the Kronecker delta.

## 2. LEGENDRE NONLINEAR FILTERS

In this section, we first review the Stone-Weierstrass theorem, which is used to build the proposed filters, and the Legendre polynomials. Then we derive the class of LN filters and discuss their properties. The proposed approach can be applied also to other classes of orthogonal polynomials, originating other similar families of nonlinear filters.

### 2.1. The Stone-Weierstrass theorem

The input-output relationship of a time-invariant, finite-memory, causal, continuous, nonlinear system can be expressed by a nonlinear function  $f$  of the  $N$  most recent input samples,

$$y(n) = f[x(n), x(n-1), \dots, x(n-N+1)], \quad (1)$$

where the input signal  $x(n)$  is assumed to take values in the range  $\mathbb{R}_1 = \{x \in \mathbb{R}, \text{ with } |x| \leq 1\}$ ,  $y(n) \in \mathbb{R}$  is the output signal, and  $N$  is the system memory.

Equation (1) can be interpreted as a multidimensional function in the  $\mathbb{R}_1^N$  space, where each dimension corresponds to a delayed input sample. This representation has been already exploited, for example, to represent truncated Volterra filters, where the nonlinearity is mapped to multidimensional kernels that appear linearly in the input-output relationship [1].

In the proposed approach, the nonlinear function  $f[x(n), x(n-1), \dots, x(n-N+1)]$  is expanded with a series of basis functions  $f_i$ ,

$$f[x(n), x(n-1), \dots, x(n-N+1)] = \sum_{i=1}^{+\infty} c_i f_i[x(n), x(n-1), \dots, x(n-N+1)], \quad (2)$$

where  $c_i \in \mathbb{R}$ , and  $f_i$  is a continuous function from  $\mathbb{R}_1^N$  to  $\mathbb{R}$ , for all  $i$ . Every choice of the set of basis functions  $f_i$  defines a different kind of nonlinear filters, which can be used to approximate the nonlinear systems in (1). We are particularly interested in nonlinear filters able to arbitrarily well approximate every time-invariant, finite-memory, continuous, nonlinear system. To this purpose, we apply the well known Stone-Weierstrass theorem [27]:

“Let  $\mathcal{A}$  be an algebra of real continuous functions on a compact set  $K$ . If  $\mathcal{A}$  separates points on  $K$  and if  $\mathcal{A}$  vanishes at no point of  $K$ , then the uniform closure  $\mathcal{B}$  of  $\mathcal{A}$  consists of all real continuous functions on  $K$ ”.

According to the Stone-Weierstrass theorem any algebra of real continuous functions on the compact  $\mathbb{R}_1^N$  which separates points and vanishes at no point is able to arbitrarily well approximate the continuous function  $f$  in (1). A family  $\mathcal{A}$  of real functions is said to be an algebra if  $\mathcal{A}$  is closed under addition, multiplication, and scalar multiplication, i. e., if (i)  $f+g \in \mathcal{A}$ , (ii)  $f \cdot g \in \mathcal{A}$ , and (iii)  $cf \in \mathcal{A}$ , for all  $f \in \mathcal{A}$ ,  $g \in \mathcal{A}$  and for all real constants  $c$ .

**Table 1.** Legendre polynomials

$\text{leg}_0(x)$	$= 1$
$\text{leg}_1(x)$	$= x$
$\text{leg}_2(x)$	$= \frac{1}{2}(3x^2 - 1)$
$\text{leg}_3(x)$	$= \frac{1}{2}x(5x^2 - 3)$
$\text{leg}_4(x)$	$= \frac{1}{8}(35x^4 - 30x^2 + 3)$
$\text{leg}_5(x)$	$= \frac{1}{8}x(63x^4 - 70x^2 + 15)$

### 2.2. Legendre polynomials

In this paper, we are interested in developing a class of nonlinear filters based on Legendre polynomials. These are orthogonal polynomial in  $\mathbb{R}_1$ , i.e.,

$$\int_{-1}^1 \text{leg}_i(x) \text{leg}_j(x) dx = \frac{2}{2i+1} \delta_{ij}. \quad (3)$$

The Legendre polynomials can be obtained with the following recursive relation

$$\text{leg}_{i+1}(x) = \frac{2i+1}{i+1} x \text{leg}_i(x) - \frac{i}{i+1} \text{leg}_{i-1}(x), \quad (4)$$

with  $\text{leg}_0(x) = 1$  and  $\text{leg}_1(x) = x$ . The first six Legendre polynomials are listed in Table 1.

Note that according to (3),

$$\int_{-1}^1 \text{leg}_i(x) dx = \int_{-1}^1 \text{leg}_i(x) \text{leg}_0(x) dx = 0 \quad (5)$$

for all  $i > 0$ .

The product of two Legendre polynomials of order  $i$  and  $j$ , respectively, can be expressed as a linear combination of Legendre polynomials up to the order  $i+j$  [34].

It can be easily proved that the set of Legendre polynomials satisfies all the requirements of Stone-Weierstrass theorem on the compact  $\mathbb{R}_1$ . Thus, we can arbitrarily well approximate any continuous function from  $\mathbb{R}_1$  to  $\mathbb{R}$  with a linear combination of Legendre polynomials.

### 2.3. LN filters

We are now interested in developing a set of Legendre basis functions that allows us to arbitrarily well approximate any nonlinear system (1). We interpret the continuous nonlinear function  $f[x(n), x(n-1), \dots, x(n-N+1)]$  as a multidimensional function in the  $\mathbb{R}_1^N$  space, where each dimension corresponds to a delayed input sample. It is then possible to give account of the Legendre basis functions in the  $N$ -dimensional case, passing from  $\mathbb{R}_1$  to  $\mathbb{R}_1^N$ . To this purpose, we first consider the 1-dimensional Legendre basis functions, i.e., the Legendre polynomials, for  $x = x(n), x(n-1), \dots, x(n-N+1)$ :

$$\begin{aligned} &1, \text{leg}_1[x(n)], \text{leg}_2[x(n)], \text{leg}_3[x(n)], \dots \\ &1, \text{leg}_1[x(n-1)], \text{leg}_2[x(n-1)], \text{leg}_3[x(n-1)], \dots \\ &\vdots \\ &1, \text{leg}_1[x(n-N+1)], \text{leg}_2[x(n-N+1)], \text{leg}_3[x(n-N+1)], \dots \end{aligned}$$

Then, to guarantee completeness of the algebra under multiplication, we multiply the terms having different variables in any possible manner, taking care of avoiding repetitions. It is easy to verify that this family of real functions and their linear combinations constitutes an algebra on the compact  $[-1, 1]$  that satisfies all the requirements of the Stone-Weierstrass theorem. Indeed, the set of functions is closed

under addition, multiplication (because of [34]) and scalar multiplication. The algebra vanishes at no point due to the presence of the function of order 0, which is equal to 1. Moreover, it separates points, since two separate points must have at least one different coordinate  $x(n-k)$  and the linear term  $\text{leg}_1[x(n-k)] = x(n-k)$  separates these points. As a consequence, the nonlinear filters exploiting these basis functions are able to arbitrarily well approximate any time-invariant, finite-memory, continuous, nonlinear system.

Let us define the order of an  $N$ -dimensional basis function as the sum of the orders of the constituent 1-dimensional basis functions. Avoiding repetitions, we thus obtain the following basis functions: The basis function of order 0 is the constant 1. The basis functions of order 1 are the  $N$  1-dimensional basis functions of the same order, i.e. the linear terms:

$$x(n), x(n-1), \dots, x(n-N+1).$$

The basis functions of order 2 are the  $N$  1-dimensional basis functions of the same order and the basis functions originated by the product of two 1-dimensional basis functions of order 1. Avoiding repetitions, the basis functions are:

$$\begin{aligned} &\text{leg}_2[x(n)], \text{leg}_2[x(n-1)], \dots, \text{leg}_2[x(n-N+1)], \\ &x(n)x(n-1), \dots, x(n-N+2)x(n-N+1) \\ &x(n)x(n-2), \dots, x(n-N+3)x(n-N+1) \\ &\vdots \\ &x(n)x(n-N+1). \end{aligned}$$

Thus, we have  $N \cdot (N+1)/2$  basis functions of order 2.

Similarly, the basis functions of order 3 are the  $N$  1-dimensional basis functions of the same order, the basis functions originated by the product between an 1-dimensional basis function of order 2 and an 1-dimensional basis function of order 1, and the basis functions originated by the product of three 1-dimensional basis functions of order 1. This constructive rule can be iterated for any order  $P$ .

The basis functions of order  $P$  can also be obtained by (i) multiplying in every possible way the basis functions of order  $P-1$  by those of order 1, (ii) deleting repetitions, and (iii) applying the following substitution rule for products between factors having the same time index:

$$\text{leg}_i(x)\text{leg}_1(x) = \text{leg}_i(x)x \longrightarrow \text{leg}_{i+1}(x).$$

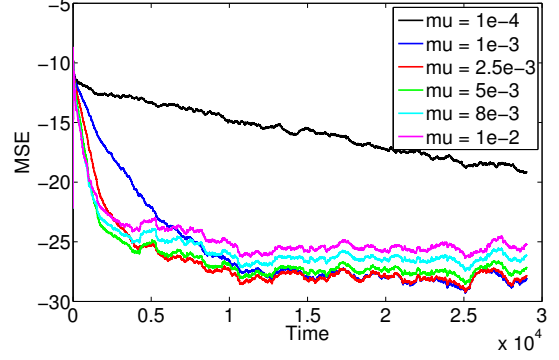
In the last passage, the property that the product between two Legendre polynomials is a linear combination of Legendre polynomials has been exploited. This rule for generating the basis functions of order  $P$  from those of order  $P-1$  is the same applied for Volterra filters, thus the two classes of filters have the same number of basis functions of order  $P$ , memory  $N$ . In our case, the linear combination of all the Legendre basis functions of the same order  $P$  defines an LN filter of uniform order  $P$ , whose number of terms is

$$\binom{N+P-1}{P}, \quad (6)$$

with  $N$  is the memory length. The linear combination of all the basis functions with order ranging from 0 to  $P$  and memory length of  $N$  samples defines an LN filter of nonuniform order  $P$ , whose number of terms is

$$\binom{N+P}{N}. \quad (7)$$

By exploiting the orthogonality property of the Legendre polynomials, it can be verified that the basis functions are orthogonal in



**Fig. 1.** Learning curves of LN filters with strong nonlinearity for different values of the step-size.

$\mathbb{R}_1^N$ . Taking two different basis functions  $f_i$  and  $f_j$ , the orthogonality condition is written as

$$\begin{aligned} &\int_{-1}^{+1} \dots \int_{-1}^{+1} f_i[x(n), \dots, x(n-N+1)] \cdot \\ &f_j[x(n), \dots, x(n-N+1)] \cdot dx(n) \dots dx(n-N+1) = 0, \end{aligned} \quad (8)$$

which immediately follows since the basis functions are product of Legendre polynomials which satisfy (3) and (5). As a direct consequence of this orthogonality property, the expansion of  $f[x(n), \dots, x(n-N+1)]$  with the proposed basis functions is a generalized Fourier series expansion [35]. Moreover, the basis functions are orthogonal for a white uniform distribution of the input signal in  $\mathbb{R}_1$ ,

$$\begin{aligned} &\int_{-1}^{+1} \dots \int_{-1}^{+1} f_i[x(n), \dots, x(n-N+1)] \cdot f_j[x(n), \dots, x(n-N+1)] \cdot \\ &p[x(n), \dots, x(n-N+1)] \cdot dx(n) \dots dx(n-N+1) = 0, \end{aligned} \quad (9)$$

where  $p[x(n), \dots, x(n-N+1)]$  is the probability density of the  $N$ -tuple  $[x(n), \dots, x(n-N+1)]$ , equal to the constant  $1/2^N$ . As a consequence, as done for FN and EMFN filters, it is possible to devise for LN filters simple identification algorithms using input signals with white uniform distributions in the range  $[-1, +1]$ . Moreover, a fast convergence of the gradient descent adaptation algorithms, used for nonlinear systems identification, is expected in this situation.

In contrast to FN and EMFN filters, the LN filters have the linear terms  $x(n), x(n-1), \dots, x(n-N+1)$  among the basis functions, and this property makes these filters better fitted to model weak or medium nonlinearities, providing efficient models in all those situations where Volterra filters give efficient models. Thus, the LN filter class combines the best characteristics of Volterra filters (universal approximation, presence of a linear term, polynomial basis functions) and of EMFN nonlinear filters (orthogonality property, good approximation of strong nonlinearities).

### 3. SIMULATION RESULTS

To show the potentialities of LN filters, we consider the identification of a real-world nonlinear device, i.e., a Presonus TubePRE singlechannel microphone/instrument tube preamplifier. The preamplifier provides a drive potentiometer that controls the amount of tube saturation, i.e., the amount of applied distortion on the output signal.

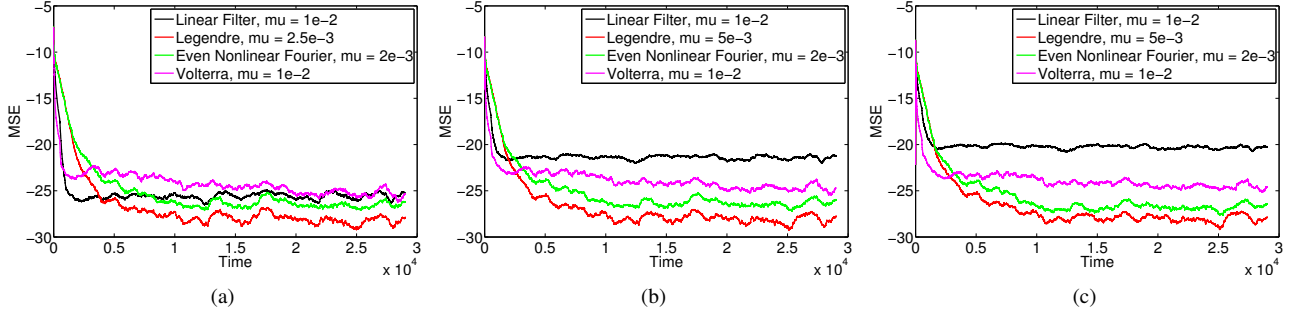


Fig. 2. Learning curves with (a) weak, (b) medium, and (c) strong nonlinearity.

The device was connected to a computer running NU-Tech Framework [36] by means of a professional sound card (MOTU 8pre). A white uniform input signal in the interval  $[-1, +1]$  at 48 kHz sampling frequency was applied at the preamplifier input and the corresponding output was recorded on computer. By acting on the drive control, three different degrees of nonlinearity have been generated and used in the test: a weak, a medium, and a strong nonlinearity. At the maximum used volume, the amplifier introduces on a 1 kHz sinusoidal input a second order and a third order harmonic distortion respectively of 11.9% and 3.7% for the weak nonlinearity, 26.2% and 6.8% for the medium nonlinearity, 40.2% and 1.5% for the strong nonlinearity. The harmonic distortion is defined as the ratio, in percent, between the magnitude of each harmonic and that of the fundamental frequency. In all conditions, the nonlinear system had memory length of around 20 samples and, thus, the system was identified using i) a linear filter of 20 sample memory, and ii) an LN, iii) an EMFN, and iv) a Volterra filter all having memory of 20 samples, order 3, and 1771 coefficients. An LMS algorithm adapting all coefficients with the same step-size has been used in the identification.

When dealing with different nonlinear filter structures we must take into account that the filters have different modeling abilities, which translate to different steady state Mean-Square-Errors (MSE), and different convergence properties. A difficult choice is that of the step-size of the adaptation algorithm that guarantees a fair comparison between the different filters. In the past, this problem was addressed [26, 25] by choosing the step-size that guarantees similar initial convergence speed for all nonlinear filters. In this paper, a different approach is proposed. Specifically, the learning curves of the different filters are compared by choosing for each filter the step-size that obtains the minimum steady-state MSE with the fastest convergence speed. Indeed, we must take into account that the steady-state MSE is the sum of three contributors: i) the additive noise, ii) the modelling error, and iii) the excess MSE generated by the gradient noise. The latter depends on the choice of the step-size and for a sufficiently small step-size is negligible compared to the first two contributors. Thus, using the acquired signals, for each filter the nonlinear system has been identified with different step-sizes. The corresponding learning curves have been plot on the same diagram and the largest step-size that reaches the minimum steady-state MSE (apart from a fraction of dB error) has been annotated. For example, Figure 1 shows the learning curves of MSE for the LN filter in the identification of the preamplifier with strong nonlinearity using the LMS algorithm with different step-sizes. In Figure 1 (and also in Figure 2) each learning curve is the ensemble average of 20 simulations of the LMS algorithm applied to non-overlapping data segments. Moreover, the learning curves have been smoothed using a box filter of 1000 sample memory length. As we can notice for a

step-size  $\mu \geq 8 \cdot 10^{-3}$  the steady-state MSE is larger than the minimum one, while for  $\mu \leq 5 \cdot 10^{-3}$  almost the same steady-state MSE is obtained for all curves.

Using the annotated step-sizes, the learning curves of the four filters have been compared at the different nonlinear conditions. Figure 2 shows the result of the comparison. The step-size used for each learning curve is reported in the legend. For a weak nonlinearity, the linear filter provides fairly good results with steady-state MSE similar to those of the nonlinear filters. In contrast, for medium and strong nonlinearities, the linear filter appears inadequate. The linear filter and the EMFN and LN filter have orthogonal basis functions for white uniform input signals, and thus provide a fast convergence speed of the LMS algorithm. The Volterra filter does not share this orthogonality property and, indeed, its convergence speed is much slower than that of the other filters. In the time interval of Figure 2, the Volterra filter does not reach the steady state conditions and provides a larger MSE than EMFN and LN filters. In this experiment, the LN filter provides in all conditions the best performances. However, extensive simulations on devices affected by third order nonlinearities stronger than those used in the previous experiment showed that quite often EMFN filters are able to give better results than LN filters.

#### 4. CONCLUSIONS

In this paper, a novel sub-class of polynomial, finite-memory LIP nonlinear filters, the LN filters, has been introduced. It has been shown that the LN filters are universal approximators, according to the Stone-Weierstrass theorem, for causal, time invariant, finite-memory, continuous, nonlinear systems, as well as the Volterra filters and the EMFN filters. The basis functions of LN filters are mutually orthogonal for white uniform input signals, as those of EMFN filters. Thanks to this orthogonality property, gradient descent algorithms with fast convergence speed and efficient nonlinear system identification algorithms can be devised. Since the basis functions of LN filters include the linear terms, these filters are better fitted than EMFN filters for modelling weak or medium nonlinearities. Thus, the proposed filters combine the best characteristics of Volterra and EMFN filters.

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