A NOVEL CLASS OF BIBO STABLE RECURSIVE NONLINEAR FILTERS

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ABSTRACT

In this paper, we introduce a novel class of nonlinear filters characterized by two interesting properties: (i) according to the Stone-Weierstrass approximation theorem, the proposed nonlinear filters are able to arbitrarily well approximate any discrete-time, timeinvariant, causal, infinite-memory, continuous, nonlinear system; (ii) the filters are always stable according to the bounded-inputbounded-output criterion. This novel class includes as a subclass the finite-memory even mirror Fourier nonlinear filters, which have been recently introduced as an useful tool for modeling strong saturation nonlinearities.

Index Terms— Nonlinear systems, linear-in-the-parameters nonlinear filters, universal approximators, BIBO stability.

1. INTRODUCTION

A model, which is often used in adaptive identification of nonlinear systems, is that resorting to linear-in-the-parameters (LIP) nonlinear filters. A discrete-time LIP filter is characterized by the linearity property of its output with respect to the filter coefficients. The class of LIP nonlinear filters is broad and includes causal, shift-invariant, finite-memory or infinite-memory filters. The most popular nonlinear models with finite memory are Volterra filters [1] and functional link artificial neural networks (FLANN) [2] exploiting trigonometric functions. Interest for these filters is still relevant with reference in particular to efficient implementations [3], [4], [5], [6], [7], [8], [9]. Applications can be found in the areas of active noise control [10], [11], [12], [13], channel equalization [14], [15], and echo cancellation [16], [17], [18], [19]. Recently, a new member of the class of finite-memory LIP nonlinear filters has been introduced in [20], [21]. It has been called even mirror Fourier nonlinear (EMFN) filter since its basis functions are even mirror symmetric, in analogy to the waveforms defining the discrete cosine transform. It has been shown in [20], [21] that the resulting family of real trigonometric functions and their linear combinations constitute an algebra on the interval [-1,1] that satisfies all the requirements of the Stone-Weierstrass theorem [22]. As a consequence, EMFN filters are universal approximators for causal, time invariant, finite-memory, continuous, nonlinear systems, as the well-known Volterra filters. However, in contrast to Volterra filters, the basis functions of the EMFN filters are mutually orthogonal for white uniform input signals in the interval [-1, +1].

Infinite-memory LIP nonlinear filters have also been considered in the literature since they can represent real-world systems with fewer coefficients than finite-memory filters. Examples include recursive second-order Volterra filters (RSOV) [23], [24], [25], [26], recursive functional link artificial neural networks (RFLANN) [27], Alberto Carini

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[28], [29] and bilinear filters [1]. The aim of our contribution is to present the general theory that permits the introduction of a novel class of infinite-memory LIP nonlinear filters. Within simple conditions, the recursive nonlinear system to be modeled can be approximated using a linear combination of trigonometric basis functions. These basis functions are derived from those of the EMFN filter, using finite sets of input and past output samples. We also prove that, according to the assumed conditions and the Stone-Weierstrass theorem, it is possible, by using a sufficiently large number of basis functions, to arbitrarily well approximate a recursive nonlinear system. Since the resulting nonlinear filter is a recursive version of the EMFN filter in [20], [21], it is named recursive even mirror Fourier nonlinear (REMFN) filter. Moreover, it is shown in the paper that the REMFN filter has the relevant property of being always stable according to the bounded-input-bounded-output (BIBO) criterion, in contrast to other currently used recursive nonlinear filters such as RSOV, bilinear and RFLANN filters.

The adaptive version of the REMFN filter is also considered. An output-error adaptation algorithm is derived, exploiting the pseudolinear regression approximation introduced for linear IIR filters in [30]. The proposed method has been applied with good results to a variety of situations. Here, the results of the approximation of a real-world nonlinear system are presented.

The paper is organized as follows. In Section 2, the problem of the approximation of the input-output relationship of an unknown nonlinear system is considered. The basis functions of the REMFN filter are derived in Section 2.1 and its BIBO stability is proved in Section 2.2. The adaptive REMFN filter is introduced in Section 3, together with an output-error adaptation algorithm. A simulation experiment on the identification of a real-world nonlinear system is presented in Section 4. Conclusions follow in Section 5.

2. RECURSIVE EVEN MIRROR FOURIER NONLINEAR FILTER

In this section, we consider the problem of the identification or approximation of the input-output relationship of a discrete-time, timeinvariant, infinite-memory, causal, continuous, nonlinear system. In our case, the unknown system is assumed to be nonlinear and its input-output relationship is expressed as

$$y(n) = f[x(n), x(n-1), \dots, x(n-N), y(n-1), \dots, y(n-M)],$$
(1)

where f is a real continuous function and x(n), y(n) are real valued sequences. We also assume two stability conditions on the system in (1):

Assumption 1: The system (1) is BIBO stable for $|x(n)| \le R$ with $|y(n)| \le A$ for all n.

Without any limitation we can assume R = 1 and A = 1. Indeed, in case $R \neq 1$ and/or $A \neq 1$, we scale x(n) and y(n) as

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follows,

$$x'(n) = \frac{x(n)}{R}$$
 and $y'(n) = \frac{y(n)}{A}$

and then consider the problem of identifying the equivalent system

$$y'(n) = f'[x'(n), \dots, x'(n-N), y'(n-1), \dots, y'(n-M)] = \frac{1}{A} f[Rx'(n), \dots, Rx'(n-N), Ay'(n-1), \dots, Ay'(n-M)]$$

Moreover, we assume that, if we apply a small perturbation to the system in (1), the output remains close to y(n): Assumption 2: Given the perturbed system

$$\widetilde{y}(n) = f[x(n), x(n-1), \dots, x(n-N), \widetilde{y}(n-1), \dots, \widetilde{y}(n-M)] + \nu(n),$$
(2)

for any $\theta > 0$ there exist $\epsilon > 0$ such that when $|\nu(n)| < \epsilon \forall n$ it is

$$|y(n) - \widetilde{y}(n)| < \theta \quad \forall n, \tag{3}$$

with y(n) the output of the system in (1).

2.1. The REMFN filter as universal approximator

In what follows, we prove that if the conditions of Assumption 1 and 2 are satisfied, then the recursive nonlinear system in (1) can be approximated using a linear combination of appropriate trigonometric basis functions. To prove this result, we consider the multivariate continuous function

$$\eta = f(\xi_0, \xi_1, \dots, \xi_{M+N}), \tag{4}$$

with $|\xi_i| < 1$ for $0 \le i \le M + N$, and we apply the well known Stone-Weierstrass theorem [22]:

"Let \mathcal{A} be an algebra of real continuous functions on a compact set K. If \mathcal{A} separates points on K and if \mathcal{A} vanishes at no point of K, then the uniform closure \mathcal{B} of \mathcal{A} consists of all real continuous functions on K".

According to the Stone-Weierstrass theorem, every algebra of real continuous functions on the compact $[-1,+1]^{M+N+1}$ which separates points and vanishes at no point is able to arbitrarily well approximate the continuous function f in (4). A family \mathcal{A} of real functions is said to be an algebra if \mathcal{A} is closed under addition, multiplication, and scalar multiplication, i.e., if (i) $f + g \in \mathcal{A}$, (ii) $f \cdot g \in \mathcal{A}$, and (iii) $cf \in \mathcal{A}$, for all $f \in \mathcal{A}$, $g \in \mathcal{A}$ and for all real constants c.

In [20], [21] it has been shown that for univariate functions

$$\eta = f[\xi],\tag{5}$$

an algebra that satisfies all the requirements of the Stone-Weierstrass theorem relies on the following even-mirror basis functions

$$1, \sin[\frac{1}{2}\pi\xi], \cos[\pi\xi], \sin[\frac{3}{2}\pi\xi], \dots, \cos[k\pi\xi], \sin[\frac{2k+1}{2}\pi\xi], \dots$$
(6)

The cosine and sine functions are the basis functions of order P = 2k and P = 2k + 1, respectively, where k is a positive integer or zero.

For multivariate functions, an algebra that satisfies all the requirements of the Stone-Weierstrass theorem and can arbitrarily well approximate (4) can be formed by writing the basis functions in (6) for $\xi = \xi_0, \xi = \xi_1, \dots, \xi = \xi_{M+N}$, and by considering all the possible products, without repetitions, between basis functions with different variables, as shown in [20], [21]. In this case, the order P of the basis functions is defined as the sum of the orders of the constituent 1-dimensional basis functions. By applying this rule and taking in our case $\xi_0 = x(n)$, $\xi_1 = x(n-1), \ldots, \xi_N = x(n-N)$, $\xi_{N+1} = \hat{y}(n-1), \ldots, \xi_{N+M} = \hat{y}(n-M)$, where $\hat{y}(n)$ is the output of the modeling filter, the basis functions of order P = 0, 1, 2, 3 become those given in Table 1. A generalization to any order P is then possible, in analogy to what has been shown in [20], [21] for EMFN filters.

Using Assumption 2, it can now be proved that with basis functions as those given in Table 1, we can arbitrarily well approximate the system in (1). In fact, for any $\epsilon > 0$, according to the Stone-Weierstrass theorem, there is a linear combination of basis functions, shortly noted as $\tilde{f}(\xi_0, \xi_1, \ldots, \xi_{M+N})$, such that for any $\xi_0, \xi_1, \ldots, \xi_{M+N}$ in [-1, +1] it is

$$\left|f(\xi_0,\xi_1,\ldots,\xi_{M+N})-\widetilde{f}(\xi_0,\xi_1,\ldots,\xi_{M+N})\right|<\epsilon.$$
 (7)

Let us now consider the system

$$\widetilde{y}(n) = \widetilde{f}[x(n), x(n-1), \dots, x(n-N), \widetilde{y}(n-1), \dots, \widetilde{y}(n-M)].$$
(8)

According to (7) it results

$$\widetilde{y}(n) = f[x(n), x(n-1), \dots, x(n-N), \widetilde{y}(n-1), \dots, \widetilde{y}(n-M)] + \nu(n)$$
(9)

with $|\nu(n)| < \epsilon$. According to Assumption 2 and the Stone-Weierstrass theorem, for any $\theta > 0$ there is a sufficiently small $\epsilon > 0$ and a linear combination of basis functions as those in Table 1 such that the error between the output of (1) and (8) is $|y(n) - \tilde{y}(n)| < \theta$. This means that, by considering a sufficiently large number $N_{\rm T}$ of basis functions of a sufficiently high order P, it is possible to arbitrarily well approximate the recursive system in (1). More specifically, it is possible to write the output $\hat{y}(n)$ of the modeling filter as

$$\hat{y}(n) = \sum_{i=1}^{N_{\rm T}} c_i f_i(n), \tag{10}$$

where, for sake of simplicity, the short notation

$$f_i(n) = f_i[x(n), x(n-1), \dots, x(n-N), \hat{y}(n-1), \dots \hat{y}(n-M)]$$
(11)

has been used. Each $f_i(n)$ is a continuous function taking the form given in Table 1 for orders up to P = 3. The resulting nonlinear filter has been named recursive even mirror Fourier nonlinear (REMFN) filter because it is a recursive version of the EMFN filter in [20], [21].

It is worth noting that the rule applied to generate the basis functions is the same used to define a symmetric truncated Volterra filter [1] or an EMFN filter [20], [21]. Therefore, the total number of coefficients in an REMFN kernel of order P is equal to that of a Volterra kernel of the same order and memory equal to N + M + 1,

$$N_P = \binom{N+M+P}{P}.$$
(12)

In general, the number of coefficients in an REMFN filter including all the kernels of order $0, \ldots, P$, is equal to

$$N_{\rm T} = \binom{N+M+1+P}{N+M+1}.$$
 (13)

As it clearly appears from the last expression, the complexity of the filter hugely increases with the order of the filter and the number of samples considered. Often it is possible to reduce the computational complexity deleting the cross terms between input and past output samples, as done, for example, with RSOV filters [23], [24], [25], [26]. According to the specific application, it is also possible to greatly reduce the number of total coefficients, as shown for example in [31], where an FIR filter and a very simple REMFN filter of order P = 3 have been used as the controller in a nonlinear noise control environment. Another aspect that affects the computational complexity is the calculation of the trigonometric basis functions. It is worth noting that, as with the FLANN and generalized FLANN filters [13], in our experiments the REMFN filter revealed to be robust with respect to the quantization of the sine and cosine functions, so that a simple look-up table can be used for the computation of their values.

2.2. BIBO stability of the REMFN filter

It is well known that recursive filters with fixed coefficients may be unstable, according to the BIBO criterion. For example, the BIBO stability of RSOV and bilinear filters with fixed coefficients requires that the poles of their linear IIR parts be inside the unit circle of the z plane. Moreover, sufficient conditions need to be imposed on the amplitude of the input signal in order to guarantee the boundedness of the output signals [25], [26], [1]. The BIBO condition is alleviated for the RFLANN filter since, as shown in [29], no additional constraints on the boundedness of the input signal are needed. Finally, in the case of the REMFN filter, no conditions at all need to be checked, since this filter is always BIBO stable. The proof is simply based on the fact that all its nonlinear basis functions are limited by the unity. From (10) it results

$$|\hat{y}(n)| \le \sum_{i=1}^{N_{\rm T}} |c_i| |f_i(n)| \le N_{\rm T} \sum_{i=1}^{N_{\rm T}} |c_i| \le K,$$
(14)

where K is a finite number.

3. THE ADAPTIVE REMFN FILTER

In order to identify or approximate a nonlinear system, the REMFN filter should be equipped with an adaptation algorithm. In general, to adapt a recursive filter there are two fundamental methods, i.e., equation-error and output-error methods. Equation-error method forms the estimates using samples of the input and desired response signals, and thus gives biased solutions. In contrast, output-error method provides a truly recursive estimate by using the actual output samples. In this case, the mean-square error function J(n) is defined as

$$I(n) = E[e^{2}(n)] = E\left\{\left[y(n) - \hat{y}(n)\right]^{2}\right\},$$
(15)

where the symbol E indicates expectation, y(n) is the conditioning signal, i.e., the output of the unknown system, and $\hat{y}(n)$ is the output of the adaptive filter given by (10). The error function J(n) is not quadratic with respect to the filter coefficients, since the recursive part of the filter depends on the coefficients themselves. Therefore, the solution of the minimization problem is not unique, in general, and local minima may exist. To avoid this difficulty, it is possible to resort to the pseudolinear regression approximation proposed in [30] for adaptive IIR filters. In practice, in the gradient computation, the dependencies on the coefficients of the recursive part of the filter are ignored. In this case, the so-derived pseudo-LMS algorithm satisfies no stochastic gradient cost function. However, experiments have shown that the algorithm works well in most situations not only for IIR filters but also for LIP nonlinear filters [1]. Therefore, it is convenient to apply it also to the REMFN filter. The pseudo-LMS algorithm is thus derived as

$$\mathbf{c}(n+1) = \mathbf{c}(n) - \frac{1}{2}\mu\nabla_{\mathbf{c}}e^2(n), \qquad (16)$$

 Table 1. Basis functions of the REMFN filter

Order 0

1.

Order 1

 $sin[\frac{1}{2}\pi x(n)], \dots, sin[\frac{1}{2}\pi x(n-N)], \\
sin[\frac{1}{2}\pi \hat{y}(n-1)], \dots, sin[\frac{1}{2}\pi \hat{y}(n-M)].$

Order 2

$$\begin{array}{c} \cos[\pi x(n)], \dots, \cos[\pi x(n-N)],\\ \cos[\pi \hat{y}(n-1)], \dots, \cos[\pi \hat{y}(n-M)],\\ \sin[\frac{1}{2}\pi x(n)]\sin[\frac{1}{2}\pi x(n-1)], \dots\\ \dots, \sin[\frac{1}{2}\pi x(n)]\sin[\frac{1}{2}\pi \hat{y}(n-M)],\\ \sin[\frac{1}{2}\pi x(n-1)]\sin[\frac{1}{2}\pi x(n-2)], \dots\\ \dots, \sin[\frac{1}{2}\pi x(n-1)]\sin[\frac{1}{2}\pi \hat{y}(n-M)],\\ \vdots\\ \sin[\frac{1}{2}\pi x(n-N)]\sin[\frac{1}{2}\pi \hat{y}(n-M)],\\ \sin[\frac{1}{2}\pi \hat{y}(n-1)]\sin[\frac{1}{2}\pi \hat{y}(n-2)], \dots\\ \dots, \sin[\frac{1}{2}\pi \hat{y}(n-2)], \dots\\ \dots, \sin[\frac{1}{2}\pi \hat{y}(n-1)]\sin[\frac{1}{2}\pi \hat{y}(n-M)],\\ \vdots\\ \sin[\frac{1}{2}\pi \hat{y}(n-M+1)]\sin[\frac{1}{2}\pi \hat{y}(n-M)].\end{array}$$



$$\begin{split} & \sin[\frac{3}{2}\pi x(n)], \dots, \sin[\frac{3}{2}\pi x(n-N)], \\ & \sin[\frac{3}{2}\pi \hat{y}(n-1)], \dots, \sin[\frac{3}{2}\pi \hat{y}(n-M)], \\ & \cos[\pi x(n)] \sin[\frac{1}{2}\pi x(n-1)], \dots \\ & \dots, \cos[\pi x(n)] \sin[\frac{1}{2}\pi \hat{y}(n-M)], \\ & \cos[\pi x(n-1)] \sin[\frac{1}{2}\pi x(n-2)], \dots \\ & \dots, \cos[\pi x(n-1)] \sin[\frac{1}{2}\pi \hat{y}(n-M)], \\ & \vdots \\ & \cos[\pi x(n-N)] \sin[\frac{1}{2}\pi \hat{y}(n-M)], \\ & \cos[\pi \hat{y}(n-1)] \sin[\frac{1}{2}\pi \hat{y}(n-2)], \dots \\ & \dots, \cos[\pi \hat{y}(n-1)] \sin[\frac{1}{2}\pi \hat{y}(n-M)], \\ & \vdots \\ & \cos[\pi \hat{y}(n-1)] \sin[\frac{1}{2}\pi \hat{y}(n-2)], \dots \\ & \dots, \cos[\pi \hat{y}(n-1)] \sin[\frac{1}{2}\pi \hat{y}(n-M)], \\ & \vdots \\ & \cos[\pi \hat{y}(n-M+1)] \sin[\frac{1}{2}\pi \hat{y}(n-M)], \\ & \sin[\frac{1}{2}\pi x(n)] \cos[\pi x(n-1)], \dots \\ & \dots, \sin[\frac{1}{2}\pi x(n)] \cos[\pi \hat{y}(n-M)], \\ & \sin[\frac{1}{2}\pi x(n-1)] \cos[\pi x(n-2)], \dots \\ & \dots, \sin[\frac{1}{2}\pi x(n-1)] \cos[\pi \hat{y}(n-M)], \\ & \vdots \\ & \sin[\frac{1}{2}\pi x(n-1)] \cos[\pi \hat{y}(n-2)], \dots \\ & \dots, \sin[\frac{1}{2} \hat{y}(n-1)] \cos[\hat{y}(n-M)], \\ & \vdots \\ & \sin[\frac{1}{2} \hat{y}(n-1)] \cos[\pi \hat{y}(n-2)], \dots \\ & \dots, \sin[\frac{1}{2} \hat{y}(n-1)] \cos[\hat{y}(n-M)], \\ & \vdots \\ & \sin[\frac{1}{2} \pi x(n)] \sin[\frac{1}{2} \pi x(n-1)] \sin[\frac{1}{2} \pi x(n-2)], \dots, \\ & \dots, \sin[\frac{1}{2} \pi \hat{y}(n-M+1)] \sin[\frac{1}{2} \pi \hat{y}(n-M+1)] \\ & \sin[\frac{1}{2} \pi \hat{y}(n-M+2)] \sin[\frac{1}{2} \pi \hat{y}(n-M+1)] \\ & \sin[\frac{1}{2} \pi \hat{y}(n-M)]. \end{split}$$

where

$$\mathbf{c}(n) = \left[c_1(n)c_2(n)\cdots c_{N_{\mathrm{T}}}(n)\right]^T,\tag{17}$$

 $e(n) = y(n) - \hat{y}(n)$ and μ is the step size. By applying the pseudolinear regression approximation to the update of the filter coefficients, it results

$$\mathbf{c}(n+1) = \mathbf{c}(n) + \mu e(n)\mathbf{f}(n), \tag{18}$$

where, using the notation in (11), the vector $\mathbf{f}(n)$ is given by

$$\mathbf{f}(n) = [f_1(n)f_2(n)\cdots f_{N_{\mathrm{T}}}(n)]^T.$$
 (19)

In our case, it is formed, for orders up to P = 3, with the basis functions of the REMFN filter of (10) given in Table 1.

4. AN EXPERIMENTAL RESULT

We consider here the identification of a saturated amplifier, a Behringer Tube Ultragain MIC100. In the experiment settings, at the maximum used volume, the amplifier introduces on a 1 kHz sinusoidal input a second order harmonic distortion of 11% and a third order harmonic distortion of 22%. The harmonic distortion is defined as the ratio, in percent, between the magnitude of each harmonic and that of the fundamental frequency. For its identification, we applied to the amplifier a white uniform input signal between [-1, +1], sampled at 8kHz, and recorded the corresponding output. An LMS update as in (18) has been used to identify the amplifier from the input and output signals. We compared the modeling performance of a Volterra, an EMFN, and an REMFN filter with the same number of coefficients. Within these conditions, all filters provide similar steady-state MSE performance. However, the EMFN and REMFN filters have a much faster convergence speed (the results are not included due to space limitations). The situation changes if the output of the amplifier is processed with an IIR filter (e.g., used to select a certain frequency range of interest) and if we simultaneously model the cascade of the amplifier and the IIR filter. In this case, we processed the output of the amplifier with a 4-th order Butterworth passband filter with normalized cut-off frequencies [0.25, 0.75]. Figure 1 shows the learning curves of the MSE for an LMS update averaged over 100 different input-output segments for different values of the step-size. Modeling the system with a set of step-sizes allows us to identify the minimum achievable MSE with the filter under test and the maximum value of the step-size (and thus convergence speed) able to obtain that MSE with a negligible error. The step sizes are chosen in the following set

$$\{ 0.01a^2, 0.01a, 0.01, 0.01a^{-1}, 0.01a^{-2} \},\$$

where $a = 10^{1/4}$. Figure 1.(a) refers to an REMFN filter having 137 coefficients formed by an EMFN forward filter of order 3, memory of 7 samples, and an EMFN feedback filter of order 3, memory of 7 samples, without cross-terms. Figure 1.(b) refers to an EMFN filter of order 3, memory 8 having 164 coefficients. Figure 1.(c) refers to a Volterra filter of order 3, memory 8 having 164 coefficients. The superiority of the REMFN filter in these experimental conditions is clearly evident. With an EMFN filter of memory 16 we can obtain a steady-state MSE close to that of the REMFN filter of memory 7, but 968 coefficients are required.

5. CONCLUSIONS

In this paper a novel class of recursive BIBO stable LIP nonlinear filters, called REMFN filters, has been presented. This class of filters includes, as a particular case, the finite-memory EMFN filters, recently introduced to efficiently deal with large nonlinearities.



Fig. 1. Learning curves with (a) REMFN, (b) EMFN, and (c) Volterra filter.

It is proved in this paper that, according to the Stone-Weierstrass approximation theorem, the REMFN filter is able to arbitrarily well approximate any discrete-time, time-invariant, causal, infinitememory, continuous, nonlinear system. The complexity of the REMFN filter is, in general, the main drawback since it hugely increases with the number of basis functions used and their order P. However, experiments showed that quite often kernels of reduced order (up to P = 3) are sufficient to model nonlinear system, especially within specific application-oriented conditions. More in general, future research will include strategies and tools for the choice of the most relevant terms of an REMFN filter according to the nonlinear system to be modeled.

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