OPTIMUM DISCRETE SINGLE GROUP MULTICAST BEAMFORMING

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ABSTRACT

In this paper, transmit beamformer design for single group multicast scenario is considered. The problem is solved in discrete form where the beamformer phase and amplitude values are selected from finite discrete sets. Original optimization problem is converted to a linear form by introducing new variables. The solution of the equivalent optimization problem is always feasible as long as the total power is above a certain value. The problem in its linear form is guaranteed to return optimum solution. Proposed approach is very effective and the number of bits can be increased to obtain close to optimum continuous phase and amplitude beamformers.

Index Terms— Transmit beamformer, discrete beamformer, mixed integer linear programming

1. INTRODUCTION

Transmit beamformer design has found widespread applications in different fields including communications, radar, sonar, etc. In this paper, "$single\ group\ multicast$" beamforming scenario is considered where the transmitter transmits the same information to several users [1].

Transmit beamforming problem is usually investigated in continuous case where the beamformer has continuous amplitude and phase. In max-min fair beamforming design, the goal is to maximize the minimum received SNR over all users. This problem is NP hard, a near-optimal solution is usually found by semidefinite relaxation [1], [2], [3]. It is known that convex optimization with semidefinite relaxation works well but does not guarantee rank one and hence optimum solution in general [1]. When the rank condition is not satisfied, randomization is used to obtain an improved result [2].

The design of discrete phase-only beamformer is investigated in [4], [5], [6]. Unfortunately previous works only propose suboptimum solutions. In [7], optimum discrete phase-only transmit beamformer design is presented. In this paper, this work is extended to optimum discrete phase-amplitude beamformer (DPAB). The problem is considered for the max-min fair approach. Optimum solution is obtained by applying a number of nontrivial transformations to the original optimization problem. Fortunately, discrete case allows one to generate known finite set and vectors for discrete phase and amplitudes. Optimization problem is converted to a linear form by introducing some additional variables. In this process, multiplication of optimization variables is converted to a linear addition expression in terms of new optimization variables. The performance of the proposed solution is good even for small number of quantization levels.

2. DISCRETE TRANSMIT BEAMFORMER DESIGN

Consider a base station equipped with M transmit antennas to transmit a common signal to N receivers, each having a single antenna. Assume that the antennas are identical. Transmitted signal is narrowband and propagation is nondispersive. The transmit beamforming "max-min" problem is to choose beamforming weight vector, $\mathbf{w}$, in order to maximize the minimum power transmitted to any user. When the phase angles and the amplitudes of the beamformer are selected from a discrete set, it is possible to find an optimum solution for the transmit beamformer. The discrete beamformer vector can be written as $\mathbf{w} = [\alpha_1 e^{j\psi_1}\ \alpha_2 e^{j\psi_2}\ ...\ \alpha_M e^{j\psi_M}]^T$ where α_i is the positive discrete amplitude and ψ_i is the discrete phase angle. Each antenna has a rated power and cannot transmit beyond this power. Considering $P_{an_{max}}$ as the maximum antenna power and P_{tot} as the total power, discrete *max-min* problem can be written as follows,

$$\max_{\psi_i,\alpha_i}\ t$$

$$s.t.\quad \mathbf{w}^H \mathbf{R}_k \mathbf{w} \geq t\gamma_k\sigma_k^2, \quad k = 1, ..., N \tag{1}$$

$$|w_i|^2 \leq P_{an_{max}} \quad i = 1, ..., M \tag{2}$$

$$\mathbf{w^H w} \leq P_{tot} \tag{3}$$

$$\psi_i \in \{0, \Delta\theta, 2\Delta\theta, ..., (2^n - 1)\Delta\theta\},\ \Delta\theta = \frac{360^\circ}{2^n} \tag{4}$$

$$\alpha_i \in \{\ \Delta_a, 2\Delta_a, ..., 2^m\Delta_a\},\ \Delta_a = \frac{\sqrt{P_{an_{max}}}}{2^m} \tag{5}$$

where γ_k is the power proportion for the kth target and $\mathbf{R}_k = \mathbf{h}_k\mathbf{h}_k^H$ by assuming full channel state information (CSI). n and m are the number of bits to represent the discrete phase and amplitude values respectively. $\Delta\theta$ and Δ_a are the discrete step size for phase and amplitude respectively. Using the fact that $\mathbf{R}_k$ is a Hermitian symmetric matrix and trigonometric identity, the optimization problem can be written as,

$$\max_{\psi_i,\alpha_i,\beta_{i,p},\mu_{i,p}} \quad t$$

$$s.t. \sum_{i=1}^{M-1}\sum_{p=i+1}^{M} 2\mu_{i,p}(cos(\angle R_k(i,p))cos\beta_{i,p} - sin(\angle R_k(i,p))sin\beta_{i,p}) + \sum_{i=1}^{M}\alpha_i^2 \geq t\gamma_k\sigma_k^2, \quad k=1,...,N \tag{6}$$

$$\sum_{i=1}^{M}\alpha_i^2 \leq P_{tot} \tag{7}$$

$$\psi_i \in \{0, \Delta\theta, 2\Delta\theta, ...(2^n-1)\Delta\theta\},\ \Delta\theta = \frac{360^\circ}{2^n} \tag{8}$$

$$\alpha_i \in \{\ \Delta_a, 2\Delta_a, ..., 2^m\Delta_a\},\ \Delta_a = \frac{\sqrt{P_{an_{max}}}}{2^m} \tag{9}$$

$$\beta_{i,p} = -\psi_i + \psi_p \tag{10}$$

$$\mu_{i,p} = \alpha_i\alpha_p, \quad i=1,2,...,M-1,\ p=i+1,...,M \tag{11}$$

The problem setting described for DPAB in (6-11) is not convex. Conversion of this problem to a linear form is presented in the next section.

3. DISCRETE OPTIMIZATION IN LINEAR FORM

In this part, DPAB expressions in (6), (7), (10) and (11) are converted into linear expressions of optimization variables.

Let the first and second parts of the left hand side of the inequality in (6) be represented as A and B respectively. The known vectors $\mathbf{c}$ and $\mathbf{s}$ are defined. $\mathbf{c}$ and $\mathbf{s}$ are composed of $cos\beta_{i,p}$ and $sin\beta_{i,p}$ terms including all the $\beta_{i,p}$ values in the finite discrete set, i.e.,

$$\mathbf{c} = [\ cos(-(2^n-1)\cdot\Delta\theta)\ ...\ cos((2^n-1)\cdot\Delta\theta)] \tag{12}$$

$$\mathbf{s} = [\ sin(-(2^n-1)\cdot\Delta\theta)\ ...\ sin((2^n-1)\cdot\Delta\theta)] \tag{13}$$

In order to access each summation term in A, $\mathbf{u}_{i,p}$ vectors which are all zero except a single element are defined. $\mathbf{u}_{i,p}$'s are the new variables of optimization and they are special ordered sets of type 1 (SOS1) vectors which are common in integer optimization [8].

$\mathbf{u}_{i,p}$ is a $(2^{n+1}-1)\times 1$ vector and carries both phase and amplitude information of the beamformer vector. The nonzero element index in $\mathbf{u}_{i,p}$ stands for the phase angle difference, $\beta_{i,p}$. Since values of $\beta_{i,p}$ range from $-(2^n-1)\Delta\theta$ to $(2^n-1)\Delta\theta$, the index of the $\mathbf{u}_{i,p}$ vector starts from $-(2^n-1)$ and end at (2^n-1). The index value of the nonzero element is $\frac{\beta_{i,p}}{\Delta\theta}$. The element amplitude is $\frac{\alpha_i\alpha_p}{\Delta_a^2}$.

A in (6) can be expressed in terms of $\mathbf{u}_{i,p}$ as,

$$A = \sum_{i=1}^{M-1}\sum_{p=i+1}^{M} 2\Delta_a^2(cos(\angle R_k(i,p))\mathbf{c} - sin(\angle R_k(i,p))\mathbf{s})\cdot\mathbf{u}_{i,p} \tag{14}$$

The phase and amplitude relationship between $\mathbf{u}_{i,p}$ vectors should be established and used during the optimization. Such relationships can be established over some vectors, $\mathbf{v}_i$. $2^n\times 1$ vector, $\mathbf{v}_i$, carries the phase and amplitude information of the beamformer vector. They are SOS1 vectors similar to $\mathbf{u}_{i,p}$. The index value of the nonzero element of $\mathbf{v}_i$ stands for ψ_i. Since values of ψ_i range from 0 to $(2^n-1)\Delta\theta$, the index of $\mathbf{v}_i$ vector starts from 0 and end at (2^n-1). The index value of the nonzero element is $\frac{\psi_i}{\Delta\theta}$. The element amplitude is $\frac{\alpha_i}{\Delta_a}$.

$\mathbf{u}_{i,p}$ and $\mathbf{v}_i$ are SOS1 vectors carrying different forms of phase and amplitude information. In order to convert the nonlinear optimization problem into a linear one, we need vectors which carry only the phase and amplitude information. In other words, phase and amplitude information on variable vectors should be separated. This can be achieved by defining SOS1 operators T_1, T_2 and T_3 respectively. T_1 generates the phase coded *"binary"* SOS1 vectors $\mathbf{u}'_{i,p}$ and $\mathbf{v}'_i$ from $\mathbf{u}_{i,p}$ and $\mathbf{v}_i$ respectively. T_2 and T_3 generate $\mathbf{v}''_i$ and $\mathbf{u}''_{i,p}$ binary SOS1 vectors respectively. These two vectors have only the coded amplitude information and are described in the following parts.

The function of $T_1\{.\}$ operator is the normalization, i.e.,

$$\mathbf{v}'_i = T_1\{\mathbf{v}_i\} = T_1\{[\ 0\ ...\ \frac{\alpha_i}{\Delta_a}\ ...\ 0\ ]^T\} = \frac{\mathbf{v}_i\Delta_a}{\alpha_i} = [\ 0\ ...\ 1\ ...\ 0\ ]^T \tag{15}$$

and therefore it generates only phase coded vector $\mathbf{v}'_i$. T_1 operates similarly on $\mathbf{u}_{i,p}$.

The equation in (10) can now be written in terms of new variables $\mathbf{v}'_i$ and $\mathbf{u}'_{i,p}$ respectively. In order to do this, (10) is first normalized with $\Delta\theta = \frac{360^\circ}{2^n}$. Let $\mathbf{d} = [\ 0\ 1\ 2\ ...\ (2^n-1)\ ]$, and $\mathbf{e} = [\ -(2^n-1)\ -(2^n-2)\ ...\ (2^n-1)\ ]$ be known row vectors of integers. After normalization, (10) can be written as,

$$\mathbf{d}\cdot(-\mathbf{v}'_i + \mathbf{v}'_p) = \mathbf{e}\cdot\mathbf{u}'_{i,p} \tag{16}$$

To convert (11) and B into the desired form, multiplication of two variables is converted to addition through a mapping operation. Let α_i and α_p be two amplitude values in multiplication, i.e, $\alpha_i\cdot\alpha_p = \mu_{i,p}$, where $\frac{\alpha_i}{\Delta_a}$, $\frac{\alpha_p}{\Delta_a}$ and $\frac{\mu_{i,p}}{\Delta_a^2}$ belong to positive integers, i.e., $\frac{\alpha_i}{\Delta_a}, \frac{\alpha_p}{\Delta_a}, \frac{\mu_{i,p}}{\Delta_a^2} \in \mathbb{Z}^+$. $\frac{\alpha_i}{\Delta_a}$ and $\frac{\alpha_p}{\Delta_a}$ belong to a finite discrete set, i.e., $\frac{\alpha_i}{\Delta_a}, \frac{\alpha_p}{\Delta_a} \in S$ where $S = \{1,2,3,...,2^m\}$ for m bits as can be seen in (5). There are $2^m + \binom{2^m}{2}$ multiplication couples since multiplication is commutative. When the multiplication of $\frac{\alpha_i}{\Delta_a}$ and $\frac{\alpha_p}{\Delta_a}$ in S is considered, the discrete set of couples can be given as,

$$P = \{1\cdot 1,\ 1\cdot 2,\ ...\ ,1\cdot 2^m,\ 2\cdot 2,\ 2\cdot 3,\ ...\ ,2\cdot 2^m,\ 3\cdot 3,\ ...\ ,2^m\cdot 2^m\} \tag{17}$$

Let $\mathbf{q}$ be a row vector whose elements are unique and ordered values of multiplication results corresponding to couples in

P, i.e.,

$$\mathbf{q} = [\ 1\ \ 2\ \ 3\ \ ...\ \ 2^m \cdot 2^m\ \ ] \quad (18)$$

The values in the discrete set S should be coded by some g_i in order to generate a linear expression and one-to-one mapping between multiplication and addition. This can be easily done by using logarithm operation since $\alpha_i \cdot \alpha_p = \mu_{i,p}$ can be equivalently written as $log\alpha_i + log\alpha_p = log\mu_{i,p}$. Note that logarithm does not solve the ultimate problem alone since $log\alpha_i$ is a nonlinear function of the variable α_i. Let $\mathbf{g}$ be a vector whose elements are the logarithm of the elements in S, i.e., $\mathbf{g} = [\ g_1\ g_2\ ...\ g_{2^m}\ ]$ where $g_i = logi$. In a similar manner, the logarithm of the values in $\mathbf{q}$ are used to obtain $\mathbf{h}$ as $\mathbf{h} = [\ h_1\ h_2\ ...\ ]$ where $h_i = logq_i$. In order to select or access each element of $\mathbf{g}$ and $\mathbf{h}$, we need to define binary SOS1 vectors $\mathbf{v}''_i$ and $\mathbf{u}''_{i,p}$ from $\mathbf{v}_i$ and $\mathbf{u}_{i,p}$ respectively. This is possible by defining $T_2\{.\}$ and $T_3\{.\}$ operators. These two operators map a SOS1 vector into a binary SOS1 vector. Note that these operators generate SOS1 vectors which carry only the amplitude information.

$T_2\{\}$ operates on $\mathbf{v}_i$ and $T_3\{\}$ operates on $\mathbf{u}_{i,p}$ only. $T_2\{\mathbf{v}_i\}$ sums the elements of $\mathbf{v}_i$ and results a binary vector of $2^m \times 1$ size with a nonzero element index being equal to the summed value.

$T_3\{\mathbf{u}_{i,p}\}$ takes the nonzero element value in $\mathbf{u}_{i,p}$ and finds its index value in $\mathbf{q}$. It generates a binary vector whose only nonzero value is at this index. $\mathbf{u}''_{i,p} = T_3\{\mathbf{u}_{i,p}\}$ is a column vector and has the same length as the $\mathbf{q}$ vector.

Now consider the multiplication,

$$\frac{\alpha_i}{\Delta_a} \cdot \frac{\alpha_p}{\Delta_a} = \frac{\mu_{i,p}}{\Delta_a^2} \quad (19)$$

which is equivalent to

$$log\frac{\alpha_i}{\Delta_a} + log\frac{\alpha_p}{\Delta_a} = log\frac{\mu_{i,p}}{\Delta_a^2} \quad (20)$$

The equation in (19) can be written in terms of known vectors, $\mathbf{g}$, $\mathbf{h}$, and binary SOS1 vectors $\mathbf{v}''_i$ and $\mathbf{u}''_{i,p}$ as,

$$\mathbf{g} \cdot (\mathbf{v}''_i + \mathbf{v}''_p) = \mathbf{h} \cdot \mathbf{u}''_{i,p} \quad (21)$$

As a result (19) is converted into a linear additive expression of optimization variables, $\mathbf{v}''_i$ and $\mathbf{u}''_{i,p}$.

In the following part, B part of the inequality in (6) is converted to a linear form. Note that this term is the same as the left hand side of the inequality in (7).

B is the sum of squared amplitudes of beamformer elements. Define a row vector $\mathbf{r}$ which is composed of the squared values in S, i.e.,

$$\mathbf{r} = [\ 1^2\ 2^2\ ...\ (2^m)^2\ ] \quad (22)$$

B can be written as,

$$B = \Delta_a^2 \mathbf{r} \cdot \sum_{i=1}^{M} \mathbf{v}''_i \quad (23)$$

Now the expressions in (6), (7), (10) and (11) are converted to linear expressions of optimization variables, $\mathbf{v}_i$, $\mathbf{u}_{i,p}$, $\mathbf{v}'_i$, $\mathbf{u}'_{i,p}$, $\mathbf{v}''_i$ and $\mathbf{u}''_{i,p}$ in (14), (23), (16) and (21) respectively. All of the variables in these expressions are integer variables. Furthermore $\mathbf{v}'_i$, $\mathbf{v}''_i$, $\mathbf{u}'_{i,p}$ and $\mathbf{u}''_{i,p}$ are binary SOS1 vectors. The final variable of optimization, t, corresponds to the $min\{\frac{SNR_k}{\gamma_k}\}$ and hence it is real, i.e., $t \in \mathbb{R}$. Therefore the problem can be solved using mixed integer linear programming [9]. The new optimization problem is written as,

$$\max_{\mathbf{v}_i, \mathbf{u}_{i,p}, \mathbf{v}'_i, \mathbf{u}'_{i,p}, \mathbf{v}''_i, \mathbf{u}''_{i,p}} \quad t$$

$$s.t. \sum_{i=1}^{M-1} \sum_{p=i+1}^{M} 2\Delta_a^2 (cos(\angle R_k(i,p))\mathbf{c} - sin(\angle R_k(i,p))\mathbf{s}) \cdot \mathbf{u}_{i,p} + \Delta_a^2 \mathbf{r} \cdot \sum_{i=1}^{M} \mathbf{v}''_i \geq t\gamma_k\sigma_k^2 \quad k = 1, ..., N \quad (24)$$

$$\Delta_a^2 \mathbf{r} \cdot \sum_{i=1}^{M} \mathbf{v}''_i \leq P_{tot} \quad (25)$$

$$\mathbf{d} \cdot (-\mathbf{v}'_i + \mathbf{v}'_p) = \mathbf{e} \cdot \mathbf{u}'_{i,p}$$

$$i = 1, 2, ..., M-1,\ p = i+1, ..., M \quad (26)$$

$$\mathbf{g} \cdot (\mathbf{v}''_i + \mathbf{v}''_p) = \mathbf{h} \cdot \mathbf{u}''_{i,p} \quad (27)$$

$$T_1\{\mathbf{u}_{i,p}\} = \mathbf{u}'_{i,p} \quad (28)$$

$$T_3\{\mathbf{u}_{i,p}\} = \mathbf{u}''_{i,p} \quad (29)$$

$$T_1\{\mathbf{v}_i\} = \mathbf{v}'_i \qquad i = 1, 2, ..., M \quad (30)$$

$$T_2\{\mathbf{v}_i\} = \mathbf{v}''_i \quad (31)$$

where $\mathbf{v}_i$ and $\mathbf{u}_{i,p}$ are vectors of special ordered sets of type 1 and known vectors $\mathbf{c}$, $\mathbf{s}$, $\mathbf{r}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{q}$ and $\mathbf{h}$ are given as in the previous parts. The discrete optimization problem in (24-31) is always feasible as long as $P_{tot} \geq M\Delta_a^2$ and optimum solution is guaranteed for DPAB. Note that the above problem is solved using mixed integer linear programming with branch and cut procedure which is known to return the global optimum [10], [11], [12], [13].

The T_1, T_2 and T_3 operators can be easily implemented in integer programming. For example, T_1 can be realized by using the following constraints, i.e.,

$$\mathbf{v}_i + \mathbf{v}'_i,\ \mathbf{u}_{i,p} + \mathbf{u}'_{i,p} \in \text{ SOS1} \quad (32)$$

$$\mathbf{v}'_i,\ \mathbf{u}'_{i,p} \in \text{ SOS1}_b \quad (33)$$

where SOS1_b stands for binary SOS1.

Operator T_2 can be realized by using the following constraints, i.e.,

$$[\ 1\ 1\ ...\ 1\ ]\mathbf{v}_i = \mathbf{s}\mathbf{v}''_i \quad (34)$$

$$\mathbf{v}''_i \in \text{ SOS1}_b \quad (35)$$

where $\mathbf{s}$ is a row vector composed of the elements in S, i.e., $\mathbf{s} = [\,1\;2\;3\;...\;2^m\,]$. Similarly T_3 can be realized by using the following constraints, i.e.,

$$[\,1\;1\;...\;1\,]\mathbf{u}_{i,p} = \mathbf{q}\mathbf{u}''_{i,p} \tag{36}$$

$$\mathbf{u}''_{i,p} \in \text{ SOS1}_b \tag{37}$$

Once the solution for $\mathbf{v}_i$'s are found, the phase angles and the amplitudes of the beamformer vector are obtained as,

$$\psi_i = \mathbf{f}_\psi^T \mathbf{v}'_i, \qquad \alpha_i = \Delta_a \mathbf{f}_\alpha^T \mathbf{v}_i, \qquad i = 1, ..., M \tag{38}$$

where $\mathbf{f}_\psi = [\,0\;\Delta\theta\;...\;(2^n - 1)\Delta\theta\,]^T$ and $\mathbf{f}_\alpha = [\,1\;1\;...\;1\,]^T$ respectively.

4. SIMULATIONS

In this paper, "Gurobi" [9] which is an efficient mixed integer linear programming solver is used by employing branch and cut strategy.

In the first experiment, a uniform linear array (ULA) with $M = 4$ elements and $N = 4$ users are used. $m = 2$ and $n = 3$ bits are used for the amplitude and phase respectively. The user locations are varied randomly for 100 trials. Fig. 1 shows the results obtained with the proposed DPAB method and brute force search approach. As it is seen from the figure both return the same solution as expected since DPAB is the optimum method. While worst case complexity of the mixed integer linear programming is exponential, usually much better efficiency is achieved since the algorithm finds the optimal solution by pruning and linear programming which is known to have polynomial complexity [10], [11], [13], [14]. Table 1 shows the computational times for DPAB and brute force over an average of 10 trials for each case. As it is seen from this table, DPAB is twofold efficient than brute force search for these experiments. As the dimension of the problem increases DPAB becomes more efficient.

In the second experiment, DPAB is compared with its continuous counterpart continuous phase amplitude beamformer (CPAB) and convex optimization with randomization (CPAB+). There are three targets. The first and second targets are fixed at $(60°, 90°)$, $(100°, 90°)$ respectively. The third target is variable. Its elevation angle is fixed at $90°$ degrees and azimuth angle is varied between $0°$ to $180°$ degrees. γ_k coefficients are selected as $\gamma_1 = 1$, $\gamma_2 = 2$ and $\gamma_3 = 4$ respectively. $P_{an_{max}} = 1W$ and $P_{tot} = 2W$ are used. $n = 4$ and $m = 2$ bits are selected. The comparison for three techniques is given in Fig. 2. CPAB cannot return rank one solution for certain azimuth angles, performance degradation is large. Randomization, and hence CPAB+, improves the solution significantly in such cases [2]. However there are still certain angular sectors such as between $140°$ and $160°$ where randomization is insufficient. The performance of DPAB is consistent and good for all azimuth angles. The difference in minimum transmit powers is small even for $n = 4$ and $m = 2$ selection. Therefore DPAB can be seen as an effective solution for any number of receiver and receiver locations.

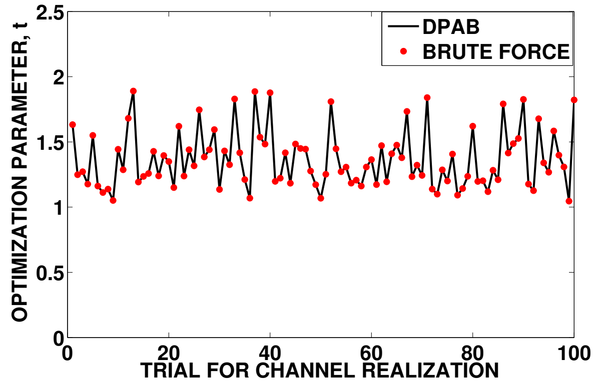

Fig. 1. Optimization parameter "t" for 100 channel realizations for DPAB and brute force search.

Table 1. Computational time of DPAB and brute force search

	$P_{tot} = 2P_{an_{max}}$		$P_{tot} = 4P_{an_{max}}$	
	$m = 2$		$m = 0$	
	$n = 3$	$n = 4$	$n = 4$	$n = 5$
DPAB	10.44 s	165.25 s	1.93 s	25.08 s
BRUTE FORCE	24.40 s	382.15 s	2.61 s	40.98 s

5. CONCLUSION

Discrete amplitude and phase transmit beamformer design problem is investigated. It is shown that optimum solution is guaranteed as long as $P_{tot} \geq M\Delta_a^2$. This result is made possible by converting the optimization problem into a linear set of equality and inequality expressions. This conversion is achieved by defining new optimization variables and replacing multiplication with addition. Discrete solutions perform very well even with small number of quantization bits. The transmit power and beampatterns approach to the continuous optimum beamformers. Practical radar, sonar and communications systems have discrete phase and amplitude hardware and hence proposed solution is a good fit for practical systems in general.

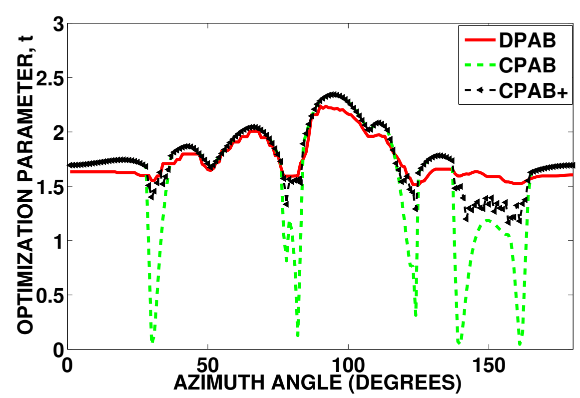

Fig. 2. Optimization parameter "t" versus azimuth angle of the third target for CPAB, CPAB+ and DPAB.

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