GENERALIZED QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING FOR SIGNAL PROCESSING

Arash Khabbazibasmenj[†]   *and*   *Sergiy A. Vorobyov*[†,*]

[†]Dept. of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada
[*] Department of Signal Processing and Acoustics, Aalto University, Finland
Emails: `khabbazi@ualberta.ca, svor@ieee.org`

ABSTRACT

In this paper, we introduce and solve a particular generalization of the quadratically constrained quadratic programming (QCQP) problem which is frequently encountered in different fields of signal processing and communications. Specifically, we consider such generalization of the QCQP problem that comprises compositions of one-dimensional convex and quadratic functions in the constraint and the objective functions. We show that this class of problems can be precisely or approximately recast as the difference-of-convex functions (DC) programming problem. Although the DC programming problem can be solved through the branch-and-bound methods, these methods do not have any worst-case polynomial-time complexity guarantees. Therefore, we develop a new approach with worst-case polynomial-time complexity that can solve the corresponding DC problem of a generalized QCQP problem. It is analytically guaranteed that the point obtained by this method satisfies the Karsuh-Kuhn-Tucker (KKT) optimality conditions. Furthermore, the global optimality can be proved analytically under certain conditions. The new proposed method can be interpreted in terms of the Newton's method as applied to a non-constrained optimization problem.

Index Terms— Generalized QCQP problem, DC programming, polynomial-time algorithms, array processing, cooperative communications.

1. INTRODUCTION

Convex optimization problems form the largest known class of optimization problems that can be efficiently addressed. As opposed to the convex optimization problems, the non-convex problems are usually extremely hard to deal with. Although the non-convex optimization problems are inherently very challenging, it is still possible to solve some of these problems by means of convex optimization techniques. Specifically, it is sometimes possible to relax a non-convex problem into a set of convex problems and then extract the optimal solution of the original problem from the solution of the convexly relaxed problems [1]–[3].

Quadratically constrained quadratic programming (QCQP) problem is one of the important classes of non-convex optimization problems which is very frequently encountered in different applications. Despite being exceedingly difficult, QCQP problems can be approximately solved using the semi-definite programming relaxation (SDR) techniques [1], [4]–[7]. SDR is a powerful and computationally efficient method which relaxes the non-convex problem into a convex problem and then extracts a suboptimal solution of the QCQP problem from the optimal solution of the convexly relaxed problem.

Despite the profound importance of the QCQP optimization problem and its approximate solution in the related fields, the more general form of these problems have not been studied thoroughly. Specifically, the QCQP problems can be generalized to include the composition of one-dimensional convex and quadratic functions in the objective and the constraint functions. In this paper, we show that such generalized QCQP problem can be precisely or approximately represented as the difference-of-convex functions (DC) programming problems which appear often in signal processing applications [8]. The existing most typical approaches developed for addressing DC programming problems are based on the so-called branch-and-bound methods [9] and [10]–[15]. However, these methods do not have any (worst-case) polynomial-time complexity guarantees which considerably limits or often prohibits their applicability in signal processing practice. Accordingly, methods with guaranteed polynomial-time complexity that can solve such DC programming problems at least suboptimally are of great importance. Thus, we develop a new approach with (worst-case) polynomial-time complexity that can solve (optimally under some conditions) the corresponding DC problem of a generalized QCQP problem.

2. GENERALIZED QCQP

We are mostly interested in the following generalization of the QCQP which includes the composition of one-dimensional convex and quadratic functions in the objective and the con-

straint functions

$$\min_{\mathbf{x},\mathbf{y}} \quad f_0(\mathbf{x}^H\mathbf{A}_0\mathbf{x}) + h_0(\mathbf{y})$$
$$\text{s.t.} \quad \alpha_{2i-1}f_{2i-1}(\mathbf{x}^H\mathbf{A}_{2i-1}\mathbf{x}) - \alpha_{2i}f_{2i}(\mathbf{x}^H\mathbf{A}_{2i}\mathbf{x})$$
$$+ h_i(\mathbf{y}) \le 0, \qquad i = 1,\cdots,M \qquad (1)$$

where $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{R}^n$, the sets $\mathbb{C}^m$ and $\mathbb{R}^n$ denote, respectively, the m-dimensional complex space and the n-dimensional Euclidean space, and $f_i : \mathcal{D}_{f_i} \subset \mathbb{R} \longrightarrow \mathbb{R}, i = 0,\cdots,2M$ are one-dimensional convex differentiable functions. The function f_0 is assumed to be a monotonic function which is bounded from the below over the feasible set of the problem. The matrices $\mathbf{A}_i \in \mathcal{H}^m, i = 0,\cdots,2M$ are Hermitian matrices that can be indefinite, $h_i : \mathcal{D}_{h_i} \subset \mathbb{R}^n \longrightarrow \mathbb{R}, i = 0,\cdots,M$ are convex differentiable functions, and $\alpha_i \in \{0,1\}, i = 1,\cdots,2M$, depending whether the function $f_i(\mathbf{x}^H\mathbf{A}_i\mathbf{x})$ is present or not. Similar to the function f_0, the convex function h_0 is also assumed to be lower-bounded.

The motivation behind the generalized QCQP formulation (1) is the fact that many important quantities in signal processing or communications have such composition forms. For instance, such composite functions are frequently encountered in the resource allocation problems. Particularly, since the transmit power of a multi antenna system and/or the received power of a single antenna receiver have quadratic forms with respect to the transmit beamforming vector/precoding matrix [16], the transmission rate, and the signal-to-interference-plus-noise ratio (SINR) can be recast in the forms of proposed compositions. Moreover, the corresponding objective function of the rate allocation schemes based on different criteria such as, for example, the sum-rate maximization [17], proportional and max-min rate fairness [18], and resource allocation based on mean square error (MSE) [19] are one form or another of the composition of one-dimensional convex and quadratic functions. Due to the aforementioned quadratic form of the transmit/received power, the robust beamforming problems can usually be recast in the form the proposed generalized QCQP [20], [21], and [22].

3. REFORMULATION

By defining the additional variables $\delta_i, i = 1,2,\cdots,2M$ and the set $C \triangleq \{k \mid \alpha_k = 1, 1 \le k \le 2M\}$, the problem (1) can be equivalently expressed as

$$\min_{\mathbf{y},\boldsymbol{\delta},\mathbf{x}} \quad f_0(\mathbf{x}^H\mathbf{A}_0\mathbf{x}) + h_0(\mathbf{y})$$
$$\text{s.t.} \quad \mathbf{x}^H\mathbf{A}_i\mathbf{x} = \delta_i, \quad i \in C \qquad (2)$$
$$\alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - \alpha_{2i}f_{2i}(\delta_{2i})$$
$$+ h_i(\mathbf{y}) \le 0, \quad i = 1,\cdots,M.$$

Since the function $h_0(\mathbf{y})$ as well as the constraint functions $\alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - \alpha_{2i}f_{2i}(\delta_{2i}) + h_i(\mathbf{y}) \le 0, i = 1,\cdots,M$

do not depend on $\mathbf{x}$, the problem (2) can be recast as

$$\min_{\mathbf{y},\boldsymbol{\delta}} \quad h_0(\mathbf{y}) + \left[\overbrace{\min_{\mathbf{x} \mid \mathbf{x}^H\mathbf{A}_i\mathbf{x}=\delta_i, \; i\in C} f_0(\mathbf{x}^H\mathbf{A}_0\mathbf{x})}^{\text{Inner Optimization Problem}} \right]$$
$$\text{s.t.} \quad \alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - \alpha_{2i}f_{2i}(\delta_{2i}) + h_i(\mathbf{y}) \le 0,$$
$$i = 1,\cdots,M. \qquad (3)$$

Then the problem (3) can be further expressed as

$$\min_{\mathbf{y},\boldsymbol{\delta}} \quad f_0(k(\boldsymbol{\delta})) + h_0(\mathbf{y})$$
$$\text{s.t.} \quad \alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - \alpha_{2i}f_{2i}(\delta_{2i}) + h_i(\mathbf{y}) \le 0,$$
$$i = 1,\cdots,M \qquad (4)$$

where $k(\boldsymbol{\delta})$ is an optimal value function (OVF) which is defined based on the inner optimization problem in (3) for a fixed value of $\boldsymbol{\delta} \in \mathcal{D}$. Here, $\mathcal{D}$ is defined as the set of all $\boldsymbol{\delta}$ such that the corresponding optimization problem obtained from $k(\boldsymbol{\delta})$ for fixed $\boldsymbol{\delta}$ is feasible. If the function f_0 is increasing, i.e., $f_0(x_1) \le f_0(x_2)$, for $x_1 \le x_2$, the OVF $k(\boldsymbol{\delta})$ is defined as

$$k(\boldsymbol{\delta}) \triangleq \left\{ \min_{\mathbf{x}} \mathbf{x}^H\mathbf{A}_0\mathbf{x} \mid \mathbf{x}^H\mathbf{A}_i\mathbf{x} = \delta_i, \; i \in C \right\}. \qquad (5)$$

In the rest of this paper, we assume that the function f_0 is increasing.[1]

4. PROPOSED SOLUTION

We first consider the case of $\text{card}\{C\} \le 3$ where $\text{card}\{\cdot\}$ is the cardinality operator. In other words, we first consider the case when the total number of composite functions $f_i(\mathbf{x}^H\mathbf{A}_i\mathbf{x})$ in the constraints of the problem (1) does not exceed three.

4.1. Number of quadratic functions in constraints is less than or equal to three

In this case, by introducing the matrix $\mathbf{X} \triangleq \mathbf{x}\mathbf{x}^H$ and observing that for any arbitrary matrix $\mathbf{Y}$, the relationship $\mathbf{x}^H\mathbf{Y}\mathbf{x} = \text{tr}\{\mathbf{Y}\mathbf{x}\mathbf{x}^H\}$ holds, the OVF $k(\boldsymbol{\delta})$ in (5) can be equivalently recast as

$$k(\boldsymbol{\delta}) = \left\{ \min_{\mathbf{X}} \text{tr}\{\mathbf{A}_0\mathbf{X}\} \mid \text{tr}\{\mathbf{A}_i\mathbf{X}\} = \delta_i, \; i \in C, \right.$$
$$\left. \text{rank}\{\mathbf{X}\} = 1, \; \mathbf{X} \succeq \mathbf{0} \right\}, \boldsymbol{\delta} \in \mathcal{D} \qquad (6)$$

where $\text{rank}\{\cdot\}$ denotes the rank of a matrix and $\mathcal{D}$ is the domain of the OVF $k(\boldsymbol{\delta})$. Using the SDP relaxation, the optimization problem of $k(\boldsymbol{\delta})$, when $\boldsymbol{\delta}$ is fixed, can be relaxed

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[1] The OVF $k(\boldsymbol{\delta})$ is similarly defined if the function f_0 is decreasing and all the following discussions are valid in this case as well.

by dropping the rank-one constraint. Then the following new OVF can be obtained

$$h(\boldsymbol{\delta}) \triangleq \left\{ \min_{\mathbf{X}} \operatorname{tr}\{\mathbf{A}_0\mathbf{X}\} \mid \operatorname{tr}\{\mathbf{A}_i\mathbf{X}\} = \delta_i,\ i \in C, \right.$$
$$\left. \mathbf{X} \succeq \mathbf{0} \right\}, \boldsymbol{\delta} \in D' \qquad (7)$$

where $\mathcal{D}'$ is the domain of the newly defined OVF. For brevity, we will refer to the optimization problems corresponding to the functions $k(\boldsymbol{\delta})$ and $h(\boldsymbol{\delta})$ when $\boldsymbol{\delta}$ is fixed simply as the optimization problems of $k(\boldsymbol{\delta})$ and $h(\boldsymbol{\delta})$, respectively. The following lemma finds the relationship between the domains of the functions $k(\boldsymbol{\delta})$ and $h(\boldsymbol{\delta})$.

Lemma 1: The domains of the functions $k(\boldsymbol{\delta})$ and $h(\boldsymbol{\delta})$ are the same, i.e., $\mathcal{D} = \mathcal{D}'$ if $\operatorname{card}\{C\} \leq 3$. Moreover, The OVFs $k(\boldsymbol{\delta})$ and $h(\boldsymbol{\delta})$ are equivalent, i.e., $k(\boldsymbol{\delta}) = h(\boldsymbol{\delta})$, $\boldsymbol{\delta} \in \mathcal{D}$ if $\operatorname{card}\{C\} \leq 3$ and some mild conditions are satisfied. Additionally, based on the optimal solution of the problem $h(\boldsymbol{\delta})$ the optimal solution of the problem $k(\boldsymbol{\delta})$ can be extracted. This Lemma is an extension of similar lemma proved for a particular generalized QCQP optimization problem in [21].

Although the OVFs $k(\boldsymbol{\delta})$ and $h(\boldsymbol{\delta})$ are equal, however, compared to the optimization problem of $k(\boldsymbol{\delta})$ which is non-convex, the optimization problem of $h(\boldsymbol{\delta})$ is convex. Using this fact and replacing $k(\boldsymbol{\delta})$ by $h(\boldsymbol{\delta})$ in the original optimization problem (2), this problem can be simplified as

$$\min_{\mathbf{y},\boldsymbol{\delta},\mathbf{X}} \quad f_0(\operatorname{tr}\{\mathbf{A}_0\mathbf{X}\}) + h_0(\mathbf{y})$$
$$\text{s.t.} \quad \operatorname{tr}\{\mathbf{A}_i\mathbf{X}\} = \delta_i, \quad i \in C, \quad \mathbf{X} \succeq \mathbf{0},$$
$$\alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - \alpha_{2i}f_{2i}(\delta_{2i}) + h_i(\mathbf{y}) \leq 0,$$
$$i = 1, \cdots, M. \qquad (8)$$

Therefore, instead of the original optimization problem (2), we can solve the simplified problem (8) in which the quadratic functions have been replaced with their corresponding linear functions. It is noteworthy to mention that in the simplified problem, the non-convex functions $f_i(\mathbf{x}^H\mathbf{A}_i\mathbf{x})$, $i = \{0\}\bigcup C$ are replaced by the convex functions $f_i(\operatorname{tr}\{\mathbf{A}_i\mathbf{X}\})$, $i = \{0\}\bigcup C$. The latter is due to the fact that the composition of a convex function with a linear function is also a convex function. Based on the optimal solution of the simplified problem, denoted as $\mathbf{X}_{\text{opt}}$, $\boldsymbol{\delta}_{\text{opt}}$, and $\mathbf{y}_{\text{opt}}$, the optimal solution of the original problem can be found. The optimal values of $\boldsymbol{\delta}$ and $\mathbf{y}$ are equal to the corresponding optimal values of the simplified problem, while, the optimal value of $\mathbf{x}$ can be constructed based on $\mathbf{X}_{\text{opt}}$ using rank-reduction techniques [23].

If the corresponding coefficients of the functions f_{2i}, $i = 1, \cdots, M$, i.e., α_{2i}, are all zero, then the problem (8) is convex and it can be easily solved. Particulary, in this case, the objective function and the constraint functions of the simplified problem (8) are all convex. Once this problem is solved,

the optimal $\mathbf{x}$ can be extracted using Lemma 1. However, if any of such coefficients is non-zero, the problem (8) is no longer convex and there exists a constraint which is the difference of two convex functions. Therefore, the problem (8) is a DC programming problem. Although the problem (8) boils down to the known family of DC programming problems, still there exists no solution for such problems with guaranteed polynomial-time complexity. The typical approach for solving such problems is the the branch-and-bound method and its various modifications [9], [10]−[15]. It is known to be an effective global optimization method. However, it does not have any worst-case polynomial-time complexity guarantees [11], [12]. It significantly limits or even prohibits its applicability in practical systems. Thus, methods with guaranteed polynomial-time complexity that can find at least a suboptimal solution for different types of DC programming problems are of a great importance. In what follows, we establish an iterative method for solving the problem (8) when at least one of the coefficients α_{2i}, $i = 1, \cdots, M$ is non-zero and therefore the relaxed problem is DC.

4.2. Polynomial-time DC algorithm

An iterative method for solving the DC programming problem (8) at least suboptimally is developed. The essence of the proposed method is to linearize the non-convex one-dimensional functions $-f_{2i}(\delta_{2i})$ appearing in the constraints

$$\alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - \alpha_{2i}f_{2i}(\delta_{2i}) + h_i(\mathbf{y}) \leq 0 \qquad (9)$$

around suitably selected points in different iterations. This new proposed method will be referred to as the Polynomial-time DC (POTDC). It is guaranteed that POTDC finds at least a KKT point, i.e., a point which satisfies the KKT optimality conditions. In order to explain the intuition behind this method, let us replace the non-convex functions $-f_{2i}(\delta_{2i})$, $i \in \mathcal{K} \triangleq \{i \mid \alpha_{2i} = 1\}$ by their corresponding linear approximations around the points $\delta_{2i,\text{Lin}}$, $i \in \mathcal{K}$, i.e.,

$$-f_{2i}(\delta_{2i}) \approx -f_{2i}(\delta_{2i,\text{Lin}}) - \frac{df_{2i}(\delta_{2i})}{d\delta_{2i}}\Big|_{\delta_{2i}=\delta_{2i,\text{Lin}}}(\delta_{2i}-\delta_{2i,\text{Lin}}). \qquad (10)$$

Performing such replacement results in the following optimization problem

$$\min_{\mathbf{y},\boldsymbol{\delta},\mathbf{X}} \quad f_0(\operatorname{tr}\{\mathbf{A}_0\mathbf{X}\}) + h_0(\mathbf{y})$$
$$\text{s.t.} \quad \operatorname{tr}\{\mathbf{A}_i\mathbf{X}\} = \delta_i,\ i \in C, \quad \mathbf{X} \succeq \mathbf{0}, \qquad (11)$$
$$\alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) + h_i(\mathbf{y}) \leq 0, \quad i = 1,\cdots,M,$$
$$i \notin \mathcal{K}$$
$$\alpha_{2i-1}f_{2i-1}(\delta_{2i-1}) - f_{2i}(\delta_{2i,\text{Lin}})$$
$$-\frac{df_{2i}(\delta_{2i})}{d\delta_{2i}}\Big|_{\delta_{2i}=\delta_{2i,\text{Lin}}}(\delta_{2i}-\delta_{2i,\text{Lin}}) + h_i(\mathbf{y}) \leq 0,$$
$$i \in \mathcal{K}.$$

As compared to the original problem (8), the relaxed problem (11) is convex and can be efficiently solved up to a desired accuracy using the interior point-based numerical methods. For the fixed values of $\delta_{2i}, i \in \mathcal{K}$ denoted as $\boldsymbol{\Delta}$, let us define the OVFs $f(\boldsymbol{\Delta})$ and $g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$ as the optimal value of the optimization problems (8) and (11), respectively, in which $\boldsymbol{\Delta}_{\text{Lin}}$ denotes the set of linearizing points, i.e., $\delta_{2i,\text{Lin}}, i \in \mathcal{K}$. Since the optimization problem (11) is convex, its corresponding OVF $g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$ is also convex with respect to $\boldsymbol{\Delta}$ [21]. Furthermore, the OVF $g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$ provides an upper-bound for the OVF $f(\boldsymbol{\Delta})$, i.e., $f(\boldsymbol{\Delta}) \leq g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$. The latter is due to the fact that the feasible set of the optimization problem (11) is a subset of the feasible set of the problem (8). Besides, with the assumption that the aforementioned OVFs are differentiable, it can be proved that the OVF $g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$ is a tangent to $f(\boldsymbol{\Delta})$ at $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\text{Lin}}$. Since the aforementioned OVFs are tangents at the linearizing point, i.e., $\boldsymbol{\Delta}_{\text{Lin}}$, and $f(\boldsymbol{\Delta})$ is upper-bounded by $g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$, the optimal minimizer of the function $g(\boldsymbol{\Delta}, \boldsymbol{\Delta}_{\text{Lin}})$ denoted as $\boldsymbol{\Delta}_{\text{opt}}$ is a decreasing point for $f(\boldsymbol{\Delta})$, that is, $f(\boldsymbol{\Delta}_{\text{Lin}}) \geq f(\boldsymbol{\Delta}_{\text{Opt}})$. Based on this observation, the POTDC method first solves the problem (11) for the arbitrary chosen initial point.

Once the optimal solution of this problem, denoted in the first iteration as $\mathbf{y}_{\text{opt}}^{(1)}$, $\mathbf{X}_{\text{opt}}^{(1)}$, and $\boldsymbol{\delta}_{\text{opt}}^{(1)}$ is found, the algorithm proceeds to the second iteration by replacing the functions $-f_{2i}(\delta_{2i}), i \in \mathcal{K}$ by their linear approximations around $\delta_{\text{opt},2i}^{(1)}, i \in \mathcal{K}$, respectively, found from the previous (initially first) iteration. In the second iteration, the resulting optimization problem has the same structure as the problem (11) in which $\delta_{2i,\text{Lin}}, i \in \mathcal{K}$ has to be set to $\delta_{\text{opt},2i}^{(1)}, i \in \mathcal{K}$ obtained from the first iteration. This process continues, and kth iteration consists of replacing $-f_{2i}(\delta_{2i}), i \in \mathcal{K}$ by their linearizations of type (10) using $\boldsymbol{\delta}_{\text{opt}}^{(k-1)}$ found at the iteration $k-1$. It can be shown that the sequence of the optimal values which are generated by proposed POTDC method, i.e., $f_0(\text{tr}\{\mathbf{A}_0 \mathbf{X}_{\text{opt}}^{(k)}\}) + h_0(\mathbf{y}_{\text{opt}}^{(k)}), k = 1, 2, \cdots$, are non-increasing and convergent. Moreover, if the proposed POTDC method converges to a regular point, that point is a KKT point, i.e., a point which satisfies the KKT optimality conditions.

The proposed POTDC method can be interpreted in terms of the Newton's method when it is utilized to minimize a non-constrained multi-dimensional function. Particularly, the Newton's method minimizes a sequence of quadratic approximations of the non-constrained objective function successively. In a similar way, our proposed method also minimizes a convex approximation of the optimal value function $f(\boldsymbol{\Delta})$ successively over the iterations. However as opposed to the Newton's method, the convex approximations of the OVF $f(\boldsymbol{\Delta})$ are not necessarily quadratic. We have utilized this similarity for establishing a polynomial-time complexity guarantee proof for the proposed POTDC method under some conditions.

5. GENERAL FORM OF THE PROPOSED SOLUTION

When the total number of the composite functions $f_i(\mathbf{x}^H \mathbf{A}_i \mathbf{x})$ in the constraints of the optimization problem (1) or equivalently $\text{card}\{C\}$ is greater than or equal to four, the OVFs $k(\boldsymbol{\delta})$ in (5) and $h(\boldsymbol{\delta})$ in (7) may not be equal in general. In this case, the OVF $h(\boldsymbol{\delta})$ is a lower-bound of $k(\boldsymbol{\delta})$, i.e., $h(\boldsymbol{\delta}) \leq k(\boldsymbol{\delta})$ ($h(\boldsymbol{\delta})$ is an upper-bound of $k(\boldsymbol{\delta})$ if f_0 is decreasing). By replacing the OVF $k(\boldsymbol{\delta})$ by $h(\boldsymbol{\delta})$ in the original optimization problem (4), this problem can be approximated by the following DC programming problem

$$\min_{\mathbf{y},\boldsymbol{\delta}} \quad f_0(h(\boldsymbol{\delta})) + h_0(\mathbf{y})$$
$$\text{s.t.} \quad \alpha_{2i-1} f_{2i-1}(\delta_{2i-1}) - \alpha_{2i} f_{2i}(\delta_{2i}) + h_i(\mathbf{y}) \leq 0,$$
$$i = 1, \cdots, M. \tag{12}$$

Since f_0 is a monotonic function, it can be concluded that the objective function of the optimization problem (12) is a lower-bound of the original problem. Thus, instead of the original objective function, the problem (12) aims at minimizing a lower-bound of the objective function. The problem (12) can be similarly solved by using the POTDC method. The extraction of (in general suboptimal) solution of the original problem from the optimal solution of the approximate problem (12) can be done through the well known standard randomization techniques.

6. CONCLUSION

A specific generalization of the QCQP optimization problem which comprises compositions of one-dimensional convex and quadratic functions in the constraints and the objective function has been introduced. Moreover, it has been explained that many important practical problems boil down to generalized QCQP which explains the significant importance of this generalization. In order to solve this class of problems, we have shown that the generalized QCQP can be precisely or approximately recast as a DC progrmming problem by means of SDR relaxation. Then we proposed a method for solving the resulted DC optimization problems at least suboptimally with polynomial-time complexity guarantees. The new proposed method can be interpreted in terms of the Newton's method as applied to a non-constrained optimization problem.

7. RELATION TO PRIOR WORK

In our previous works [17], [18], [20], and [21], we have observed that some of the important problems in signal processing and communications, such as, resource allocation or robust adaptive beamforming problems involve a generalized form of a QCQP optimization. In this paper, we introduce and solve such generalization of QCQP optimization which comprises compositions of one-dimensional convex and quadratic functions in the constraints and the objective function.

8. REFERENCES

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