ON THE ESTIMATION OF GRID OFFSETS IN CS-BASED DIRECTION-OF-ARRIVAL ESTIMATION

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ABSTRACT

Compressed Sensing (CS) has been recently applied to direction of arrival (DOA) estimation, leveraging the fact that a superposition of planar wavefronts corresponds to a sparse angular power spectrum. However, to apply the CS framework we need to construct a finite dictionary by sampling the angular domain with a predefined sampling grid. Therefore, the target locations are almost surely not located exactly on a subset of these grid points. This leads to a model mismatch which deteriorates the performance of the estimators. In this paper we take an analytical approach to investigate the effect of such grid offsets on the recovered spectra. We show that each off-grid source can be well approximated by the closest two neighboring points on the grid. We propose a simple and efficient scheme to estimate the grid offset for a single source or multiple well-separated sources. We also discuss a numerical procedure for the joint estimation of the grid offsets of closer sources. Simulation results demonstrate the effectiveness of the proposed methods.

1. INTRODUCTION

Direction of arrival (DOA) estimation has been an active field of research for many decades [1]. Estimated DOAs are used in various applications like localization of the transmitting sources, channel modeling, tracking and surveillance in Radar, and many others. If the field is modeled as a superposition of a few planar wave-fronts, the DOA estimation problem can be expressed as a sparse recovery problem and the Compressed Sensing (CS) framework can be applied.

Many powerful CS-based DOA estimation algorithms have been proposed in recent years [2, 4, 6, 5]. CS based DOA estimation may be an attractive solution with respect to the hardware complexity of the receiving arrays and the complexity of the numerical solution (compared to Maximum Likelihood algorithms) while being insensitive to source correlation and allowing arbitrary array geometries (as opposed to most subspace-based estimators).

However, they all face one common problem. Although the model is sparse in a continuous angular domain, to apply the CS framework we need to construct a finite dictionary by sampling this domain with a predefined sampling grid. Therefore, the target locations are almost surely not located exactly on a subset of these grid points. This leads to a model mismatch that results in a degradation of the performance.

It may seem that the solution is to make the grid as fine as possible. However, this violates the restricted isometry property (RIP) [14] of the system and deteriorates the CS recovery process. Early approaches for CS based DOA estimation suggest to tackle this off-grid problem by simply refining the grid adaptively around the candidate targets found with an initial, mismatched grid [2] or taking centroids of the dominant coefficients as the exact location [13]. One type of more sophisticated solutions models the mismatch error explicitly and fits it to the observed data statistically [3, 7]. Other approaches deal with the continuous problem directly and propose some modifications to the recovery algorithm to deal with such scenario, i.e., interpolating between grid points [18], atomic norm minimization [17], or perturbed OMP [8]. Note that this typically increases the computational complexity significantly.

In this paper we take an analytical approach to investigate the effect of recovering the spectrum of sources not contained in the dictionary. Unlike earlier works that have provided a quantitative analysis of the approximation error [16, 20], we examine the specific shape of the resulting spectrum. We show that for one off-grid source the recovered spectrum is not sparse but it can be well approximated by the closest two dictionary atoms on the grid and their coefficients can be exploited to estimate the grid offset. We then extend our model to consider multiple sources. When they are sufficiently separated, the offset estimation strategy can be applied separately. For closely spaced sources we propose an efficient joint estimation strategy and demonstrate its performance in numerical simulations.

2. DATA MODEL

Consider K narrow-band planar waves impinging on an array of M elements. At the array side, the observations are given by

$$\boldsymbol{y}(t) = \sum_{k=1}^{K} \boldsymbol{a}(\theta_k) \cdot s_k(t) + \boldsymbol{n}(t), \tag{1}$$

This work was supported by the International Graduate School on Mobile Communications (MOBICOM), Ilmenau, Germany.

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where $\boldsymbol{a}(\theta) \in \mathbb{C}^M$ is the array manifold, θ_k is the DOA vector, $\boldsymbol{s}_k(t)$ are the amplitudes of the impinging waves at time t, and $\boldsymbol{n}(t)$ is the additive white Gaussian noise (AWGN) contaminating the observations.

One way to interpret the scenario in (1) is that the power received at the array sensors concentrates at few locations θ_k from all possible DOAs θ, which means that the received power is "sparse" in the angular domain. This sparsity motivates the use of compressed sensing for DOA estimation. The CS based DOA estimation problem is formulated as [2]

$$\boldsymbol{y}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t), \tag{2}$$

where $\boldsymbol{A} \in \mathbb{C}^{M\times N}$ is the sampled array manifold (dictionary). The number of grid points is given as $N = M \cdot P$, $P > 1$ where P is the oversampling factor representing how fine the grid is. Note that P is sometimes also referred as the "super resolution factor" [15]. We consider this term misleading since increasing P does not actually improve the resolution (which is limited by M, as also shown in [15]).

In this paper we assume a uniform linear array (ULA) of isotropic sensors[1]. Moreover, in order to achieve a uniform mutual coherence between columns of $\boldsymbol{A}$, we sample the manifold uniformly in the spatial frequency domain instead of the angular domain. For a ULA with half-wavelength inter-element spacing, the spatial frequency μ is defined as $\mu = \pi \cdot \sin\theta$. The dictionary is then sampled at the points $\mu_n = \Delta \cdot (n-1)$, $n = 1, 2, \ldots, N$, where $\Delta = \frac{2\pi}{N}$.

3. ANALYTICAL STUDY OF THE OFF GRID PROBLEM

3.1. One source off the grid

As mentioned earlier, no matter how fine the grid is, there will always be sources that do not lie exactly on one of the grid points. In this section, we analyze this problem both qualitatively and quantitatively. For simplicity, let us start by one source off the grid. In the absence of noise, this simplifies (1) into $\boldsymbol{y} = \tilde{\boldsymbol{a}} \cdot s$, where $\tilde{\boldsymbol{a}} = \boldsymbol{a}(\mu_L + \epsilon \cdot \Delta)$. Here, $L \in \mathbb{N}$, i.e., μ_L represents the next "left" grid point. Moreover, $0 \le \epsilon < 1$ is the grid offset, expressed as a fraction of Δ.

In general, $\boldsymbol{y}$ is not 1-sparse in $\boldsymbol{A}$. In fact, an exact representation of $\boldsymbol{y}$ in $\boldsymbol{A}$ requires M non-zero coefficients. Moreover, an arbitrary subset of M out of N coefficients could be used to find such a representation.

In the CS framework, we often employ an ℓ_1-type regularization to find a sparse solution. For instance, the Basis Pursuit method [9] solves the following problem

$$\min \|\boldsymbol{\alpha}\|_1 \quad \text{s.t.} \quad \boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{\alpha}. \tag{3}$$

[1]This assumption is made for simplicity and should be seen as a first step only. CS-based DOA estimation can only be put to practice if $\boldsymbol{A}$ is constructed by considering the physical effects for a realistic array [21]. We have shown an extension to a practical scenario with a circular array under a polarimetric setting in a more recent study [22].

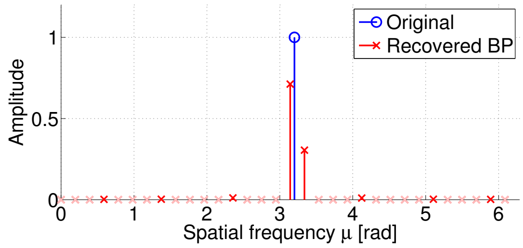

Fig. 1. Recovered spectrum using for one source off the grid ($M = 8, P = 4, \epsilon = 0.3$) using BP [9].

The purpose of this regularization is to concentrate the energy on as few coefficients as possible. This suggests that the solution to (3) chooses the closest neighboring atoms on the grid to represent the off-grid source. To support this intuition, Fig. 1 demonstrates the resulting spectrum when solving (3) using BP [9] for $M = 8$, $P = 4$ (i.e., $N = 32$), and $\epsilon = 0.3$. We observe that, as expected, most of the energy is concentrated on the two closest grid points.

Motivated by this finding, we investigate the approximation of $\tilde{\boldsymbol{a}}$ using the two neighboring atoms on the grid, i.e.,

$$\boldsymbol{a}(\mu_L + \epsilon \cdot \Delta) = \left[\boldsymbol{a}(\mu_L), \boldsymbol{a}(\mu_{L+1})\right] \cdot \begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} + \boldsymbol{a}_{\mathrm{Res}} \tag{4}$$

where $\boldsymbol{a}_{\mathrm{Res}}$ is the residual that is not representable by the neighbors. The coefficients α_1 and α_2 are found by solving

$$\min_{\alpha_1 \alpha_2} \left\| \tilde{\boldsymbol{a}} - \left[\boldsymbol{a}(\mu_L), \boldsymbol{a}(\mu_{L+1})\right] \cdot \begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} \right\|_2^2. \tag{5}$$

After some algebraic manipulations, we can express the explicit solution to (5) as

$$\alpha_1(\epsilon) = \frac{D(0)\cdot D(\epsilon) - D(1)\cdot D(1-\epsilon)}{D(0)^2 - D(1)^2} \cdot \mathrm{e}^{\jmath\cdot\epsilon\cdot\pi\frac{(M-1)}{M\cdot P}}, \tag{6}$$

$$\alpha_2(\epsilon) = \frac{D(0)\cdot D(1-\epsilon) - D(1)\cdot D(\epsilon)}{D(0)^2 - D(1)^2} \cdot \mathrm{e}^{-\jmath\cdot(1-\epsilon)\cdot\pi\frac{(M-1)}{M\cdot P}}, \tag{7}$$

$$\text{where} \quad D(x) = \frac{\sin\left(\frac{\pi\cdot x}{P}\right)}{\sin\left(\frac{\pi\cdot x}{M\cdot P}\right)}. \tag{8}$$

Fig. 2 shows the behavior of the calculated coefficients with varying the offset ϵ for $P = 2$ and 4. It can be shown that

$$\lim_{P\to\infty} \alpha_1(\epsilon) = 1 - \epsilon, \tag{9}$$

$$\lim_{P\to\infty} \alpha_2(\epsilon) = \epsilon, \tag{10}$$

i.e., as P increases (and so N), α_1 and α_2 becomes more linear with ϵ.

To assess the accuracy of our two term approximation, the relative approximation error (AE) has been examined. We define

$$\mathrm{AE}(\epsilon, M, P) \doteq \frac{\|\boldsymbol{a}_{\mathrm{Res}}\|_2^2}{\|\tilde{\boldsymbol{a}}\|_2^2}. \tag{11}$$

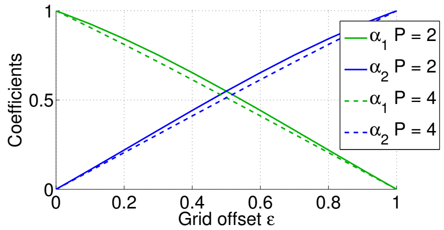

Fig. 2. Behavior of the approximation coefficients α_1 and α_2 for $M = 8$.

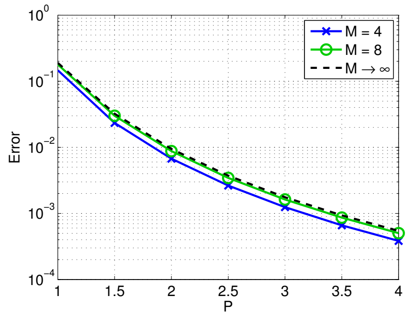

Fig. 3. Worst-case approximation error $\mathrm{AE}(1/2, M, P)$ vs. M and P.

It can be shown that the AE is convex and symmetric in ϵ and that $\mathrm{AE}(\epsilon, M, P) \leq \mathrm{AE}(1/2, M, P)$, i.e., as expected, the worst case error is at $\epsilon = 0.5$. Moreover, a closed form expression for this worst case error is given by

$$\mathrm{AE}(1/2, M, P) = 1 - \frac{4\sin^2\left(\frac{\pi}{2P}\right)\cot\left(\frac{\pi}{2MP}\right)}{M^2\sin\left(\frac{\pi}{MP}\right) + M\sin\left(\frac{\pi}{P}\right)}, \quad (12)$$

which we depict in Fig. 3. In fact, Fig. 3 shows that (12) increases mildly with M and decreases rapidly with increasing P (0.01 at $P = 2$ and 0.001 at $P = 3$). In the limits we have

$$\lim_{P\to\infty} \mathrm{AE}(1/2, M, P) = 0, \quad (13)$$

$$\lim_{M\to\infty} \mathrm{AE}(1/2, M, P) = 1 - \frac{4\left(1 - \cos\left(\frac{\pi}{P}\right)\right)P^2}{\pi\left(P\sin\left(\frac{\pi}{P}\right) + \pi\right)}. \quad (14)$$

From the results of (6) and (7), we were inspired[2] to define a simple estimator for ϵ given by

$$\hat{\epsilon} = \frac{\alpha_2}{\alpha_1 + \alpha_2}. \quad (15)$$

Note that in the absence of noise we have from (9) and (10)

$$\lim_{P\to\infty} \hat{\epsilon} = \epsilon. \quad (16)$$

3.2. Multiple Sources

So far, we have discussed a single source only. When multiple sources are present, their mutual influence depends on the correlation between the array steering vectors. As long as the distance between the sources is $\gg P$ grid points they are almost orthogonal and hence the mutual influence is very low. In this case, they can be treated independently and the estimator (15) can be applied separately. This is exemplified in Fig. 4 which depicts such a case ($M = 8$, $P = 4$, $\mu_1 = 6.3\Delta$, $\mu_2 = 15.3\Delta$).

This strategy fails for any pair of sources that has a distance which is close to (or even below) P grid points. Note that it has recently been shown that the CS-based recovery of the correct support can only be guaranteed when the sources have a spacing that is $\geq P$ grid points [15][3]. Let us assume that the support has been estimated correctly, i.e., for two sources located at $\mu_k = (L_k - 1 + \epsilon_k)\cdot\Delta$, $k = 1, 2$ we have found the left neighboring grid points L_1 and L_2. Then, the best two-term approximation for one source shown in (5) can be extended to the joint estimation of the grid offsets for two sources by considering the four neighboring grid points $L_1, L_1 + 1, L_2$, and $L_2 + 1$. These provide four coefficients $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^{\mathrm{T}} \in \mathbb{C}^{4\times1}$ which depend on both offsets ϵ_1 and ϵ_2. Although we have not found a closed-form solution like (15) we propose a simple numerical procedure to estimate ϵ_1 and ϵ_2 from $\boldsymbol{\alpha}$.

To this end, let $\boldsymbol{d}(x) = [D(x), D(x-1), D(x-d), D(x-d-1)]^{\mathrm{T}} \in \mathbb{R}^{4\times1}$, $\boldsymbol{D}_0 = [\boldsymbol{d}(0), \boldsymbol{d}(1), \boldsymbol{d}(d), \boldsymbol{d}(d+1)] \in \mathbb{R}^{4\times4}$, and $\boldsymbol{D}_1 = [\boldsymbol{d}(\epsilon_1), \boldsymbol{d}(\epsilon_2 + d)] \in \mathbb{R}^{4\times2}$, where $d = L_2 - L_1$. Moreover, we define $\bar{\boldsymbol{\alpha}} = \boldsymbol{D}_0 \cdot \boldsymbol{\Phi} \cdot \boldsymbol{\alpha}$, where $\boldsymbol{\Phi} = \mathrm{diag}\left\{\left[1 \quad e^{\jmath\pi\frac{M-1}{M\cdot P}} \quad e^{\jmath\pi d\frac{M-1}{M\cdot P}} \quad e^{\jmath\pi(d+1)\frac{M-1}{M\cdot P}}\right]\right\}$. Then it can be shown that in the absence of noise, $\bar{\boldsymbol{\alpha}}$ is a linear combination of $\boldsymbol{d}(\epsilon_1)$ and $\boldsymbol{d}(\epsilon_2 + d)$. Therefore, we can obtain ϵ_1 and ϵ_2 by minimizing

$$J(\epsilon_1, \epsilon_2) = \left\|\bar{\boldsymbol{\alpha}} - \boldsymbol{D}_1 \cdot \boldsymbol{D}_1^{+} \cdot \bar{\boldsymbol{\alpha}}\right\|_2^2, \quad (17)$$

i.e., tuning ϵ_1, ϵ_2 such that the overlap of $\bar{\boldsymbol{\alpha}}$ with the column space of $\boldsymbol{D}_1$ is maximized. Note that the optimization problem (17) is easy to solve: the search region is bounded to $[0, 1] \times [0, 1]$ and by visual inspection it appears to be a smooth and convex shape with a clear unique minimum within this range. Therefore, the solution to (17) takes considerably less time than the sparse recovery algorithm that is run before to estimate the support.

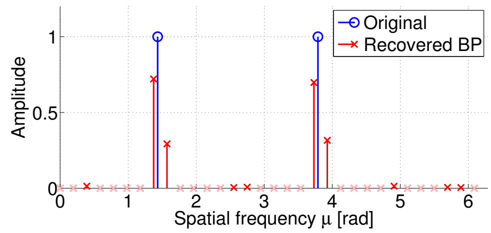

Fig. 4. Recovered spectrum for two far sources ($M = 8, P = 4, \epsilon_1 = \epsilon_2 = 0.3$).

[2] Note that similar estimators are used in the literature for frequency interpolation [10, 11]. However, they have been derived in a completely different context and it was not expected that such techniques are applicable to a spectrum recovered by ℓ_1-minimization.

[3] In fact, a spacing of P grid points corresponds to a distance in spatial frequency of $2\pi/M$ radians, which is referred to as the "Rayleigh resolution limit" [1].

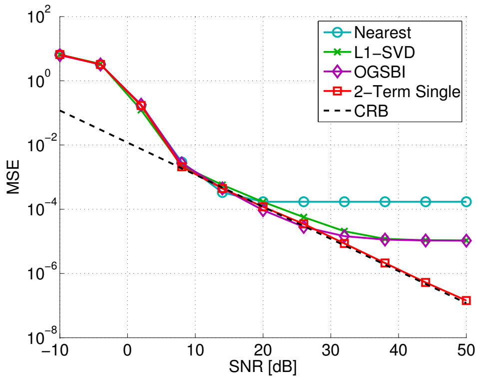

Fig. 5. MSE vs. SNR for $P = 6$, one source at $\mu = 7.1 \cdot \Delta$

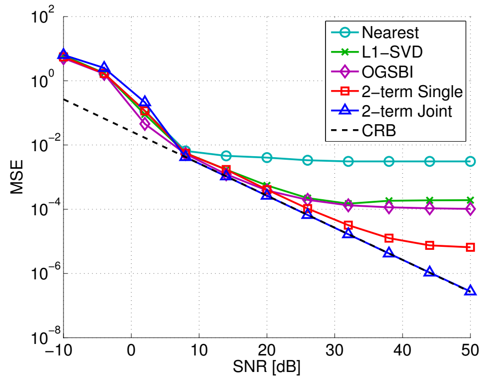

Fig. 6. MSE vs. SNR for $P = 8$, two sources at $\mu_1 = 12.4 \cdot \Delta$, $\mu_2 = 28.4 \cdot \Delta$, i.e., $\mu_2 - \mu_1 = 2 \cdot P \cdot \Delta$

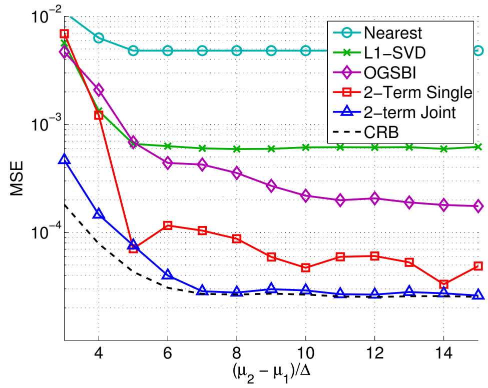

Fig. 7. MSE vs. $\Delta\mu$ for $P = 4$, $\mu_1 = 0.75 \cdot \Delta$, $\mu_2 = \mu_1 + \Delta\mu$

For more than two sources, the extension is straightforward. From a first CS-based recovery with N grid points, we obtain an initial coarse estimate of the sources' locations. Based on it we identify clusters of sources that are closely spaced where the different clusters are sufficiently far apart so that they can be treated independently. For each cluster, we apply a joint estimation of the grid offsets using the single-source strategy shown in (15), the two-source strategy from (17), or an appropriately extended K-source strategy, depending on the number of sources per cluster.

4. SIMULATION RESULTS

In this section, we evaluate our algorithm for CS-based DOA estimation off the grid. We consider $M = 8$ sensors and a single snapshot t. The noise samples are drawn from a zero mean circularly symmetric complex Gaussian distribution with variance P_{N}. The symbols s_k are generated according to $s_k = \mathrm{e}^{\jmath\varphi_k}$, where φ_k are i.i.d. uniformly distributed random variables in $[0, 2\pi]$. We depict the mean square estimation error of the spatial frequencies vs. the SNR = $1/P_{\mathrm{N}}$.

We compare the following strategies: "Nearest", "2-term single" and "2-term joint" refer to choosing the nearest grid point, applying (15), and solving (17), respectively, where the support has been estimated using the BP algorithm [9]. For reference, we depict the performance of the OGSBI algorithm [7] and the ℓ1-SVD [2] using three refinement steps. Note that the computational complexity of both reference schemes is higher than our proposed solutions. We also show the deterministic Cramér-Rao Bound (CRB).

Fig. 5 shows the performance for a single source at $7.1 \cdot \Delta$ and $P = 6$. We observe that the estimator (15) successfully finds the grid offset and the resulting estimator achieves the CRB.

In the case of two sources, Fig. 6 shows the MSE vs. SNR for a case where the spatial separation $\mu_2 - \mu_1 = 2 \cdot P \cdot \Delta$, i.e., the sources are relatively far from each other. In this case, applying the estimator (15) separately provides accurate estimates with a small bias becoming visible only at very high SNRs. Moreover, the solution of (17) achieves the CRB. Fig. 7 depicts the MSE vs. the spatial separation $\Delta\mu$ where the SNR is fixed to 30 dB and we have $\mu_1 = 0.75 \cdot \Delta$, $\mu_2 = \mu_1 + \Delta\mu$. We observe that the single-source approximation scheme works reasonably well until a separation of $P \cdot \Delta$ which is the lower limit derived in [15]. Below it, the mutual influence becomes too strong. On the other hand, the joint estimator still works very well for distances below this limit and outperforms the more complex ℓ1-SVD and the OGSBI algorithm.

5. CONCLUSION

In this paper we address the problem of CS-based DOA estimation for off-grid sources. We study the spectrum in the case of off-grid sources qualitatively and find that most of the energy of the off-grid source after reconstruction is concentrated in the two neighboring grid points. Based on this observation, we derive the best two-term approximation coefficients explicitly and show that the approximation error is very small for $N > M$. Moreover, based on the asymptotically linear behavior of the coefficients with the grid offset, we propose a very simple scheme to estimate the grid offset based on the observed coefficients. For multiple sources we show that this simple scheme still works well when they are sufficiently spaced. For closely spaced sources, we propose a numerical procedure for the joint estimation of their offsets from the recovered spectra at their neighboring grid points. Numerical simulations demonstrate the effectiveness of the proposed schemes.

6. REFERENCES

[1] H. Karim, and M. Viberg, "Two Decades of Array Signal Processing Research: The Parametric Approach", *IEEE Signal Processing Magazine*, 13(4), Jul 1996.

[2] D. M. Malioutov, M. Cetin, and A.S. Willsky, "Sparse signal reconstruction perspective for source localization with sensor arrays", *IEEE Transactions on Signal Processing*, 53(8), Aug 2005.

[3] H. Zhu, G. Leus, and G. B. Giannakis, "Sparsity-Cognizant Total Least-Squares for Perturbed Compressive Sampling", *IEEE Transactions on Signal Processing*, 59(5), May 2011.

[4] J. F. Gu, W. P. Zhu, and M. N. Swamy, "Compressed Sensing for DOA Estimation with Fewer Receivers than Sensors", *IEEE International Symposium on Circuits and Systems (ISCAS)*, May 2011.

[5] C. Austin, J. Ash, and R. Moses, "Parameter Estimation using Sparse Reconstruction with Dynamic Dictionaries," Proc. Intl. Conf. Acoustics, Speech, and Signal Processing (ICASSP 2011), Prague, Czech Republic, May 2011.

[6] A. C. Gurbuz, V. Cevher, and J. H. McClellan, "Bearing Estimation via Spatial Sparsity using Compressive Sensing", *IEEE Transactions on Aerospace and Electronic Systems*, 48(2), Apr 2012.

[7] Z. Yang, L. Xie, and C. Zhang, "Off-Grid Direction of Arrival Estimation Using Sparse Bayesian Inference", *IEEE Transactions on Signal Processing*, 61(1), Jan 2013.

[8] O. Teke, A. C. Gurbuz, and O. Arikan, "Perturbed Orthogonal Matching Pursuit", *IEEE Transactions on Signal Processing*, vol. 61, no. 24, pp. 6220–6231, Dec. 2013.

[9] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic Decomposition by Basis Pursuit", *SIAM Journal on Scientific Computing*, 20(1), Jan 1998.

[10] V. K. Jain, W. L. Collins, and D. C. Davis, "High Accuracy Analog Measurements via Interpolated FFT," *IEEE Transactions on Instrumentation and Measurement*, vol. 28, no. 2, pp. 113-122, Jun 1979.

[11] E. Jacobsen, P. Kootsookos, "Fast, Accurate Frequency Estimators," *IEEE Signal Processing Magazine*, vol.24, no.3, pp.123-125, May 2007.

[12] Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad, "Orthogonal Matching Pursuit: Recursive Function Approximation with Applications to Wavelet Decomposition", *Conference on Signals, Systems and Computers*, Nov 1993.

[13] C. Feng, S. Valaee, and Z. Tan, "Multiple Target Localization Using Compressive Sensing", *IEE Global Telecommunications Conference*, Nov 2009.

[14] E. J. Candes, J. Romberg, and T. Tao, "Robust Uncertainty Principles: Exact Signal Reconstruction from Highly Incomplete Frequency Information", *IEEE Transactions on Information Theory*, 52(2), Feb 2006.

[15] E. J. Candes, and C. F. Granda, "Towards a Mathematical Theory of Super-Resolution", *CoRR Journal*, Mar 2012.

[16] M. A. Hermann, and T. Strohmer, "General Deviants: An Analysis of Perturbations in Compressed Sensing", *IEEE Journal of Selected Topics in Signal Processing*, 4(2), Apr 2010.

[17] G. Tang, B. Bhaskar, P. Shah, and B. Recht, "Compressed Sensing Off the Grid", *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7465–7490, Aug 2013.

[18] C. Ekanadham, D. Tranchina, and E. P. Simoncelli, "Recovery of sparse translation-invariant signals with continuous basis pursuit," *IEEE Transactions on Signal Processing*, 59(10), Oct 2011.

[19] M. F. Duarte, and R. G. Baraniuk "Recovery of Frequency-Sparse Signals from Compressive Measurements", *48th Annual Allerton Conference on Communication, Control, and Computing*, Sep 2010.

[20] Y. Chi, L. L. Scharf, A. Pezeshki, and A. R. Calderbank "Sensitivity to Basis Mismatch in Compressed Sensing", *IEEE Trans. on Signal Processing*, 59(5), May 2011.

[21] M. Landmann, M. Käske, and R.S. Thomä, "Impact of Incomplete and Inaccurate Data on High Resolution Parameter Estimation in Multidimensional Channel Sounding," *IEEE Trans. on Antennas and Propagation*, vol. 60, no 2, Feb 2012, pp. 557-573.

[22] F. Römer, M. Ibrahim, R. Alieiev, M. Landmann, R.S. Thomä, and G. Del Galdo "Polarimetric Compressive Sensing Based DOA Estimation," *Workshop on Smart Antennas*, Erlangen, Germany, Mar 2014.