

# DISTRIBUTED QUANTIZATION FOR COMPRESSED SENSING

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## ABSTRACT

We study distributed coding of compressed sensing (CS) measurements using vector quantizer (VQ). We develop a distributed framework for realizing optimized quantizer that enables encoding CS measurements of correlated sparse sources followed by joint decoding at a fusion center. The optimality of VQ encoder-decoder pairs is addressed by minimizing the sum of mean-square errors between the sparse sources and their reconstruction vectors at the fusion center. We derive a lower-bound on the end-to-end performance of the studied distributed system, and propose a practical encoder-decoder design through an iterative algorithm.

**Index Terms**— Compressed sensing, distributed source coding, vector quantization, correlation, mean square error.

## 1. INTRODUCTION

With the rapid growth of sensor networks and their popularity to accomplish autonomous tasks such as sensing and computation, distributed coding of correlated sources has attracted much attention. In short, distributed coding considers separately encoding of correlated sources (from different terminals), and decoding the coded symbols jointly at a fusion center (FC). Distributed lossless and lossy source coding are well-developed fields and various theoretical results on this topic have been established [1–7]. The characteristics of sensor networks motivate the development of new techniques and algorithms which need to be aware of the systems' limited resources, computational complexity and power consumption. In this regard, compressed sensing (CS) [8] can be considered as an emerging tool for signal compression and acquisition that significantly reduces costs due to sampling and reconstruction, leading to lower power consumption and computations. By exploiting signal's sparsity in a known domain, CS allows the signal to be retrieved from relatively few measurements through a non-linear procedure.

The problem of source coding (through quantization) for CS measurements in a point-to-point setup has gained significant attention recently, and a wide range of interesting problems regarding the design and analysis of CS reconstruction and quantization algorithms have been formulated, see, e.g., [9–17, 17–22]. Considering communication channel errors, in [23], we have studied joint source-channel coding for CS through optimal design of vector quantizer (VQ).

All of these works are dedicated to point-to-point (single-terminal) quantized transmission of CS measurements through either noiseless or noisy communication channel. In a distributed fashion, Bajwa *et al.* [24] proposed a distributed joint source-channel communication architecture by transmitting uncoded (analog) random projections of sensor data to the FC. Feizi *et al.* [25] have proposed a low-complexity and distributed lossless source compression scheme by providing a link between analog network coding and CS. This idea has been generalized in [26] as well. In this paper, similar to the above mentioned works, we consider a distributed framework for CS. However, an important difference in our work is that on

the contrary to [24–26], we focus on *lossy compression* (using VQ) and *encoded transmission* of CS measurements followed by *joint decoding* at the FC. To the best of our knowledge, the distributed design and analysis of VQ for CS measurements—which can be of great importance in sensor networks—have not been addressed yet.

In this work, we consider, without loss of generality, two linear CS systems where they obtain few number of measurements from two correlated sparse vectors. The low-dimensional (possibly noisy) measurements are encoded using VQ, and then transmitted to a FC for joint reconstruction of correlated sparse sources. As a performance criterion in this distributed setup, we are interested in the sum of mean square error (MSE) distortions between the sparse source vectors and their reconstruction vectors at the FC. By applying the end-to-end MSE criterion, our contributions are as following: (1) *Establishing (necessary) conditions for optimality of VQ encoder-decoder pairs.* (2) *Deriving analytical expressions for the minimum mean square error (MMSE) estimators of correlated sparse sources from noisy CS measurement vectors; the MMSE estimation is an integral part of optimum VQ scheme in distributed CS setup.* (3) *Providing a theoretical lower-bound on the MSE performance.* (4) *Developing a practical VQ encoder-decoder design algorithm through an iterative algorithm.*

## 2. DISTRIBUTED MMSE ESTIMATION FOR CS

In this section, we introduce the CS system setup, and study the MMSE estimation of correlated sparse sources from noisy CS measurements in a distributed framework. We consider an exact  $K$ -sparse (in a known basis) vector  $\Theta \in \mathbb{R}^N$  comprised of  $K$  non-zero coefficients ( $K \ll N$ ). We define the (random) support set of the vector  $\Theta \triangleq [\Theta_1, \dots, \Theta_N]^T$  as  $\mathcal{S} \triangleq \{n \in \{1, 2, \dots, N\} : \Theta_n \neq 0\}$  with  $|\mathcal{S}| = \|\Theta\|_0 = K$ . Here,  $\|\cdot\|_0$  and  $|\cdot|$  denote the  $\ell_0$  norm and cardinality, respectively. The non-zero coefficients of  $\Theta$  are drawn according to  $\Theta_n \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\theta^2)$ ,  $n \in \mathcal{S}$ . We assume a distributed CS system with two correlated sources  $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^N$  in which the correlation is established according to

$$\mathbf{X}_l = \Theta + \mathbf{Z}_l, \quad l \in \{1, 2\}, \quad (1)$$

where  $\mathbf{Z}_l \triangleq [Z_{1,l}, \dots, Z_{N,l}]^T \in \mathbb{R}^N$  is an exact  $K$ -sparse vector with a common support set  $\mathcal{S}$  whose non-zero components are drawn as  $Z_{n,l} \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma_{z_l}^2)$ ,  $n \in \mathcal{S}$ ; thus  $\|\mathbf{Z}_l\|_0 = K$ ,  $l \in \{1, 2\}$ . We assume that  $\mathbf{Z}_1$  and  $\mathbf{Z}_2$  are uncorrelated with each other and with the common signal  $\Theta$ . This joint sparsity model (also known as JSM-2) for distributed CS was first introduced in [27].

Here, we stress that although the extension to arbitrary number of correlated sources is straightforward, the assumption of a distributed system with two sources simplifies the presentation throughout the paper. We assume, without loss of generality, that  $\sigma_{z_1}^2 = \sigma_{z_2}^2 \triangleq \sigma_z^2$  and  $\sigma_\theta^2 + \sigma_z^2 = 1$ . To measure the amount of correlation between sources, we define the *correlation ratio* as

$$\rho \triangleq \sigma_\theta^2 / \sigma_z^2. \quad (2)$$

Hence,  $\sigma_\theta^2 = \frac{\rho}{1+\rho}$  and  $\sigma_z^2 = \frac{1}{1+\rho}$ . Also, note that  $\rho \rightarrow \infty$  implies that the sources are highly correlated, whereas  $\rho \rightarrow 0$  means that they are highly uncorrelated. The remote sparse sources  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are measured by CS-based sensors, leading to measurement vectors  $\mathbf{Y}_1 \in \mathbb{R}^{M_1}$  and  $\mathbf{Y}_2 \in \mathbb{R}^{M_2}$  described as

$$\mathbf{Y}_l = \Phi_l \mathbf{X}_l + \mathbf{W}_l, \quad l \in \{1, 2\}, \quad \|\mathbf{X}_l\|_0 = K, \quad (3)$$

where  $\Phi_l \in \mathbb{R}^{M_l \times N}$  is a fixed sensing matrix of the  $l^{th}$  sensor, and  $\mathbf{W}_l \in \mathbb{R}^{M_l}$  is an additive measurement noise vector independent of other sources. Without loss of generality, we assume that  $M_1 = M_2 \triangleq M$ , and in the spirit of CS, we consider that  $K < M < N$ .

In order to reconstruct an unknown sparse source from noisy under-sampled measurement vectors in a distributed fashion, we define CS reconstruction distortion as

$$D_{cs} \triangleq \frac{1}{2K} \sum_{l=1}^2 \mathbb{E}[\|\mathbf{X}_l - \tilde{\mathbf{X}}_l\|_2^2], \quad (4)$$

where  $\tilde{\mathbf{X}}_l \in \mathbb{R}^N$  ( $l \in \{1, 2\}$ ) is an estimation vector of the sparse source  $\mathbf{X}_l$  from noisy CS measurements  $\mathbf{Y}_1$  and  $\mathbf{Y}_2$ . Here,  $\|\cdot\|_2$  denotes  $\ell_2$  norm. In this work, we focus on the Bayesian minimum mean square error (MMSE) framework for distributed reconstruction since we are interested in the MSE performance of random sparse vectors with known priors. Later in Section 4, it will be shown that the MMSE estimator is central in developing distributed quantizer design procedure which is the main focus of this paper. The following proposition provides an analytical expression for the MMSE estimator. We omit the proof (given in [28]) due to space limitation.

**Proposition 1** Consider the following assumptions: (i) the  $K$ -sparse sources  $\mathbf{X}_1, \mathbf{X}_2$  are correlated based on the model (1) with ratio  $\rho$ . (ii) The  $K$  elements of the support set are uniformly drawn from all  $\binom{N}{K}$  possibilities. (iii) The measurement noise vector is distributed as  $\mathbf{W}_l \sim \mathcal{N}(\mathbf{0}, \sigma_{w_l}^2 \mathbf{I}_M)$ ,  $l \in \{1, 2\}$ , which is uncorrelated with the CS measurements and sources. Then, the MMSE estimator of  $\mathbf{X}_l$  given the noisy CS measurement vector  $\mathbf{y} = [\mathbf{y}_1^\top \mathbf{y}_2^\top]^\top$  that minimizes  $D_{cs}$  in (4), is obtained by  $\tilde{\mathbf{x}}_l^*(\mathbf{y}) \triangleq \mathbb{E}[\mathbf{X}_l|\mathbf{y}]$  which has the following closed form expression

$$\tilde{\mathbf{x}}^*(\mathbf{y}) \triangleq [\tilde{\mathbf{x}}_1^*(\mathbf{y})^\top \tilde{\mathbf{x}}_2^*(\mathbf{y})^\top]^\top = \frac{\sum_{S \subset \Omega} \beta_S \cdot \tilde{\mathbf{x}}^*(\mathbf{y}, S)}{\sum_{S \subset \Omega} \beta_S}, \quad (5)$$

where  $\tilde{\mathbf{x}}^*(\mathbf{y}, S) \triangleq \mathbb{E}[\mathbf{X}|\mathbf{y}, S]$  and within its support

$$\tilde{\mathbf{x}}^*(\mathbf{y}, S) = \begin{bmatrix} \mathbf{I}_K & \mathbf{I}_K & \mathbf{0}_K \\ \mathbf{I}_K & \mathbf{0}_K & \mathbf{I}_K \end{bmatrix} \mathbf{C}^\top \mathbf{D}^{-1} \mathbf{y}, \quad (6)$$

and otherwise zeros. Further,

$$\beta_S = e^{\frac{1}{2}(\mathbf{y}^\top (\mathbf{N}^{-1} \mathbf{F}^\top (\mathbf{E}^{-1} + \mathbf{F}^\top \mathbf{N}^{-1} \mathbf{F})^{-1} \mathbf{F} \mathbf{N}^{-1}) \mathbf{y} - \ln \det(\mathbf{E}^{-1} + \mathbf{F}^\top \mathbf{N}^{-1} \mathbf{F}))}$$

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} \frac{\rho}{1+\rho} \Phi_{1,S} & \frac{1}{1+\rho} \Phi_{1,S} & \mathbf{0}_{M \times K} \\ \frac{\rho}{1+\rho} \Phi_{2,S} & \mathbf{0}_{M \times K} & \frac{1}{1+\rho} \Phi_{2,S} \end{bmatrix}, \\ \mathbf{D} &= \begin{bmatrix} \Phi_{1,S} \Phi_{1,S}^\top + \sigma_{w_1}^2 \mathbf{I}_M & \frac{\rho}{1+\rho} \Phi_{1,S} \Phi_{2,S}^\top \\ \frac{\rho}{1+\rho} \Phi_{2,S} \Phi_{1,S}^\top & \Phi_{2,S} \Phi_{2,S}^\top + \sigma_{w_2}^2 \mathbf{I}_M \end{bmatrix}, \\ \mathbf{N} &= \begin{bmatrix} \sigma_{w_1}^2 \mathbf{I}_M & \mathbf{0}_M \\ \mathbf{0}_M & \sigma_{w_2}^2 \mathbf{I}_M \end{bmatrix}, \\ \mathbf{E} &= \begin{bmatrix} \frac{\rho}{1+\rho} \mathbf{I}_K & \mathbf{0}_K & \mathbf{0}_K \\ \mathbf{0}_K & \frac{1}{1+\rho} \mathbf{I}_K & \mathbf{0}_K \\ \mathbf{0}_K & \mathbf{0}_K & \frac{1}{1+\rho} \mathbf{I}_K \end{bmatrix}, \\ \mathbf{F} &= \begin{bmatrix} \Phi_{1,S} & \Phi_{1,S} & \mathbf{0}_{M \times K} \\ \Phi_{2,S} & \mathbf{0}_{M \times K} & \Phi_{2,S} \end{bmatrix}, \end{aligned}$$

and  $\Phi_{l,S}$  ( $l \in \{1, 2\}$ ) is formed by choosing the columns of  $\Phi_l$  indexed by the elements of a possible support set  $\mathbf{s}$ . Also,  $\Omega$  denotes the set of all possible supports.

Note that the MSE of the MMSE estimator (5) can be empirically computed using Monte-Carlo simulations. It should be also mentioned that as  $N$  increases, the implementation of the MMSE estimator may not be feasible since the summation in (5) is taken over all possible supports. Here, we do not focus on approximation methods for the MMSE estimator because it is beyond the scope of the paper. Interested readers are referred to [29] for the clues regarding the implementation of approximated MMSE estimator.

In the next section, we consider the main scenario of this work which is distributed quantization of CS measurements.

### 3. SYSTEM DESCRIPTION

In this section, we give an account for the basic assumptions and models made about the studied distributed coding system.

We consider that VQ encoders at terminals 1 and 2 (without any collaboration) be fed by the noisy CS measurements  $\mathbf{Y}_1$  and  $\mathbf{Y}_2$  (under model (3)), respectively. The encoder mapping  $\mathbf{E}_l$  ( $l \in \{1, 2\}$ ) encodes  $\mathbf{Y}_l$  to a transmission index  $i_l$ , i.e.,  $\mathbf{E}_l : \mathbb{R}^M \rightarrow \mathcal{I}_l$  where  $i_l \in \mathcal{I}_l$ , and  $\mathcal{I}_l$  is a finite index set defined as  $\mathcal{I}_l \triangleq \{0, 1, \dots, 2^{R_l} - 1\}$  with  $|\mathcal{I}_l| \triangleq \mathfrak{R}_l = 2^{R_l}$ . Here,  $R_l$  is the assigned quantization rate for each encoder in bits/vector. We fix the total quantization rate at  $R_1 + R_2 \triangleq R$  bits/vector. Denoting the encoded index by  $I_l$ , the encoders are specified by the regions  $\{\mathcal{R}_{i_l}\}_{i_l=0}^{\mathfrak{R}_l-1}$  such that when  $\mathbf{Y}_l \in \mathcal{R}_{i_l}$ , the encoder outputs  $\mathbf{E}_l(\mathbf{Y}_l) = i_l \in \mathcal{I}_l$ .

Now, we consider the VQ decoder at a FC which uses both indexes  $i_1 \in \mathcal{I}_1$  and  $i_2 \in \mathcal{I}_2$  in order to make the estimate of the sparse source vector, denoted by  $\hat{\mathbf{X}}_l \in \mathbb{R}^N$ . Given the received indexes  $i_1$  and  $i_2$ , the decoder is a mapping  $\mathbf{D}_l : \mathcal{I}_1 \times \mathcal{I}_2 \rightarrow \mathcal{C}_l$ , where  $\mathcal{C}_l$ , with  $|\mathcal{C}_l| = 2^{R_1+R_2}$ , is a finite discrete codebook set containing all reproduction codevectors. The decoder's functionality is described by a look-up table;  $(I_1 = i_1, I_2 = i_2) \Rightarrow (\hat{\mathbf{X}}_1 = \mathbf{D}_1(i_1, i_2), \hat{\mathbf{X}}_2 = \mathbf{D}_2(i_1, i_2))$ .

We assess the end-to-end performance of our studied system by

$$D \triangleq \frac{1}{2K} \sum_{l=1}^2 \mathbb{E}[\|\mathbf{X}_l - \hat{\mathbf{X}}_l\|_2^2]. \quad (7)$$

Note that the MSE depends on CS reconstruction distortion and quantization error. Our goal is to design robust VQ encoder-decoder pairs against all these kinds of error.

### 4. DISTRIBUTED QUANTIZER DESIGN AND ANALYSIS

We note that the joint design of the encoder and decoder mappings,  $\mathbf{E}_l$  and  $\mathbf{D}_l$  ( $l \in \{1, 2\}$ ), is generally not an easy task. Therefore, we optimize each mapping (with respect to minimizing the MSE in (8)) by fixing the other mappings. The resulting mappings fulfil necessary conditions for optimality, and can be implemented in practice. Since the system is symmetric, we only show the optimization method for the first encoder and then the decoder.

Keeping the mappings  $\mathbf{E}_2$ ,  $\mathbf{D}_1$  and  $\mathbf{D}_2$  fixed, we have

$$\begin{aligned} D &= \frac{1}{2K} \sum_{i_1=0}^{\mathfrak{R}_1-1} \int_{\mathbf{y}_1 \in \mathcal{R}_{i_1}} \underbrace{\mathbb{E}[\|\mathbf{X}_1 - \mathbf{D}_1(I_1, I_2)\|_2^2 | \mathbf{y}_1, i_1]}_{\triangleq D_1(\mathbf{y}_1, i_1)} \\ &\quad + \underbrace{\mathbb{E}[\|\mathbf{X}_2 - \mathbf{D}_2(I_1, I_2)\|_2^2 | \mathbf{y}_1, i_1]}_{\triangleq D_2(\mathbf{y}_1, i_1)} p(\mathbf{y}_1) d\mathbf{y}_1, \end{aligned} \quad (8)$$

$$\begin{aligned}
D_1(\mathbf{y}_1, i_1) &\triangleq \mathbb{E}[\|\mathbf{X}_1 - \mathbf{D}_1(I_1, I_2)\|_2^2 | \mathbf{y}_1, i_1] \stackrel{(a)}{=} \mathbb{E}[\|\mathbf{X}_1\|_2^2 | \mathbf{y}_1] + \sum_{i_2=0}^{\mathfrak{R}_2-1} P(i_2 | \mathbf{y}_1) \left[ \|\mathbf{D}_1(i_1, i_2)\|_2^2 - 2\mathbb{E}[\mathbf{X}_1^\top | \mathbf{y}_1, i_2] \mathbf{D}_1(i_1, i_2) \right] \\
&\stackrel{(b)}{=} \mathbb{E}[\|\mathbf{X}_1\|_2^2 | \mathbf{y}_1] + \sum_{i_2=0}^{\mathfrak{R}_2-1} \left[ \int_{\mathbf{y}_2 \in \mathcal{R}^{i_2}} \left( \|\mathbf{D}_1(i_1, i_2)\|_2^2 - 2\mathbb{E}[\mathbf{X}_1^\top | \mathbf{y}_1, \mathbf{y}_2] \mathbf{D}_1(i_1, i_2) \right) p(\mathbf{y}_2 | \mathbf{y}_1) d\mathbf{y}_2 \right] \\
i_1^* &= \arg \min_{i_1 \in \mathcal{I}_1} \left\{ \sum_{i_2=0}^{\mathfrak{R}_2-1} \int_{\mathcal{R}^{i_2}} \left[ \|\mathbf{D}(i_1, i_2)\|_2^2 - 2\tilde{\mathbf{x}}^*(\mathbf{y}_1, \mathbf{y}_2)^\top \mathbf{D}(i_1, i_2) \right] p(\mathbf{y}_2 | \mathbf{y}_1) d\mathbf{y}_2 \right\},
\end{aligned} \tag{10}$$


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$$i_1^* = \arg \min_{i_1 \in \mathcal{I}_1} \left\{ \sum_{i_2=0}^{\mathfrak{R}_2-1} \int_{\mathcal{R}^{i_2}} \left[ \|\mathbf{D}(i_1, i_2)\|_2^2 - 2\tilde{\mathbf{x}}^*(\mathbf{y}_1, \mathbf{y}_2)^\top \mathbf{D}(i_1, i_2) \right] p(\mathbf{y}_2 | \mathbf{y}_1) d\mathbf{y}_2 \right\}, \tag{11}$$


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where  $p(\mathbf{y}_1)$  is the  $M$ -fold probability density function (pdf) of the measurement vector  $\mathbf{Y}_1$ . Since  $p(\mathbf{y}_1)$  is a non-negative value, in order to optimize the mapping  $\mathbf{E}_1$  in the sense of minimizing  $D$ , it suffices to minimize the expression inside the braces in (8). Thus, the optimized encoding index  $i_1^*$  is obtained by

$$i_1^* = \arg \min_{i_1 \in \mathcal{I}_1} \{D_1(\mathbf{y}_1, i_1) + D_2(\mathbf{y}_1, i_1)\}. \tag{9}$$

Now,  $D_1(\mathbf{y}_1, i_1)$  can be rewritten as (10) on top of the page, where (a) follows by expanding the conditional expectation and the fact that  $\mathbf{X}_1$  and  $\mathbf{D}_1(I_1, I_2)$  are independent conditioned on  $\mathbf{y}_1, i_1, i_2$ . Further, (b) follows from marginalization of the expression inside the brackets in (a) over  $i_2$  and  $\mathbf{y}_2$ . In a same fashion,  $D_2(\mathbf{y}_1, i_1)$  can be parameterized similar to (10) with the only difference that  $\mathbf{X}_1$  and  $\mathbf{D}_1(i_1, i_2)$  are replaced with  $\mathbf{X}_2$  and  $\mathbf{D}_2(i_1, i_2)$ , respectively. Following (9) and (10), the MSE-minimizing encoding index  $i_1^*$  is given by (11), where  $\tilde{\mathbf{x}}^*(\mathbf{y}_1, \mathbf{y}_2) \triangleq [\tilde{\mathbf{x}}_1^*(\mathbf{y}_1, \mathbf{y}_2)^\top \tilde{\mathbf{x}}_2^*(\mathbf{y}_1, \mathbf{y}_2)^\top]^\top$  and  $\mathbf{D}(i_1, i_2) \triangleq [\mathbf{D}_1(i_1, i_2)^\top \mathbf{D}_2(i_1, i_2)^\top]^\top$ . Here, the codevectors  $\mathbf{D}_l(i_1, i_2)$ , ( $l \in \{1, 2\}$ ), are given, and the vector  $\tilde{\mathbf{x}}_l^*(\mathbf{y}_1, \mathbf{y}_2)$  denotes the MMSE estimators derived in Proposition 1. It should be mentioned that although the observation at terminal 2,  $\mathbf{y}_2$ , appears in the formulation of the optimized encoder at terminal 1, it is finally integrated out.

Assuming all encoders are fixed, it can be shown that the MSE-minimizing decoder is given by

$$D_l^*(i_1, i_2) = \mathbb{E}[\mathbf{X}_l | i_1, i_2] = \frac{\int_{\mathcal{R}^{i_1}} \int_{\mathcal{R}^{i_2}} \tilde{\mathbf{x}}_l^*(\mathbf{y}_1, \mathbf{y}_2) p(\mathbf{y}_1, \mathbf{y}_2) d\mathbf{y}_1 d\mathbf{y}_2}{\int_{\mathcal{R}^{i_1}} \int_{\mathcal{R}^{i_2}} p(\mathbf{y}_1, \mathbf{y}_2) d\mathbf{y}_1 d\mathbf{y}_2}, \tag{12}$$

where the second equality can be shown by marginalizing of the conditional expectation over  $\mathbf{Y}_1$  and  $\mathbf{Y}_2$  and using the Bayes' rule.

When there is no correlation between sources ( $\rho \rightarrow 0$ ), then it can be shown that the optimized encoder (11) and the optimized decoder in (12) boil down to the optimized encoder and decoder in the point-to-point source coding of CS measurements, cf. [23].

We emphasize that we do not assume any sparse structure on the reconstructed vectors at the receiving-ends in contrast to conventional  $\ell_1$ -norm reconstruction methods. The reason is due to the fact that we are interested in the final reconstruction MSE, where considering any kind of sparse structure might degrade the performance.

Now, we analyze the end-to-end MSE for our studied distributed system. Recall the MMSE estimation of the correlated sources  $\tilde{\mathbf{X}}_l^* \triangleq \mathbb{E}[\mathbf{X}_l | \mathbf{Y}_1, \mathbf{Y}_2]$ ,  $l \in \{1, 2\}$ , then we rewrite the end-to-end MSE as

$$D \stackrel{(a)}{=} \frac{1}{2K} \sum_{l=1}^2 \mathbb{E}[\|\mathbf{X}_l - \tilde{\mathbf{X}}_l^*\|_2^2] + \frac{1}{2K} \sum_{l=1}^2 \mathbb{E}[\|\tilde{\mathbf{X}}_l^* - \hat{\mathbf{X}}_l\|_2^2] \triangleq D_{cs} + D_q, \tag{13}$$

where (a) can be shown by the definition of the MMSE estimator  $\tilde{\mathbf{X}}_l^*$  and by using the Markov property  $\mathbf{X}_l \rightarrow (I_1, I_2) \rightarrow \hat{\mathbf{X}}_l$ ,  $l \in \{1, 2\}$ .

Interestingly, (13) implies that, without loss of optimality, the end-to-end MSE, denoted by  $D$ , can be summed up as CS reconstruction MSE (of the MMSE estimator), denoted by  $D_{cs}$ , and quantization MSE, denoted by  $D_q$ . This property together with information theoretic results in [5] can be used to develop a lower-bound on  $D$  provided by the following theorem. We omit the proof (given in [28]) due to lack of space.

**Theorem 2** Consider the assumptions given in Proposition 1. Let the total quantization rate be  $R = R_1 + R_2$  bits/vector where  $R_l$  is the assigned quantization rate at terminal  $l \in \{1, 2\}$ , then the asymptotic (in quantization rate) end-to-end MSE (8) is lower-bounded as

$$D \geq \max \{D_q^{(lb)}, D_{cs}\}, \tag{14}$$

where  $D_q^{(lb)} =$

$$\sqrt{\left(1 - \frac{\rho^2}{(1+\rho)^2}\right) 2^{\frac{-2(R - \log_2(\frac{N}{K}))}{K}} + \frac{\rho^2}{(1+\rho)^2} 2^{\frac{-4(R - \log_2(\frac{N}{K}))}{K}}}. \tag{15}$$

and  $D_{cs}$  is calculated by (4) corresponding to the distortion due to MMSE estimation of correlated sparse sources  $\mathbf{X}_1$  and  $\mathbf{X}_2$  from noisy CS measurements derived in (5) of Proposition 1.

When CS measurements are noisy, i.e.  $\sigma_{w_1}^2, \sigma_{w_2}^2 \neq 0$ , it can be verified as the quantization rate  $R$  increases, the end-to-end MSE saturates to  $D_{cs}$  since  $D_q^{(lb)}$  decays exponentially, but  $D_{cs}$  becomes constant by quantization rate. The source correlation ratio  $\rho$  also plays an important role on the level of the lower-bound (14). By taking the first derivative of  $D_q^{(lb)}$  in (15) with respect to  $\rho$ , it can be verified that the derivative is always negative which means as the source correlation  $\rho$  increases, the lower-bound decreases.

## 5. PRACTICAL DESIGN ALGORITHM

The necessary optimal conditions in (11) (and its equivalence  $i_1^*$ ) and (12) can be combined in an *alternate-iterate* training algorithm in order to design optimized encoder-decoder pairs for distributed quantization of CS measurements which converges to locally optimum encoder regions and decoder codevectors. A natural order to optimize the mappings is: 1) the first encoder, 2) the first decoder, 3) the second encoder and 4) the second decoder.

To implement the algorithm, we make some modifications. The integral in (11) (for calculating the optimized encoder) cannot be solved in closed form in general. Hence, we modify the integral, and compute it numerically. The integral in (11) can be approximated as

$$\|\mathbf{D}(i_1, i_2)\|_2^2 P(i_2 | \tilde{\mathbf{y}}_1) - 2P(i_2 | \tilde{\mathbf{y}}_1) \mathbb{E}[\mathbf{X}^\top | \tilde{\mathbf{y}}_1, i_2] \mathbf{D}(i_1, i_2) \tag{16}$$

where we have only approximated  $\mathbf{y}_1$  by its scalar-quantized representation, denoted by  $\tilde{\mathbf{y}}_1$ , using  $r_y$ -bit nearest-neighbor coding (using, e.g., LBG algorithm [30]). Hence,  $\mathbf{y}_1$  is discretized, and

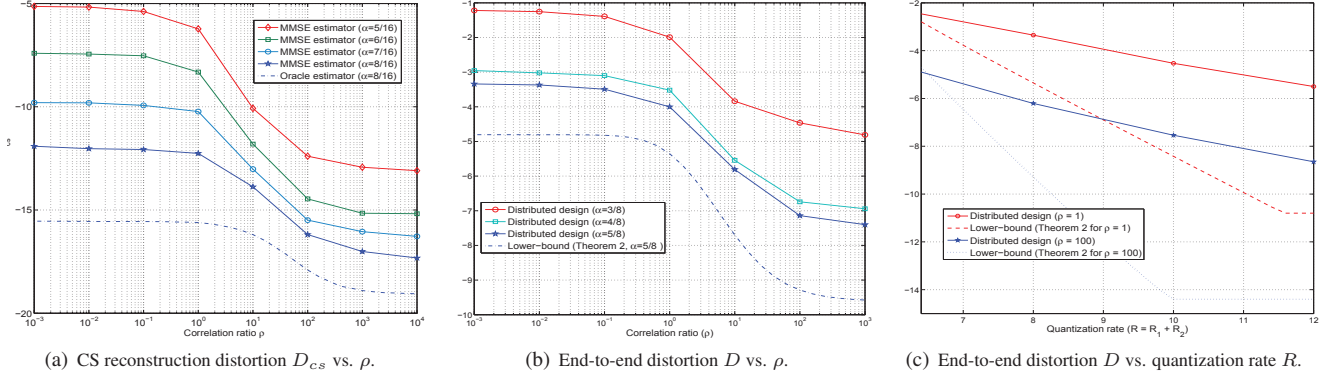


Fig. 1. Simulation results for the distributed design methods are compared with corresponding lower-bounds.

$P(i_2|\tilde{\mathbf{y}}_1) \triangleq \Pr\{I_2 = i_2 | \tilde{\mathbf{Y}}_1 = \tilde{\mathbf{y}}_1\}$  indicates an element of a transition probability matrix whose transitions can be numerically computed. In order to evaluate the conditional mean  $\mathbb{E}[\mathbf{X}^\top | \tilde{\mathbf{y}}_1, i_2]$  in (16), we generate samples of  $\mathbf{X}_1$  and  $\mathbf{X}_2$ , and then take average over those samples that have resulted in the quantized value  $\tilde{\mathbf{y}}_1$  and the quantization index  $i_2$ . Using these modifications, the encoder computational complexity grows at most like  $\mathcal{O}(2^{R_1+R_2})$ .

Moreover, the calculation of the codevectors  $\mathbf{D}_l(i_1, i_2)$  derived in (12) requires massive integrations of highly non-linear functions. Therefore, we calculate  $\mathbb{E}[\mathbf{X}_l | i_1, i_2]$  empirically by generating Monte-Carlo samples of  $\mathbf{X}_l$ , and then take average over those samples which have led to the quantized indexes  $i_1$  and  $i_2$ .

## 6. EXPERIMENTS

We assess the performance using CS reconstruction MSE,  $D_{cs}$ , and end-to-end MSE,  $D$ , characterized in (4) and (8), respectively. The correlated sources with correlation ratio  $\rho$  (defined in (2)) are randomly generated according to the models described in Section 2. The sensing matrices  $\Phi_1$  and  $\Phi_2$  are produced by choosing the first (indexed from the first row downwards) and the last (indexed from the last row upwards)  $M$  rows of a  $N \times N$  discrete cosine transform (DCT) matrix. Then, the columns of the resulting matrices are normalized to unit-norm. To measure the level of under-sampling, we define the *measurement rate*  $0 < \alpha \leq 1$  as  $\alpha \triangleq M/N$ . We define signal-to-measurement noise ratio (SMNR) at terminal  $l \in \{1, 2\}$  as  $\text{SMNR}_l \triangleq \mathbb{E}[\|\mathbf{X}_l\|_2^2] / \mathbb{E}[\|\mathbf{W}_l\|_2^2] = K / (M\sigma_{w_l}^2)$ . All simulations are performed by generating  $3 \times 10^5$  realizations of the source vectors.

In our first experiment, we study the impact of source correlation  $\rho$  and measurement rate  $\alpha$  on the CS reconstruction MSE,  $D_{cs}$ . We consider  $(N = 16, K = 2, \text{SMNR}_l = 10 \text{ dB})$ , and empirically compute  $D_{cs}$  for the MMSE estimator derived in (5) of Proposition 1. The results are illustrated in Figure 1(a) as a function of  $\rho$  for measurement rates  $\alpha = 5/16, \dots, 8/16$ . The empirical *oracle estimator* lower-bound corresponding to the measurement rate  $\alpha = 8/16$  is also demonstrated. Note that the *ideal* oracle estimator is calculated from (6) given the *a priori* known support for each source realization. From Figure 1(a), we observe that increasing number of CS measurements improve the performance which is expected since the sources are estimated from more amount of information. Another point is that  $D_{cs}$  varies significantly by changing the correlation ratio  $\rho$  which is also reflected from the oracle lower-bound. This is due to the fact that at low correlation, the measurement vectors become uncorrelated, therefore there is no gain obtained by, e.g., estimation of  $\mathbf{X}_1$  from observations at the second terminal, i.e.,  $\mathbf{y}_2$ . On the other hand, when the sources are highly correlated, the estimation procedure tends to estimating a single source  $\Theta$  from  $2M$  observa-

tions, i.e.,  $\mathbf{y}_1$  and  $\mathbf{y}_2$ .

In our second experiment, we demonstrate the effect of  $\rho$  and  $\alpha$  on the end-to-end performance  $D$ . We use  $(N = 8, K = 2, R = R_1 + R_2 = 8 \text{ bits/vector with } R_1 = R_2)$ , and assume clean measurements. Further, The vectors  $\mathbf{y}_1$  and  $\mathbf{y}_2$  are pre-quantized using  $r_y = 3$  bits per measurement entry. We vary the correlation ratio from very low  $\rho = 10^{-3}$  to very high values  $\rho = 10^3$ , and compare the simulation results with the lower-bound derived in (14) of Theorem 2. The results are shown in Figure 1(b) for measurement rates  $\alpha = \frac{3}{8}, \frac{4}{8}, \frac{5}{8}$ . As would be expected, at a fixed quantization rate  $R$  and correlation ratio  $\rho$ , increasing  $\alpha$  improves the performance since the sources are reconstructed from larger sets of information. Hence, the end-to-end MSE decreases, and the curves approach the lower-bound. The correlation is a useful factor to reduce the end-to-end distortion which is also reflected from the lower-bound. This behavior can be interpreted as follows. When the sources are fully correlated,  $\rho \rightarrow \infty$ , the two sources can be viewed as a single source, and the FC is able to reconstruct the source jointly from two sets of received quantized indexes. Hence, the performance is maximized. On the other hand, when the sources are uncorrelated,  $\rho \rightarrow 0$ , the FC reconstructs the sources from two sets of independent quantized indexes. Thus, there is no gain in joint decoding at the FC.

Now, we investigate how the performance varies by quantization rate. We use the simulation parameter set  $(N = 8, K = 2, \alpha = 5/8, \text{SMNR}_1 = \text{SMNR}_2 = 10) \text{ dB}$ . In Figure 1(c), we illustrate the end-to-end MSE of the proposed design method as a function of total quantization rate  $R = R_1 + R_2$  (with  $R_1 = R_2$ ) for two values of correlation ratios:  $\rho = 1$  (low-correlated sources) and  $\rho = 100$  (high-correlated sources). The simulation curves are compared with the lower-bound in (14) corresponding to  $\rho = 1, 100$ . From Figure 1(c), we observe that the performance improves by increasing quantization rate. Moreover, increasing correlation between sources reduces the MSE as observed from other experiments too. We also note that as quantization rate increases, all the curves converge to their respective MSE floors, specified by  $D_{cs}$  that can be seen from the lower-bounds in Figure 1(c).

## 7. CONCLUSIONS

We studied the design and analysis of the distributed vector quantization of CS measurements. We derived necessary conditions for optimality of encoder-decoder pairs by minimizing end-to-end MSE. We analyzed the MSE and proved that it is the sum of CS reconstruction MSE (of MMSE estimator) and quantization MSE. This result helped us to derive a lower-bound on the end-to-end MSE. Simulations revealed that correlation between sources, besides compression resources e.g. measurement and quantization rates, is an effective factor on CS reconstruction MSE as well as quantization MSE.

## 8. REFERENCES

- [1] D. Slepian and J. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. 19, no. 4, pp. 471–480, 1973.
- [2] R. Zamir and T. Berger, "Multiterminal source coding with high resolution," *IEEE Trans. Inf. Theory*, vol. 45, no. 1, pp. 106–117, 1999.
- [3] Z. Xiong, A. Liveris, and S. Cheng, "Distributed source coding for sensor networks," *IEEE Sig. Proc. Magazine*, vol. 21, no. 5, pp. 80–94, 2004.
- [4] Y. Oohama, "Rate-distortion theory for gaussian multiterminal source coding systems with several side informations at the decoder," *IEEE Trans. Inf. Theory*, vol. 51, no. 7, pp. 2577–2593, 2005.
- [5] A. Wagner, S. Tavildar, and P. Viswanath, "Rate region of the quadratic Gaussian two-encoder source-coding problem," *IEEE Trans. Inf. Theory*, vol. 54, no. 5, pp. 1938–1961, 2008.
- [6] N. Wernersson, J. Karlsson, and M. Skoglund, "Distributed quantization over noisy channels," *IEEE Trans. Commun.*, vol. 57, no. 6, pp. 1693–1700, 2009.
- [7] J. Sun, V. Misra, and V. Goyal, "Distributed functional scalar quantization simplified," *IEEE Trans. Sig. Proc.*, vol. 61, no. 14, pp. 3495–3508, 2013.
- [8] E. Candes and M. Wakin, "An introduction to compressive sampling," *IEEE Sig. Proc. Magazine*, vol. 25, no. 2, pp. 21–30, Mar. 2008.
- [9] E. Candes, J. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Comm. Pure Appl. Math.*, vol. 59, no. 8, pp. 1207–1223, 2006.
- [10] C. Güntürk, M. Lammers, A. Powell, R. Saab, and O. Yilmaz, "Sigma delta quantization for compressed sensing," in *Annual Conf. Inf. Sciences and Systems*, March 2010, pp. 1–6.
- [11] A. Zymnis, S. Boyd, and E. Candes, "Compressed sensing with quantized measurements," *IEEE Sig. Proc. Lett.*, vol. 17, no. 2, pp. 149–152, Feb. 2010.
- [12] W. Dai and O. Milenkovic, "Information theoretical and algorithmic approaches to quantized compressive sensing," *IEEE Trans. Commun.*, vol. 59, no. 7, pp. 1857–1866, Jul. 2011.
- [13] L. Jacques, D. Hammond, and J. Fadili, "Dequantizing compressed sensing: When oversampling and non-Gaussian constraints combine," *IEEE Trans. Inf. Theory*, vol. 57, no. 1, pp. 559–571, Jan. 2011.
- [14] M. Yan, Y. Yang, and S. Osher, "Robust 1-bit compressive sensing using adaptive outlier pursuit," *IEEE Trans. Sig. Proc.*, vol. 60, no. 7, pp. 3868–3875, Jul. 2012.
- [15] U. S. Kamilov, V. K. Goyal, and S. Rangan, "Message-passing de-quantization with applications to compressed sensing," *IEEE Trans. Sig. Proc.*, vol. 60, no. 12, pp. 6270–6281, Dec. 2012.
- [16] J. Sun and V. Goyal, "Optimal quantization of random measurements in compressed sensing," in *IEEE Int. Symp. Inf. Theory*, Jul. 2009, pp. 6–10.
- [17] A. Shirazinia, S. Chatterjee, and M. Skoglund, "Performance bounds for vector quantized compressive sensing," in *Int. Symp. Inf. Theory and App.*, Oct. 2012, pp. 289–293.
- [18] P. Boufounos, "Universal rate-efficient scalar quantization," *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1861–1872, March 2012.
- [19] U. Kamilov, V. Goyal, and S. Rangan, "Optimal quantization for compressive sensing under message passing reconstruction," in *IEEE Int. Symp. Inf. Theory*, 31 2011–Aug. 5 2011, pp. 459–463.
- [20] A. Shirazinia, S. Chatterjee, and M. Skoglund, "Analysis-by-synthesis quantization for compressed sensing measurements," *IEEE Trans. Sig. Proc.*, vol. 61, no. 22, pp. 5789–5800, 2013.
- [21] V. Goyal, A. Fletcher, and S. Rangan, "Compressive sampling and lossy compression," *IEEE Sig. Proc. Mag.*, vol. 25, no. 2, pp. 48–56, March 2008.
- [22] J. Laska and R. Baraniuk, "Regime change: Bit-depth versus measurement-rate in compressive sensing," *IEEE Trans. Sig. Proc.*, vol. 60, no. 7, pp. 3496–3505, Jul. 2012.
- [23] A. Shirazinia, S. Chatterjee, and M. Skoglund, "Channel-optimized vector quantizer design for compressed sensing measurements," in *IEEE Int. Conf. Acoust. Speech, and Sig. Proc.*, 2013, pp. 4648–4652.
- [24] W. Bajwa, J. Haupt, A. Sayeed, and R. Nowak, "Joint source-channel communication for distributed estimation in sensor networks," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3629–3653, 2007.
- [25] S. Feizi, M. Medard, and M. Effros, "Compressive sensing over networks," in *Annu. Allerton Conf. Commun., Control and Computing*, 2010, pp. 1129–1136.
- [26] S. Feizi-Khankandi and M. Médard, "A power efficient sensing/communication scheme: Joint source-channel-network coding by using compressive sensing," *CoRR*, vol. abs/1110.0428, 2011.
- [27] D. Baron, M. F. Duarte, M. B. Wakin, S. Sarvotham, and R. G. Baraniuk, "Distributed compressive sensing," *CoRR*, vol. abs/0901.3403, 2009.
- [28] A. Shirazinia, S. Chatterjee, and M. Skoglund, "Distributed joint source-channel coding for compressed sensing," *IEEE Trans. Commun.*, submitted.
- [29] M. Elad and I. Yavneh, "A plurality of sparse representations is better than the sparsest one alone," *IEEE Trans. Inf. Theory*, vol. 55, no. 10, pp. 4701–4714, Oct. 2009.
- [30] Y. Linde, A. Buzo, and R. Gray, "An algorithm for vector quantizer design," *IEEE Trans. Commun.*, vol. 28, no. 1, pp. 84–95, Jan 1980.