ORDINAL POTENTIAL FUNCTIONS FOR NETWORK SELECTION IN HETEROGENEOUS WIRELESS NETWORKS

Liang Ze Wong, Tony Q.S. Quek*†, and Michael Padilla†*

*Institute for Infocomm Research, A*STAR, 1 Fusionopolis Way, #21-01 Connexis, Singapore 138632
†Singapore University of Technology and Design, 20 Dover Drive, Singapore 138682

ABSTRACT

We consider the distributed allocation of spectrum in a heterogeneous wireless network. We model this as a non-cooperative game, where multiple players decide which channels to transmit on. When co-channel interference between players is symmetric and channel utilities are additive, we define a generalized ordinal potential function for the game which is motivated by the energy function in Ising models. This guarantees convergence of best-response dynamics, and does not depend on the level of interference between players or on the price imposed on each channel.

***Index Terms*—** Heterogeneous networks; network selection; ordinal potential games; congestion games

1. INTRODUCTION

In heterogeneous wireless networks, multiple radio access technologies (RATs) coexist in the same network and are collectively able to provide better coverage, quality of service (QoS), and mobility support than if they were operating in isolation. To capitalize on the benefits of heterogeneous networks one needs to be able to transmit on multiple RATs simultaneously. This capability is found in multi-mode terminals (MMTs), which are able to access multiple bands simultaneously. We refer to these bands abstractly as "resource blocks (RBs)"[1].

MMTs interfere with other MMTs sharing the same RBs, causing degradation in the QoS. The utility of an RB to an MMT is modelled as a function that decreases as the congestion increases. We formulate this as a non-cooperative game with partial information, where each MMT acts in a self-interested manner in trying to maximize its own utility. We show that best-response updates can be expressed as simple threshold policies in the RB selection game. When congestion between MMTs is symmetric, we define a generalized ordinal potential function for the RB selection game. We finally show that the convergence of best-response dynamics to a pure strategy Nash equilibrium is independent of any (fixed) channel prices that may be imposed on each RB, allowing network managers to optimize the channel price without worrying about convergence issues.

An analysis of the interactions between users on a shared wireless channel with partial information has been carried out in [2]. The paper gives a detailed characterization of the equilibria achievable when 2 users share a channel, knowing only their signal-to-noise ratio. In this paper, we expand the scope of consideration to networks with larger numbers of users and channels. Congestion games [3] and its generalizations (such as graphical congestion games [4]) have been widely used to model such scenarios. Instead of approximating our game using a congestion game or exact potential game, as in [5], we show that the RB selection game restricted to one channel is an ordinal potential game, which further generalizes congestion games and graphical congestion games, and can be interpreted as a graphical congestion game on a weighted complete graph. Ordinal potential games have been used to prove convergence in wireless collision channels [6]. In that paper, a rate-alignment condition was required for the derivation of the ordinal potential function, but their simulations suggested that this condition was not necessary for convergence. Under slightly different conditions, we derive a different ordinal potential function which does not require the rate-alignment condition. We also consider multiple channels, and show that the sum of each channel's ordinal potential functions is a generalized ordinal potential function, thus giving our game the Finite Improvement Property [7] and guaranteeing convergence.

2. SYSTEM MODEL

We model our system as a non-cooperative game, which we call the RB selection game. The players are the MMTs in the network and they seek to maximize their individual utilities by transmitting on the RBs in the network. In our RB selection game we have a set $\mathcal{K}$ of K MMTs that can transmit on a set $\mathcal{R}$ of R RBs. Each MMT can either transmit or not transmit on RB r, indicated by

$$p_k^r := \begin{cases} 1 & \text{if MMT } k \text{ transmits on RB } r \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The action of MMT k is the binary vector

$$\mathbf{p}_k = (p_k^1, \ldots, p_k^R) \in \mathcal{P}_k = \{0,1\}^R, \tag{2}$$

while the combined actions of the other MMTs are denoted $\mathbf{p}_{-k}$. The combined action of all MMTs is

$$\mathbf{p} = (\mathbf{p}_k, \mathbf{p}_{-k}) = (\mathbf{p}_1, \ldots, \mathbf{p}_K) \in \mathcal{P} = \{0, 1\}^{R \times K}. \quad (3)$$

In a game with full-information, each player is able to observe the actions taken by all other players. The RB selection game is a partial-information game, where each MMT does not know the actions of all other MMTs. The only information that MMT k has regarding $\mathbf{p}_{-k} = (\mathbf{p}^1_{-k}, \ldots, \mathbf{p}^R_{-k})$ is obtained via a congestion function

$$\mathbf{c}_k(\mathbf{p}_{-k}) = \Big(c^1_k(\mathbf{p}^1_{-k}), \ldots, c^R_k(\mathbf{p}^R_{-k})\Big). \quad (4)$$

Each c^r_k reflects the amount of congestion on RB r faced by MMT k if it were to transmit on RB r, and depends only on the other transmissions that occur on RB r (i.e. we assume no cross-channel interference). The value of c^r_k is given by

$$\begin{aligned} c^r_k(\mathbf{p}^r) &= A^r_k \cdot \mathbf{p}^r \\ &= \sum_{l \in \mathcal{K}} A^r_{kl} p^r_l \end{aligned} \quad (5)$$

where A^r is a symmetric $K \times K$ matrix with entries $A^r_{kl} \geq 0$ and zeros along the diagonal, and $\mathbf{p}^r = (p^r_1, \ldots, p^r_K)$. The matrix A^r specifies how much interference MMTs cause each other on RB r. The symmetry of A^r reflects our assumption that interference between MMTs is symmetric.

Given c^r_k, MMT k can compute the utility it obtains from transmitting on RB r via the utility function

$$u^r_k : \mathbb{R} \to \mathbb{R}. \quad (6)$$

We model this channel utility function as a decreasing function i.e. for all $x < y \in \mathbb{R}$

$$u^r_k(x) > u^r_k(y).$$

Here, x and y represent different levels of congestion. The decreasing nature of u^r_k reflects the condition that as congestion levels increase, the utility of an RB decreases. The total utility experienced by MMT k is then the sum of the utilities due to each of the RBs it transmits on i.e.

$$u_k(\mathbf{p}_k, \mathbf{p}_{-k}) := \sum_{r : p^r_k = 1} u^r_k\big(c^r_k(\mathbf{p}_{-k})\big). \quad (7)$$

An ubiquitous example of a function satisfying the above conditions is the channel capacity function given by the Shannon-Hartley theorem [8] $u^r_k(x) = \beta \log_2\left(1 + \frac{s}{\sigma^2 + x}\right)$, where $\beta, s, \sigma > 0$. Here β is the channel bandwidth, s is the channel state and x is Gaussian noise with variance σ^2.

MMT k has to pay a fixed price $\pi^r_k \geq 0$ for transmitting on RB r. The total price paid for a particular action is

$$\pi_k(\mathbf{p}_k) := \sum_{r : p^r_k = 1} \pi^r_k. \quad (8)$$

We do not assume that MMT's have a power constraint. Instead, the power required for transmission can be factored into π^r_k. MMTs thus have to make a trade-off between increasing u_k and decreasing π_k. The final objective of MMT k is to maximize the function

$$f_k = u_k - \pi_k. \quad (9)$$

Assuming the RBs and MMTs are fixed, an RB selection game is defined by the objective functions f_k. We may thus specify an RB selection game $\mathcal{G}$ by just specifying the objective functions f_k, keeping in mind that f_k encapsulates both the utility and price functions

$$\mathcal{G} = \big(\{f_k\}_{k \in \mathcal{K}}\big) = \big(\{u_k - \pi_k\}_{k \in \mathcal{K}}\big). \quad (10)$$

Finally, we may restrict out attention to $\mathcal{G}^r = \big(\{f^r_k\}_{k \in \mathcal{K}}\big)$, which is the game obtained if RB r is the only RB available in the network.

3. CONVERGENCE OF BEST-RESPONSE DYNAMICS

In our RB selection game, the additive nature of the MMTs utility functions implies that best-response updates take the following form:

Definition 1 (RB best-response) *The best-response of MMT k to the combined actions of the other MMTs, $\mathbf{p}_{-k}$, is given by $(p^1_k, \ldots, p^R_k)$ where $p^r_k = 1$ if and only if $f^r_k(\mathbf{p}^r_{-k}) > 0$.*

Since the prices π^r are constant, the condition $f^r_k(\mathbf{p}^r_{-k}) = u^r_k\big(c^r_k(\mathbf{p}_{-k})\big) - \pi^r_k > 0$ is equivalent to $u^r_k\big(c^r_k(\mathbf{p}_{-k})\big) > \pi^r_k$. Since u^r_k is a decreasing function, this is the same as

$$c^r_k(\mathbf{p}_{-k}) < u^{r\,-1}_k(\pi^r_k), \quad (11)$$

where $u^{r\,-1}_k$ is the inverse of u^r_k. The RB best-response update rule coincides with the lazy best-response update rule in [4]. Written in this way, MMTs do not need to carry out an exhaustive search over all their possible actions, which may be prohibitive when R is large.

To prove convergence of RB best-response dynamics, we first define two common concepts from game theory: Nash equilibria and ordinal potential games.

Definition 2 *A combined action $\mathbf{p} = (\mathbf{p}_k, \mathbf{p}_{-k}) \in \mathcal{P}$ is a* Nash equilibrium *if*

$$f_k(\mathbf{q}_k, \mathbf{p}_{-k}) - f_k(\mathbf{p}_k, \mathbf{p}_{-k}) < 0 \quad (12)$$

for all $\mathbf{q}_k \in \mathcal{P}_k$ and $k \in \mathcal{K}$.

Nash equilibria play an important role in game theory because they are the points at which no player will want to unilaterally deviate. Any such deviation leads to a decrease in the objective function of that player. Thus Nash equilibria, if they exist, will be fixed points of iterative best-response updates. We will show that RB best-reponse dynamics will converge to a Nash equilibrium by defining an ordinal potential function for the game.

Definition 3 *A game $\mathcal{G} = \big(\{f_k\}_{k\in\mathcal{K}}\big)$ is an* ordinal potential game *if there exists a function $\phi : \mathcal{P} \to \mathbb{R}$ satisfying*

$$f_k(\mathbf{q}_k, \mathbf{p}_{-k}) - f_k(\mathbf{p}) > 0 \iff \phi(\mathbf{q}_k, \mathbf{p}_{-k}) - \phi(\mathbf{p}) > 0 \tag{13}$$

for all $k \in \mathcal{K}, \mathbf{p} \in \mathcal{P}, \mathbf{q}_k \in \mathcal{P}_k$. The function ϕ is called an ordinal potential function *of the game.*

In an ordinal potential game, players' objectives are aligned with the ordinal potential function in the sense that a player's utility function will increase (or decrease) if and only if the ordinal potential function also increases (or decreases). Further, ordinal potential games do not have weak-improvement cycles, and are guaranteed to have at least one (not necessarily unique) Nash equilibrium [9]. In particular, the combined action that maximizes ϕ will be a Nash equilibrium. The main result of this paper is the derivation of an ordinal potential function for the RB selection game, which is shown in the following theorem.

Theorem 1 *Define the following function for each RB*

$$\phi^r(\mathbf{p}^r) := -\frac{1}{2}\sum_{k,l\in\mathcal{K}} A^r_{kl}\, p^r_k\, p^r_l + \sum_{k\in\mathcal{K}} u^{r\,-1}_k(\pi^r_k)\, p^r_k, \tag{14}$$

Then ϕ^r is an ordinal potential function for $\mathcal{G}^r$,the RB selection game restricted to RB r.

Proof We need to prove the following statement for each RB r:

$$\phi^r(q^r_k, \mathbf{p}^r_{-k}) - \phi^r(\mathbf{p}^r) > 0 \iff f^r_k(q^r_k, \mathbf{p}^r_{-k}) - f^r_k(\mathbf{p}^r) > 0. \tag{15}$$

Suppose $p^r_k = 0$ and $q^r_k = 1$. Note that (14) can be rewritten as

$$\phi^r(\mathbf{p}^r) = -\frac{1}{2}\big(2\sum_{k,l:p^r_k,p^r_l=1} A^r_{kl}\big) + \sum_{k:p^r_k=1} u^{r\,-1}_k(\pi^r_k), \tag{16}$$

so the difference in ϕ^r when moving from $\mathbf{p}^r$ to $\mathbf{q}^r$ is given by

$$\phi^r(q^r_k, \mathbf{p}^r_{-k}) - \phi^r(\mathbf{p}^r) = -\frac{1}{2}\big(2\sum_{l:p^r_l=1} A^r_{kl}\big) + u^{r\,-1}_k(\pi^r_k), \tag{17}$$

which is positive if and only if

$$\begin{aligned} u^{r\,-1}_k(\pi^r_k) &> \sum_{l:p^r_l=1} A^r_{kl} \\ &= c^r_k(\mathbf{p}_{-k}). \end{aligned} \tag{18}$$

Applying the decreasing function u^r_k to both sides gives $\pi^r_k < u^r_k\big(c^r_k(\mathbf{p}_{-k})\big)$ and thus $f^r_k(\mathbf{p}^r_{-k}) > 0$. ■

Corollary 1 *The value of ϕ^r increases when MMTs iteratively carry out RB best-reponse updates on RB r.*

Corollary 2 *Let $\Phi = \sum_{r\in\mathcal{R}} \phi^r$. Then Φ increases when MMTs iteratively carry out RB best-response updates.*

Since there are only a finite number of possible actions, Φ can only take a finite number of values. The convergence of iterative best-response dynamics in the RB selection game follows directly from this result. In fact, Corollary 2 implies that Φ is a *generalized* ordinal potential function [7] of $\mathcal{G}$, i.e.

$$f_k(\mathbf{q}_k, \mathbf{p}_{-k}) - f_k(\mathbf{p}) > 0 \implies \Phi(\mathbf{q}_k, \mathbf{p}_{-k}) - \Phi(\mathbf{p}) > 0 \tag{19}$$

for all $k \in \mathcal{K}, \mathbf{p} \in \mathcal{P}, \mathbf{q}_k \in \mathcal{P}_k$. This in turn implies that the RB selection game has the Finite Improvement Property [7].

Note that Theorem 1 depends on he symmetry of A^r, not on the actual values of A^r_{kl} or π^r_k. This implies that the convergence of the RB selection does not depend on the interference between players or the price imposed on each RB. Network managers can thus set prices to meet their own objectives without worrying about the convergence of the system.

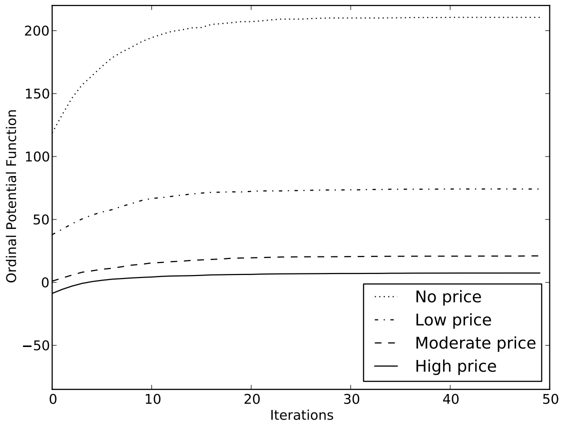
It turns out that $-\phi^r$ is analogous to the energy of a configuration in an Ising model from statistical physics or a Hopfield network [10] from artificial neural networks. Briefly, we have a set of particles $\{1, 2, \ldots, K\}$ that may be in one of two states, -1 or 1. Each particle's state depends on the states of the other particles through a symmetric correlation matrix W, as well as on an external field θ that is applied to each particle. The correlation between the states of particles k and l is represented by the value W_{kl}. Let $\mathbf{x} = (x_1, \ldots, x_K)$ represent the states of the particles at some point in time. Then the state of particle k at the next step in time depends on the quantity

$$\sum_l W_{kl} x_l - \theta_k, \tag{20}$$

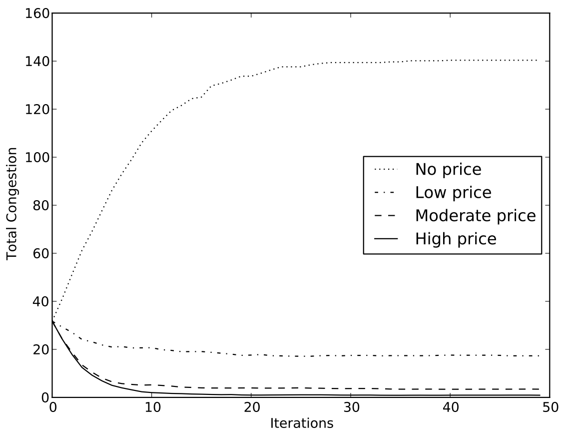
where W is a symmetric matrix with zeros along the diagonal. In a Hopfield network, $x_k = 1$ if $\sum_l W_{kl} x_l - \theta_k > 0$. If particles update their states asynchronously according to this rule, it can be shown that the energy function

$$-\frac{1}{2}\sum_{k,l} W_{k,l} x_k x_l + \sum_k \theta_k x_k \tag{21}$$

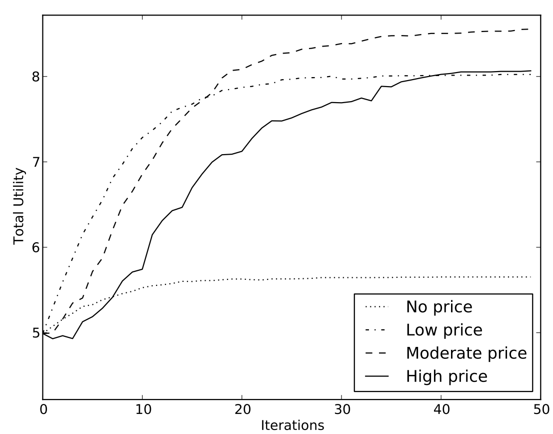
will always decrease. One can interpret the matrix W in the following way: if $W_{kl} > 0$, then the states of particles k and l are positively correlated. The RB selection game restricted to a single RB r can be though of as a Hopfield network or Ising model with matrix $W = -A^r$. Since the entries of $-A^r$ are negative, this implies that MMTs are trying to play actions that are negatively correlated to each other, which agrees with what intuition tells us would happen when MMTs mutually interfere. The price scheme is the analogue of the external field, allowing us to influence the behaviour of MMTs with appropriate prices.



(a) Generalized Ordinal Potential Function (Φ)



(b) Total Congestion



(c) Total Utility

Fig. 1. Best-response dynamics under 4 price schemes. Under best-response updates, the functions in (a) always increase, but those in (b) and (c) do not.

4. SIMULATIONS

To illustrate our results, we carry out a simulation following the setup of [1], with 8 MMTs sharing 5 RBs (i.e. $K = 8$ and $R = 5$). The interference matrix A^r is a symmetric random matrix with entries $A^r_{kl} \in [0, 1]$, $A^r_{kk} = 0$. The utility function is the channel capacity function given by the Shannon-Hartley theorem [8]

$$u^r_k(x) = \beta^r \log_2 \left(1 + \frac{s^r_k}{(\sigma^r_k)^2 + x}\right), \tag{22}$$

where β^r_k is the channel bandwidth of RB r, s^r_k is the channel state of RB r to MMT k and $(\sigma^r_k)^2$ is the noise variance of RB r to MMT k. We fix $(\sigma^r_k)^2 = 0.1$, while randomly choosing the other paramters uniformly over $s^r_k \in \{1, 2, 3, 4\}$, and $\beta^r \in \{1, 2, 3\}$. Our simulation results are averaged over 50 scenarios with varying A, β and s. For each scenario, we imposed 4 pricing schemes, ranging from no price at all, to a high price that discourages transmission. We then compare the evolution of three values as MMTs carry out RB best-response updates: a) the value of Φ, b) the total congestion faced by the MMTs, and c) the total utility of all the MMTs.

From Fig.1(a) we see that Φ always increases with each iteration, confirming Corollary 2. This is not the case for the total congestion, $\sum_{r\in\mathcal{R}} \sum_{k,l\in\mathcal{K}} A^r_{kl} p^r_k p^r_l$, or the total utility $\sum_{k\in\mathcal{K}} u_k$. These two other functions thus cannot be used as generalized ordinal potential functions for the RB selection game. Note that the moderate price scheme maximizes the total utility. A network manager seeking to maximize the total utility would thus choose this price scheme over the other three.

5. CONCLUSION

In this paper we have modelled network selection in a heterogeneous wireless network as an RB selection game. We have shown that this game is a sum of ordinal potential games by defining ordinal potential functions that are closely related to the total congestion experienced in the network. Further, the convergence of the system is not dependent on the actual price that is imposed on each RB. Future work could could consider non-additive utility functions (e.g. power constraints and cross-channel interference), as well as time-varying channel conditions, while also making further use of the physical interpretation of the ordinal potential function as the energy of a configuration in an Ising model or a Hopfield network to derive new, non-deterministic algorithms for network selection.

6. REFERENCES

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