ELLIPSE FITTING USING FINITE RATE OF INNOVATION PRINCIPLES

Satish Mulleti and Chandra Sekhar Seelamantula

Department of Electrical Engineering
Indian Institute of Science
Bangalore - 560012, Karnataka, India
Emails: satishm@ee.iisc.ernet.in, chandra.sekhar@ieee.org

ABSTRACT

We address the problem of parameter estimation of an ellipse from a limited number of samples. We develop a new approach for solving the ellipse fitting problem by showing that the x and y coordinate functions of an ellipse are finite-rate-of-innovation (FRI) signals. Uniform samples of x and y coordinate functions of the ellipse are modeled as a sum of weighted complex exponentials, for which we propose an efficient annihilating filter technique to estimate the ellipse parameters from the samples. The FRI framework allows for estimating the ellipse parameters reliably from partial or incomplete measurements even in the presence of noise. The efficiency and robustness of the proposed method is compared with state-of-art direct method. The experimental results show that the estimated parameters have lesser bias compared with the direct method and the estimation error is reduced by 5-10 dB relative to the direct method.

***Index Terms*—** Ellipse, parametric curves, finite-rate-of innovation, annihilating filter

1. INTRODUCTION

Data reduction and data classification are important tasks in computer vision and pattern recognition [1]. One of the ways to perform these operations is by curve fitting. For example, given N two-dimensional data points, we can fit a circle to the data points by minimizing a predefined objective function. Once the parameters of the circle are estimated, the N data points can be represented by the coordinates of the center of the circle and the radius of the circle. Ellipse fitting problem is also extensively employed in image analysis, computer vision, and pattern recognition problems. For example, the boundaries of images of cells are well approximated by ellipses [2]. Ellipse fitting problem is also encountered in astronomy, where the path of the motion of the heavenly bodies can be approximated by an ellipse.

1.1. Related work

Ellipse fitting or ellipse parameter estimation is a classical but still actively researched area. Ellipse fitting methods can be broadly classified into clustering-based methods or distance-based least squares (LS) methods. Clustering based methods include Hough transform [3], [4] and fuzzy clustering [5] based techniques. The advantages of these methods are that they can fit multiple ellipses simultaneously and they are robust against outliers. However, these methods are slow and require large amount of storage and the computational cost is too high. Zhang and Liu [6] proposed a low complexity Hough transform-based ellipse fitting. In the LS-based methods, the parameters of an ellipse are estimated by minimizing either the algebraic distance or geometric distance between the data points and the ellipse. In algebraic distance-based methods, the ellipse parameters from the implicit second-order polynomial are estimated by minimizing sum-of-squared algebraic distance over all the data points. To avoid trivial solutions, certain constraints are imposed on the parameters [7]- [15]. These methods may not always result in an ellipse. Fitzgibbon et al. [10] proposed an ellipse specific algebraic distance based method, which is solved using generalized eigenvalue-based approach. The algebraic distance does not have any physical interpretation and these methods produce highly biased ellipses in the presence of partial data. To address some of these issues, different approaches have been proposed in which geometric or statistical distances are minimized. In geometrical distance measure [16], [17], [18], the curve parameters are obtained by minimizing the Euclidian distance between the data point and the ellipse. These methods are iterative and require an initial estimate of the ellipse parameters. Several modifications of algebraic and geometrical distances have been proposed in [19–21]. Based on the statistical analysis of data for ellipse fitting problem, different ellipse estimation methods have been proposed in [22–24].

1.2. This paper

In this paper, we take a different viewpoint to the ellipse fitting problem, by considering the parametric equations of an ellipse:

$$x(t) = x_0 + a\,\cos(\theta)\cos(t) - b\,\sin(\theta)\sin(t), \qquad (1)$$

$$y(t) = y_0 + a\,\sin(\theta)\cos(t) + b\,\cos(\theta)\sin(t), \qquad (2)$$

where $t \in [0, 2\pi)$, (x_0, y_0) are the coordinates of the center of the ellipse, a and b are lengths of the major and minor axes,

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respectively, and θ represents the angle between major axis of the ellipse and x-axis. The parametric equations of the ellipse in (1) and (2) are defined by finite number of free parameters $\{x_0, y_0, a, b, \theta\}$ for $t \in [0, 2\pi]$, and hence they fall under the class of finite-rate-of-innovation (FRI) signals. We propose to estimate the parameters of the ellipse using the FRI signal sampling method first proposed by Vetterli et al. [25]. Subsequently, Dragotti et al. linked it with Strang-Fix conditions of wavelet theory [26]. FRI signals have a finite number of free parameters over the unit interval. The samples (with suitable sampling kernel) of FRI signals, contains the parameter information of corresponding analog signals in the form of frequencies and amplitudes of sum of weighted complex exponentials (SWCE). Hence, the signal reconstruction problem reduces to one of parameter estimation of SWCE, which is a classical problem in high-resolution spectral estimation (HRSE) [27].

1.3. Problem formulation

Suppose we are given N uniform samples of $x(t)$ and $y(t)$ with unknown sampling intervals T_x and T_y, respectively. We assume that the samples are corrupted by independent and identically distributed (i.i.d.) zero mean additive white Gaussian noise (AWGN), that is,

$$\tilde{x}(n) = x(nT_x) + w_x(n), \tag{3}$$
$$\tilde{y}(n) = y(nT_y) + w_y(n), \quad \text{for } n = 1, 2 \cdots N, \tag{4}$$

where $\{w_x(n)\}_{n=1}^{N}$ and $\{w_y(n)\}_{n=1}^{N}$ are i.i.d Gaussian random variables with zero mean and standard deviation σ_x and σ_y, respectively. The goal is to estimate the five ellipse-specifying parameters $\{x_0, y_0, a, b, \theta\}$ of the underlying ellipse as accurately as possible.

2. PROPOSED SOLUTION FOR ELLIPSE PARAMETER ESTIMATION

The problem posed in the previous section is a non-linear one, since the sampling intervals T_x and T_y are in general unknown. The proposed parameter estimation method solves the estimation problem in two stages. In the first stage, we apply HRSE methods to estimate the sampling intervals. We then use the estimated sampling intervals in the second stage, to estimate ellipse specific parameters using LS mthod. As the accuracy of estimating parameters depends on estimation accuracy of T_x and T_y, we propose a lowpass filtering based denoising approach to improve the accuracy of sampling intervals and ellipse parameters.

2.1. Ellipse fitting using high-resolution methods

Using Euler's identity $e^{j\phi} = \cos\phi + j\sin\phi$, (3) and (4) are rewritten as

$$\tilde{x}(n) = x_0 + \alpha_1 e^{jnT_x} - \alpha_2\, e^{-jnT_x} + w_x(n), \tag{5}$$
$$\tilde{y}(n) = y_0 + \beta_1 e^{jnT_y} - \beta_2\, e^{-jnT_y} + w_y(n), \tag{6}$$

where the scalars $\alpha_1, \alpha_2, \beta_1$ and β_2 are related to ellipse parameters as

$$\alpha_1 = (a\,\cos\theta + jb\,\sin\theta)/2,\ \alpha_2 = (a\,\cos\theta - jb\,\sin\theta)/2,$$
$$\beta_1 = (a\,\sin\theta - jb\,\cos\theta)/2,\ \beta_2 = (a\,\sin\theta + jb\,\cos\theta)/2.$$

In (5) and (6), the uniform samples of parametric ellipse are sequences of sum of three complex exponentials with frequencies at $[-T_x, 0, T_x]$ and $[-T_y, 0, T_y]$, respectively, corrupted with AWGN. Estimating T_x and T_y from (5) and (6) is equivalent to the classical problem of estimating frequencies of SWCE in the presence of AWGN [27]. There are several methods proposed in the literature [27] to solve this problem. However, in the ellipse fitting applications where limited number of data points $\{\tilde{x}(n), \tilde{y}(n)\}$ are given, annihilating filter method [25] is well suited.

One could apply a third-order annihilating filter separately to $\tilde{x}(n)$ and $\tilde{y}(n)$ to estimate T_x and T_y. In each of (5) and (6), we have prior information that one of the frequency (exponents in (5) and (6)) is located at zero and other two have same magnitude and opposite sign. Using this information we propose a modified annihilating filter to suit the signal model. We present the method for estimating T_x from $\tilde{x}(n)$, and a similar analysis applies for estimating T_y from $\tilde{y}(n)$.

2.2. Modified annihilating filter

In the absence of noise, the transfer function of causal annihilating filter for the sequence $\tilde{x}(n)$ is given as

$$H(z) = (1 - e^{-jT_x}z^{-1})(1 - z^{-1})(1 - e^{jT_x}z^{-1}), \tag{7}$$

and the corresponding impulse response is given by $h = [1,\ -r,\ r,\ -1]$, with $r = 1 + 2\cos T_x$. The output of the annihilating filter to the input $\tilde{x}(n)$ is given by the sequence $c_{\tilde{x}}(n) = \tilde{x}(n) * h(n) = (\tilde{x}(n) - \tilde{x}(n-3)) - r(\tilde{x}(n-1) - \tilde{x}(n-2))$ for $n > 3$. Ideally the sequence $c_{\tilde{x}}(n)$ should be zero for $n > 3$, but in the presence of noise sequence $c_{\tilde{x}}(n)$ can not be zero for any choice of real T_x. We estimate T_x that minimizes the cost $\sum_{n=4}^{N} |c_{\tilde{x}}(n)|^2$. The closed-form expression for estimate of r in terms of noisy samples is given as $\hat{r}_x = \frac{\sum_{n=3}^{N-1}(\tilde{x}(n)-\tilde{x}(n-3))(\tilde{x}(n-1)-\tilde{x}(n-2))}{\sum_{n=3}^{N-1}(\tilde{x}(n-1)-\tilde{x}(n-2))^2}$. The estimated sampling interval is given as $\hat{T}_x = \cos^{-1}(\frac{\hat{r}_x - 1}{2})$. The advantage of the modification in the annihilating filter is that it does not require root-finding procedure, which is present in conventional annihilating filter and a closed-form expression for estimated T_x is derived, which reduces the computations considerably.

2.3. Denoising using lowpass filtering (LPF)

We propose a lowpass-filtering-based denoising approach, which precedes the annihilating filter. Since $\tilde{x}(n)$ is composed of a sum of complex exponentials with frequencies at $[-T_x, 0, T_x]$, and AWGN $w_x(n)$, in the frequency spectrum of $\tilde{x}(n)$, the ellipse specific information is available in the frequency band of $[-T_x, T_x]$. We can reduce the effect of the noise outside the spectral band by applying a LPF operation with a cutoff frequency chosen as T_x. In practice, we use an approximation of T_x, given by physical constrain on the sampling method. Since we need to estimate T_x from the output sequence of the LPF operation, we need the filtered output to be in sum-of-complex-exponentials form. This condition restrict the filter to have finite impulse response (FIR) and it imposes a restriction on the length of the filter for a given cutoff frequency. We derive conditions on the filter with the help of a simple example. Suppose we are given a length-N complex exponential sequence $x_0(n) = e^{\mathrm{j}\omega_0 n}$ for $n = 0 : N-1$ and a FIR filter $\{h(n)\}_{n=0}^{M}$ of length $M+1$ with $M < N$. The convolution of the two sequences is given as

$$(x_0 * h)(n) = \begin{cases} 0, \quad \text{for} \quad n < 0, \\ \sum_{m=0}^{n} h(m) e^{-\mathrm{j}\omega_0 (m-n)}, \quad \text{for} \quad 0 \leq n \leq M-1, \\ H(\omega_0) e^{\mathrm{j}\omega_0 n}, \quad \text{for} \quad M \leq n \leq N-1, \\ \sum_{m=n-M}^{N-1} h(m) e^{-\mathrm{j}\omega_0 (m-n)}, \text{ for } N \leq n < N+M, \\ 0, \quad \text{for} \quad n \geq N+M, \end{cases}$$

where $H(\cdot)$ is the frequency response of the filter $h(n)$. This example shows that the convolution output is a complex exponential with frequency ω_0 for $M \leq n \leq N-1$. Since the filtering operation is linear, we can extend the analysis to a sum-of-complex exponentials in $\tilde{x}(n)$. Hence, by applying M-length FIR filter for denoising, we have $N - M$ output samples in the form of SWCE form, which are subsequently used in annihilating filter method to estimate the sampling interval T_x. Given a fixed cutoff frequency, longer the filter impulse response M, better is the denoising performance. However, this reduces the number of effective samples (which have SWCE form) available for the annihilating filter. Hence, the length of the denoising filter acts as a tradeoff parameter. For the sequence $\tilde{x}(n)$ (in absence of noise), the effective filtered output is given by

$$\tilde{x}_L(n) = x_0 H_L(0) + \alpha_1 H_L(T_x) e^{\mathrm{j}nT_x} - \alpha_2 H_L(-T_x) e^{-\mathrm{j}nT_x},$$

for $n \in [M, N-1]$ and $H_L(\cdot)$ is frequency response of the LPF. Once sampling interval is estimated, we can employ LS methods to estimate the ellipse parameters. These parameters should be scaled appropriately by $H_L(0), H_L(T_x)$ and $H_L(-T_x)$ (can be calculated using estimated T_x) to compensate for the amplitude scaling due to filtering.

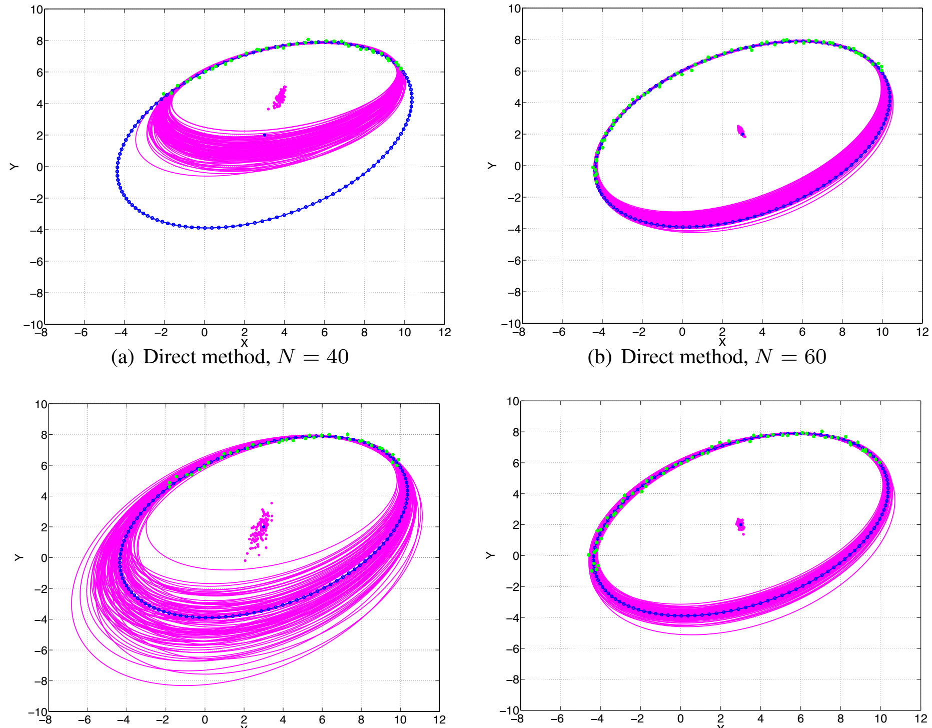
2.4. LS method to estimate the ellipse parameters

Once the sampling intervals T_x and T_y are estimated from $\tilde{x}(n)$ and $\tilde{y}(n)$ by proposed modified annihilating filter method, parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 in (5) and (6) are linearly related to $\tilde{x}(n)$ and $\tilde{y}(n)$. These four parameters are estimated from $\{\tilde{x}(n), \tilde{y}(n)\}_{n=1}^{N}$ using the LS method. Once $\alpha_1, \alpha_2, \beta_1$ and β_2 are estimated, the ellipse-specific parameters $\{x_0, y_0, a, b, \theta\}$ are derived from them.

3. SIMULATION RESULTS

In this section we present simulation results in the presence of noise and for partial ellipse data and compare them with the Fitzgibbon's direct method [10]. In all the experiments, we set the ellipse parameters to be $x_0 = 3, y_0 = 2, a = 8, b = 5, \theta = 30^o$ and $T_x = T_y = 0.05$ and the ellipse data points are generated using these parameters. The first set of experiments is designed to demonstrate the accuracy of the proposed method in estimating the parameters from partial data. We run the experiments for $N = 40$ and $N = 60$ data points, which are taken from one-third and half of the arc of the ellipse. In both the cases we applied the LPF for denoising. The LPF has a cutoff frequency of 0.01π radians and was designed using Kaiser window approach with window parameter 4. The order of the filter is $M = 28$ for $N = 40$, and $M = 48$ for $N = 60$, respectively. In Fig. 1, we show 100 independent realizations of estimated ellipses with noise standard deviation $\sigma_x = \sigma_y = 0.1$. In these plots, the blue curve shows the ground truth ellipse and the magenta colored ellipses are the estimated ones. The green points are one realization of noisy data samples used to estimate the ellipse parameters. The magenta points in the center show the estimated center of ellipses and the blue point actual center. We observe that, for $N = 40$, the estimated ellipses using the direct method are highly biased. However in the proposed method, with $N = 40$, the estimated ellipses are less biased and the bias depends on under- or over- estimation of the sampling intervals T_x and T_y. With $N = 60$, both methods perform almost similarly, but a small amount of bias is present in the direct method [10].

The second experiment is performed to study the effect of measurement noise on the performance of the proposed method. In the experiment we vary $\sigma_x = \sigma_y$ from 0.05 to 0.5 in steps of 0.05. In Fig. 2, we show the normalized mean-square error (MSE) and bias of ellipse parameters for $N = 60$. We employed LPF of order $M = 48$ and cutoff frequency of 0.01π radians. The LPF is truncated using a Kaiser window with window parameter 6. Let θ and $\hat{\theta}$ denote the actual and estimated parameters, respectively. The normalized MSE is given by $\mathcal{E}(\theta - \hat{\theta})^2/\theta^2$ and normalized bias by $(\theta - \mathcal{E}(\hat{\theta}))/\theta$, respectively, where $\mathcal{E}(\cdot)$ denotes the expectation operator. In our simulations, we approximate it by sample mean over 1000 independent realizations for each σ_x. In Fig. 2, we show that the proposed method has less bias relative to the direct method for various ellipse parameters and the bias is close to zero for different noise levels. The MSE in the proposed method is lower than that of the direct method by 5 to 10 dB, for all ellipse parameters except for the major axis.



(a) Direct method, $N = 40$ (b) Direct method, $N = 60$

(c) Proposed method, $N = 40$ (d) Proposed method, $N = 60$

Fig. 1. Ellipse fitting based on partial observations. The blue contour denotes the ground truth, whereas the magenta contours show estimated ellipses. The estimated centers are also shown in magenta and ground truth centers in blue.

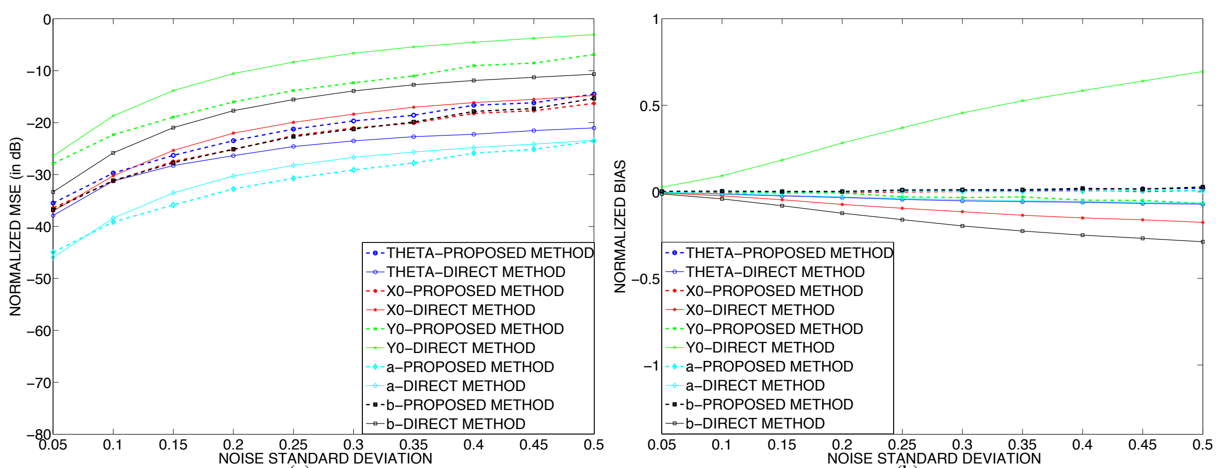


Fig. 2. (a) Normalized MSE, and (b) normalized bias, versus noise standard deviation; uniform sampling; $N = 60$.

4. CONCLUSIONS

We presented a new approach for ellipse fitting by observing that the parametric equations of an ellipse satisfy the FRI signal model. The uniformly sampled sequence of the x and y coordinate functions of ellipse are modeled as a sum of weighted complex exponentials and a modified annihilating filtering approach is proposed to estimate the ellipse parameters. The proposed technique is compared with Fitzgibbon's direct ellipse-fitting method. The FRI based method is unbiased compared with Fitzgibbon's method and the MSE in the estimated parameters is less by about 5 to 10 dB in most of the ellipse parameters over a wide range of noise levels. We are currently carrying out validations on experimental data and methods for ellipse fitting based on randomly sampled data.

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