DISTRIBUTIONAL UPPER BOUND ON THE INTERFERENCE IN SPATIAL WIRELESS MULTIUSER ULTRAWIDEBAND COMMUNICATION SYSTEMS

Gareth W. Peters[1], *Ido Nevat*[2], *Laurent Clavier*[3] *and François Septier*[4]

[1]Department of Statistical Science, University College London, UK
[2]Institute for Infocomm Research, A*STAR, Singapore
[3] Institut Mines-Télécom / Télécom Lille / IRCICA-IEMN UMR CNRS 8520, France
[4] Institut Mines-Télécom / Télécom Lille / LAGIS UMR CNRS 8219, France

ABSTRACT

We develop a novel distributional upper bound on the interference created in an ultra-wideband wireless communication systems under two general assumptions: the first is that there is an unknown number of interferers who are distributed according to a homogeneous Poisson point process randomly in space; and the second is that the frequency bands occupied by the unknown number of interferers is also a random variable in an ultra-wideband setting. Then given these two general assumptions, we derive a distributional upper bound representation of the total interference.

Index Terms— Interference Models, α-Stable, Geometric Stable

1. INTRODUCTION

Interference from undesired active users in a wireless network is a strong limitation on the performance of communication systems. In such settings, signal reception is corrupted by interference from other users that co-occupy the same spatial domain and propagation medium. Therefore, there is a need to characterize the properties of interference resulting from spatially distributed users. The interference arises at the receiver due to extraneous signals radiated by other users who are distributed in the field of transmission of the primary user. Examples include ad-hoc networks and cognitive radio. The characterization of the spatial interference in practical wireless communication systems in which several attributes of the interferers are considered unknown a-priori is an emerging field of research that combines relevant components of stochastic geometry and wireless communications, see examples in [1,2]. In such settings it is easily shown that the interference is impulsive in nature, see [3–8]. In addition, it has been shown that the resulting impulsive interference dominates over the contribution due to thermal noise.

The development of spatial impulsive interference models is of practical importance as it provides a framework for practitioners to develop algorithms in order to undertake tasks such as symbol detection, channel estimation and power allocation. The first step in such a process is a clear mathematical framework to understand the properties of the interference incident from the spatially distributed unknown number of interferers. This is precisely what we characterize in this paper.

Previous approaches that have studied spatial interference under impulsive noise have considered the following models: Laplace, Generalized Gaussian, Cauchy, α-stable, Middleton class A, see example details in [6, 7, 9]. These existing approaches that have been developed have dealt purely with scenarios in which the bandwidth occupied by the interferes is assumed fixed and known.

In this paper we generalize these frameworks to make the results widely applicable to address practical settings in which the bandwidth or the number of carrier frequencies utilized by the unknown number of interferers that are randomly distributed in space is itself also an unknown random quantity, making for the most general class of models one may consider. This scenario arises for example in settings in which multiple networks occupying the same frequency bands with the same transmission characteristics are co-located.

1.1. Multiuser Ultrawideband Communication System Model

We present a generic multi-user wireless system model without power control and develop the total interference model for a recieved signal observed in the presence of a random number of unknown spatially distributed interferers. In particular, we develop the interference model for a wireless communication network caused by users that share the same propagation medium. We treat the interfering users as spatially Poisson distributed and their transmitted signals are subject to a power-law propagation loss function. The system model is defined as follows:

1. Consider an unknown number of spatially distributed transmitters denoted by N. They are distributed on a circular domain $\Omega\left(A_{\mathcal{R}}\right) := \left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq r_T\right\}$ with area $A_{\mathcal{R}}$, at locations indexed by $\mathcal{L} = \left\{L^{(i)}\right\}_{i=1\ldots N}$ according to an homogeneous spatial Poisson process with intensity parameter λ. Therefore, the number of transmitters in $\Omega\left(A_{\mathcal{R}}\right)$ has distribution
$$\mathbb{P}\left(N=n\right) = \frac{\left(\lambda A_{\mathcal{R}}\right)^n}{n!}\exp\left(-\lambda A_{\mathcal{R}}\right).$$
2. The i-th interferer ($i \in \{1, 2, \ldots, N\}$) transmits an i.i.d wide band signal, represented in the frequency domain by:
$$\mathcal{S}^{(i)}(f) = \sum_{k=1}^{K} X_k^{(i)} G_k(f - f_k),$$
where we decompose the signal on elementary bands whose width is given by a shaping filter $G_k(f)$. Note,

the shaping filter can be the same function on each frequency band or a different function with the same bandwidth for more general settings. The central frequency of the shaping function is denoted by f_k, otherwise commonly known as a subcarrier frequency. We denote by $X_k^{(i)}$ the unknown i-th interferers transmitted symbol on subcarrier k. We consider the number of carrier frequencies to be a discrete positive valued random variable K. Such a model allows the interferers to utilize adaptive bandwidths (and consequently a changing transmission rate) depending on the channel occupancy that is given by K. Consequently, in the time domain the representation of the frequency domain signal of the i-th interferer is given by:

$$S^{(i)}(t) = \sum_{k=1}^{K} X_k^{(i)} g_k^{(i)}(t) \exp\left(2j\pi f_k t\right).$$

3. The bandwidth of each interferer, as quantified by the number K of carriers, is distributed according to a geometric distribution given by $K \sim$ Geometric (p), given by

$$\mathbb{P}\mathrm{r}\left(K_p = k\right) = p(1-p)^{k-1}, \;\; k = 1, 2, 3, \ldots,$$

with $p = \frac{1}{\lambda}$ and $\mathbb{E}[K] = \frac{1}{p}$. Practical examples of this include the following two scenarios: all interferers transmit in the same bandwidth, but this bandwidth is unknown to the receiver and modeled according to a geometric distribution; alternatively, all interferers utilize the same total bandwidth, but the frequencies occupied by any given user may not overlap, however, the total bandwidth per user is unknown to the receiver and modeled according to a geometric distribution.

4. The distance of the i-th interferer from the receiver is a random variable denoted by $R^{(i)}$ and given by:

$$R^{(i)} = \left\| L^{(i)} - l^R \right\|,$$

where $L^{(i)}$ is a random location of the i-th potential interferer and l^R is a known location of the receiver in region $\mathcal{R}$. Given $N = n$ total interferers, the location of the i-th interferer is uniformly distributed in space over the *circular interference domain* with a distribution given by:

$$f_{R^{(i)}|N}\left(r|N=n\right) = \begin{cases} \dfrac{2r}{r_T^2}, \text{ if } 0 \le r \le r_T \\ 0, \text{ otherwise,} \end{cases}$$

where r_T is the maximal distance in which an interferer can have a non-negligible contribution to the interference.

5. For the i-th interferer, the baseband representation of the channel experienced by the symbol $X_k^{(i)}$ is given by $\frac{A_k^{(i)} e^{j\Phi_k^{(i)}}}{R_i^{-\sigma/2}}$. The random variable for the phase, denoted by Φ_k, is uniformly distributed in $[0, 2\pi]$. The path loss experienced by the i-th interferer is given by $R_i^{-\frac{\sigma}{2}}$, where σ is the attenuation coefficient, a deterministic and known parameter reflecting the physical environment in which transmission is occurring. $A_k^{(i)} e^{j\Phi_k^{(i)}}$ is a complex coefficient that contains the shadowing and multipath fading. This representation is general enough to encompass all commonly encountered fading models.

6. The total interference experienced by the received signal, after applying the target users shaping filter $\left(g^{(0)}\right)$ at the receiver side is given by:

$$\begin{aligned} Y &= \sum_{i=1}^{N} \frac{1}{R_i^{-\sigma/2}} \sum_{k=1}^{K} A_k^{(i)} X_k^{(i)} c_k^{(i)} e^{j\Phi_k^{(i)}} \\ &= \sum_{k=1}^{K}\sum_{i=1}^{N} Y_I^{(k,i)} + j \sum_{k=1}^{K}\sum_{i=1}^{N} Y_Q^{(k,i)} \end{aligned} \tag{1}$$

where $c_k^{(i)}$ is a random variable resulting from the filtering on subcarrier k and depends on the system parameters:

$$c_k^{(i)} = \int_{\Re} g_k^{(i)}(u+\Delta_i) \exp\left(2j\pi(f_k - f_0)u\right) g^{(0)}(T_s - u) du$$

where $g^{(0)}(t-u)$ is the shaping filter of the reference user. It has a wider frequency response than the interferers and is time limited. Its position depends on the sampling time T_s. We only consider interferers for which $g_k^{(i)}(u)$ and $g^{(0)}(T_s - u)$ are both non zero on a common, non null, interval.

2. SPATIAL INTERFERENCE AT A SINGLE FREQUENCY

Existing results of [4–6, 10] study the spatial interference model and show that at a given frequency of transmission, the interference at the receiver experienced by an unknown number of randomly distributed interferers is given by an isotropic α-stable characteristic function (CF) in $\mathbb{C}^2$. To obtain this result one specifies the CF of the k-th transmission frequency of the i-th user.

Definition 1 *The CF of the interference at the k-th transmission frequency, from i-th potential interferer, is given by:*

$$\begin{aligned} &\varphi_{Y_I^{(k,i)}, Y_Q^{(k,i)}}\left(\omega_I^{(k,i)}, \omega_Q^{(k,i)}\right) \\ &= \mathbb{E}_{Y_I^{(k,i)}, Y_Q^{(k,i)}}\left[\exp\left(j\omega_I^{(k,i)} Y_I^{(k,i)} + j\omega_Q^{(k,i)} Y_Q^{(k,i)}\right)\right]. \end{aligned}$$

This expression can then be extended to the CF for the total interference for the k-th transmission frequency, given an unknown number of independent potential interferes N as given next for the k-th transmission frequency, for a random number of $N \in A_{\mathcal{R}}$ potential interferers:

$$\varphi_{Y_I^{(k)}, Y_Q^{(k)}}\left(\omega_I^{(k)}, \omega_Q^{(k)}\right) = \mathbb{E}_{\mathbf{R}, \mathbf{c}_k, \mathbf{A}_k, \mathbf{\Phi}_k, N}\left[\exp\left(F\left(\omega_I^{(k)}, \omega_Q^{(k)}\right)\right)\right].$$

with

$$F\left(\omega_I^{(k)}, \omega_Q^{(k)}\right) = \left(j\sqrt{A}\sum_{i=1}^{N} R_i^{-\sigma/2} A_k^{(i)} c_k^{(i)} \cos\left(\Phi_k^{(i)} - \eta\right)\right)$$

$$A = \left(\omega_I^{(k)}\right)^2 + \left(\omega_Q^{(k)}\right)^2, \;\; \eta = \arctan\left(\frac{\omega_Q^{(k)}}{\omega_I^{(k)}}\right).$$

Given this representation for the CF we can derive the CF of the total interference at the k-th frequency. The resulting log CF, for the total interference at the k-th frequency, for a random number of potential interferers N in a region $A_{\mathcal{R}}$ can be expressed in the form of a CF representing the family of isotropic bivariate α-stable distributions $\mathcal{S}(\alpha, 0, \gamma, \delta; 0)$,

$$\psi_{Y_I^{(k)}, Y_Q^{(k)}}\left(\omega_I^{(k)}, \omega_Q^{(k)}\right) = -\gamma \left| \sqrt{\left(\omega_I^{(k)}\right)^2 + \left(\omega_Q^{(k)}\right)^2} \right|^{\alpha},$$

with $\alpha = \frac{4}{\sigma} < 2$ and $\gamma = \lambda\pi\mathbb{E}_{\mathbf{A}_k, \boldsymbol{c}_k}\left[(A_k c_k)^{\frac{4}{\sigma}}\right]\int_0^\infty \frac{J_1(x)}{x^{\frac{4}{\sigma}}}\mathrm{d}x$ where $J_1(x)$ denotes the Bessel function of the first order. This result is in the form of a bivariate log characteristic function of the complex random variable $Y^{(k)} = \sum_{i=1}^N (Y_I^{(k,i)} + jY_Q^{(k,i)})$ in (1). The resulting characteristic function is a member of the elliptic family of stable distributions for all $0 < \alpha < 2$ and $\gamma > 0$.

3. TOTAL INTERFERENCE IN ULTRA-WIDEBAND SYSTEMS

In this section we extend existing results for the representation of the characteristic function for interference at a single fixed frequency to the setting of total interference for a random number of carrier frequencies, given by the compound Geometric Stable distribution of the compound random sum: $\widetilde{\mathbf{Y}}^K = \sum_{k=1}^K \mathbf{Y}^{(k)}$. Therefore the density of the total interference across all frequencies in a wideband setting is given by studying the geometrically weighted random mixture of the k-fold convolved bivariate stable distributions for the total interference on each carrier frequency discussed above. Note, in the univariate case, since the stable distribution is closed under convolution, then for i.i.d. random variables $X_i \sim \mathcal{S}_\alpha(\beta_i, \gamma_i, \delta_i; 0)$ the distribution of linear combination given k frequencies, is given by the random variable $\sum_{i=1}^k X_i \sim S(\alpha, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}; 0)$ with parameters

$$\widetilde{\gamma}^\alpha = \sum_{i=1}^k \gamma_i^\alpha, \quad \widetilde{\beta} = \frac{\sum_{i=1}^k \beta_i \gamma_i^\alpha}{\sum_{i=1}^k \gamma_i^\alpha},$$

$$\widetilde{\delta} = \begin{cases} \sum_{i=1}^k \delta_i + \tan\frac{\pi\alpha}{2}\left(\widetilde{\beta}\widetilde{\gamma} - \sum_{i=1}^k \beta_j\gamma_j\right), & \alpha \neq 1, \\ \sum_{i=1}^k \delta_i + \frac{2}{\pi}\left(\widetilde{\beta}\widetilde{\gamma}\log\widetilde{\gamma} - \sum_{i=1}^k \beta_j\gamma_j\log\gamma_i\right), & \alpha = 1. \end{cases}$$

However, we are considering in our context bi-variate stable models, therefore we utilize the following representation which uniquely and exactly characterizes the bivariate results whilst allowing us to utilize the univariate representation above. Consider a multivariate isotropic α-stable random vector $\mathbf{Y}$ of dimension d with scale γ and location $\boldsymbol{\delta}$. Its density can be represented under projection as follows [11]. For every vector $\mathbf{u} \in \mathbb{R}^d$, the one-dimensional projection $\langle \mathbf{u}, \mathbf{Y} \rangle$ is a univariate α-stable symmetric RV with stability index α. As detailed in [11,12], the projection onto vector $\mathbf{u}$ in the isotropic case is given by the stable univariate random vector:

$$\langle \mathbf{u}, \mathbf{Y} \rangle \sim \mathcal{S}_\alpha(0, \gamma(\mathbf{u}), \delta(\mathbf{u}); 0).$$

By Cramer-Wold these univariate projections characterize the joint distribution, where $\gamma(\cdot)$ and $\delta(\cdot)$ are called projection parameter functions [13, 14] and [11, Section 2.1]. In the special case of the isotropic multi-variate α-stable model we obtain $\forall \mathbf{u} \in \mathbb{R}^d$ the simplification $\gamma(\mathbf{u}) = \gamma$.

3.1. Asymptotic Distributional Upperbound on Total Interference in Ultra-Wideband Systems

We now present a novel result which shows that in ultra-wideband settings we can significantly simplify the representation of the total interference of these models by utilizing a special distributional convergence result.

We consider the setting in which the total system bandwidth is designed under an ultra-wideband framework in which we are interested in the total interference across all frequencies. In such a setting we can derive a unique representation of the upper bound for the characterization of the total interference according to the domain of attration of a Geometric-Stable distribution. We derive this result via the representation of a spatial complex isotropic stable compound process as the asymptotic limit of a Linnik law (symmetric Geometric-Stable [15]) given in Theorem 1 below.

First we note the following properties of the α-Stable model and how they relate to the relevant properties we shall utilize from the Geometric Stable model. In the case of the α-stable severity model, one has the property that if random vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are i.i.d. α-stable distributed random vectors, then for some constant functions $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$ one has that

$$\boldsymbol{S}_n \stackrel{d}{=} \boldsymbol{a}_n(\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n) + \boldsymbol{b}_n,$$

is also α-stable distributed. In the case of a Geometric stable distribution one has, that as the number of elements in the sum K_p (the number of carrier frequencies occupied by the interferers in the ultra-wideband system) is geometrically distributed with parameter p and i.i.d. interferences with any distribution $\mathbf{X}_i \sim F_X$ such that one has that the compound total (interference) given by

$$\boldsymbol{Y}_{K_p} \stackrel{d}{=} \boldsymbol{a}_{K_p}\left(\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_{K_p}\right) + \boldsymbol{b}_{K_p}$$

with $a(p) > 0$ and $b(p) \in \mathbb{R}^d$ converge weakly (as $p \to 0$) to a limit, then the limit will be the Geometric stable distribution [16].

Theorem 1 (Ultrawideband Total Interference Upper Bound) *The distributional upper bound for the total interference of the ultra-wideband system is given uniquely by the univariate Geometric-Stable model:*

$$Z(\boldsymbol{u}) \sim GS(\alpha_{GS}(\boldsymbol{u}), \beta_{GS}(\boldsymbol{u}), \gamma_{GS}(\boldsymbol{u}), \delta_{GS}(\boldsymbol{u})),$$

with tail index $\alpha_{GS} \in [0, 2]$, skewness parameter $\beta_{GS} \in [-1, 1]$, scale parameter $\gamma_{GS} > 0$ and location parameter $\delta_{GS} \in \mathbb{R}$.

The resulting distributional upper bound for the total intereference given by the Geometric-Stable model characteristic function given by:

$$\log \Phi_{Z(\boldsymbol{u})}(\omega) = \log \mathbb{E}[\exp(i\omega Z(\boldsymbol{u}))]$$

$$= \begin{cases} [1 + \gamma_{GS}(\boldsymbol{u})|\omega|\left(1 + i\beta_{GS}(\boldsymbol{u})(2\pi) sgn(\omega)\log|\omega|\right) \\ -i\delta_{GS}(\boldsymbol{u})\omega], \alpha_{GS} = 1 \\ \left[1 + \gamma_{GS}(\boldsymbol{u})^{\alpha_{GS}}|\omega|^{\alpha_{GS}}\left(1 - i\beta_{GS}(\boldsymbol{u})\, sgn(\omega)\tan\left(\frac{\pi\alpha_{GS}}{2}\right)\right)\right. \\ \left.-i\delta_{GS}(\boldsymbol{u})\omega\right], \alpha_{GS} \neq 1. \end{cases}$$

Proof: For the Ultrawideband case, we consider the asymptotic limit as $p \to 0$ and therefore $K_p \to \infty$. In this case, the compound process for the resulting total interference model converges in distribution to the bivariate Geometric-Stable distribution characterized by a spectral measure Γ_{GS} and a location vector $\boldsymbol{m}$ according to

$$\lim_{p\to 0}\sum_{k=1}^{K_p}\boldsymbol{Y}^{(k)} = \lim_{p\to 0}\sum_{k=1}^{K_p}\left[Y_I^{(k)}, Y_Q^{(k)}\right]^T \xrightarrow{d} \boldsymbol{Z} \sim GS\left(\Gamma_{GS}, \boldsymbol{m}\right),$$

where we used a general result for Geometric random sums of i.i.d. univariate random variables [16] and for the multivariate setting [17]. The case of the symmetric distribution family for the summand random variables is considered the Linnik distribution [18]. Next we modify by first projecting the complex isotropic stable distribution to a univariate stable distribution, which we then approximate the compound process random number of summands by a Geometric distribution with an upper bound achieved by taking the limit as $p \to 0$. Under projection of the total interference onto a vector $\boldsymbol{u} \in \mathbb{R}^2$ which produces a univariate random variable given by

$$Z(\boldsymbol{u}) := \lim_{p\to 0}\left\langle \sum_{k=1}^{K_p}\boldsymbol{Y}^{(k)}, \boldsymbol{u}\right\rangle.$$

■

In addition, in Theorem 2 below we relate this Geometric-Stable characteristic function to that of the α-Stable characteristic function.

Theorem 2 *This limiting distributional upper bound for the total interference is represented according to the α-stable characteristic function (Φ_X) under Zolotarev's B-Type parametrization according to the relationship*

$$\begin{aligned}&\log\Phi_{Z_{N_p}}\left(\theta;\alpha_{GS},\beta_{GS},\gamma_{GS},\delta_{GS}\right)\\ &\qquad = \left[1-\log\Phi_X\left(\theta;\alpha_B,\beta_B,\gamma_B,\delta_B\right)\right]^{-1}\end{aligned}$$

with the following relationships

$$\alpha_{GS}=\alpha_B,\quad \beta_{GS}=\begin{cases}\cot\left(\frac{\pi}{2}\alpha_B\right)\tan\left(\frac{\pi}{2}\beta_B K(\alpha_B)\right), & \alpha\neq 1\\ \beta_B, & \alpha = 1\end{cases}$$

$$\delta_{GS}=\delta_B\gamma_B,\quad \gamma_{GS}=\begin{cases}\cos\left(\frac{\pi}{2}\beta_B K(\alpha_B)\right)\gamma_B, & \alpha\neq 1\\ \frac{\pi}{2}\gamma_B, & \alpha = 1\end{cases}$$

These results hold where the actual carrier frequencies occupied by each interferer in the wideband system are different and where different numbers of carriers may be considered for each interferers occupancy.

In Fig. 1 we illustrate the result obtained in Theorem 1. We present the asymptotic convergence of the the PDF and CDF of the total interference for various values of the mean occupancy, $p = \{0.2, 0.1, 0.075, 0.01\}$.

4. CONCLUSIONS

We developed a distributional upper bound on the interference created in an ultra-wideband wireless communication systems. We considered the practical scenario in which there is an unknown number of interferers who are distributed according to a homogeneous Poisson point process randomly in space, and that the frequency bands occupied by the unknown number of interferers is also a random variable in an ultra-wideband setting. Given these two general assumptions, we

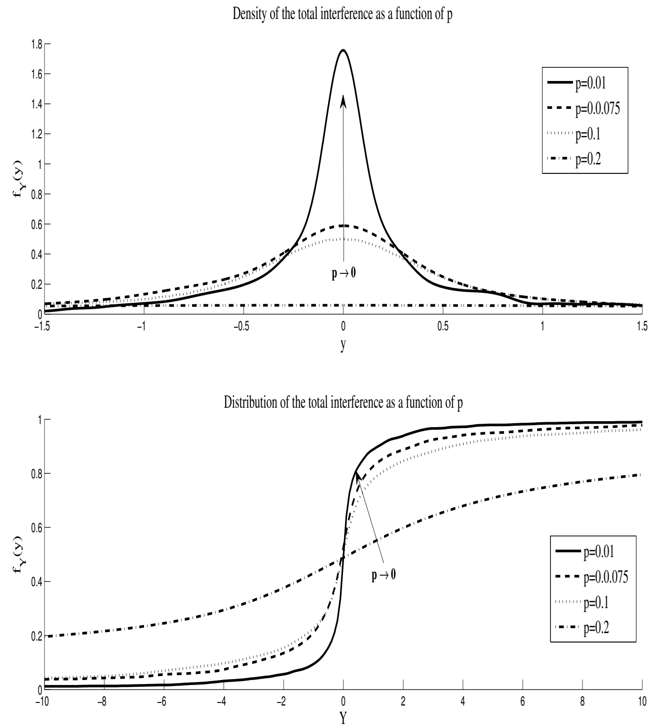

Fig. 1: Distribution of the total interference in an Ultrawideband communication system. The parameter p controls the mean number of subbcarriers occupied.

derived a distributional upper bound representation of the total interference via characterization of the resulting total interference of all interferers as a Geometric Stable distribution.

5. REFERENCES

[1] R. Ganti, F. Baccelli, and J. Andrews, "Series expansion for interference in wireless networks," vol. 58, no. 4, pp. 2194 –2205, april 2012.

[2] M. Kountouris and J. Andrews, "Downlink sdma with limited feedback in interference-limited wireless networks," vol. PP, no. 99, pp. 1 –12, 2012.

[3] E. Sousa, "Performance of a spread spectrum packet radio network in a poisson field of interferers," vol. 38, no. 6, pp. 1743–1754, Nov. 1992.

[4] M. Win, P. Pinto, and L. Shepp, "A mathematical theory of network interference and its applications," vol. 97, no. 2, pp. 205–230, Feb. 2009.

[5] H. E. Ghannudi, L. Clavier, N. Azzaoui, F. Septier, and P. Rolland, "α-stable interference modeling and cauchy receiver for an IR-UWB *ad hoc* network," vol. 58, no. 6, pp. 1748–1757, June 2010.

[6] P. Pinto and M. Win, "Communication in a poisson field of interferers-part ii: Channel capacity and interference spectrum," vol. 9, no. 7, pp. 2187–2195, "jul" 2010.

[7] P. Pinto and M. Win, "Communication in a poisson field of interferers–part i: Interference distribution and error probability," vol. 9, no. 7, pp. 2176–2186, July 2010.

[8] A. Rabbachin, T. Quek, H. Shin, and M. Win, "Cognitive network interference," vol. 29, no. 2, pp. 480–493, Feb. 2011.

[9] N. Beaulieu and D. Young, "Designing time-hopping ultrawide bandwidth receivers for multiuser interference environments," vol. 97, no. 2, pp. 255–284, Feb. 2009.

[10] P. Pinto, C.-C. Chong, A. Giorgetti, M. Chiani, and M. Win, "Narrowband Communication in a Poisson Field of Ultrawideband Iinterferers," in *IEEE International Conference on Ultra-Wideband (ICUWB)*, Sept. 2006, pp. 387–392.

[11] J. Nolan, A. Panorska, and J. McCulloch, "Estimation of stable spectral measures," *Mathematical and Computer Modelling*, vol. 34, no. 9, pp. 1113–1122, 2001.

[12] J. P. Nolan, *Stable Distributions - Models for Heavy Tailed Data*, Birkhauser, Boston, 2012, In progress, Chapter 1 online at academic2.american.edu/$\sim$jpnolan.

[13] G. Samorodnitsky and M. Taqqu, *Stable non-Gaussian processes: Stochastic models with infinite variance*, Chapman & Hall, 1994.

[14] V. Zolotarev, *Univariate stable distributions*, Moscow, Nauka, 1983.

[15] Y. V. Linnik and I. Ostrovskii, *Decomposition of random variables and vectors*, 1977.

[16] T. J. Kozubowski and S. T. Rachev, "The theory of geometric stable distributions and its use in modeling financial data," *European journal of operational research*, vol. 74, no. 2, pp. 310–324, 1994.

[17] T. J. Kozubowski and S. T. Rachev, "Multivariate geometric stable laws," *Journal of Computational Analysis and Applications*, vol. 1, no. 4, pp. 349–385, 1999.

[18] Y. V. Linnik, "Linear forms and statistical criteria, i, ii," *Selected Transl. Math. Statist. and Prob*, vol. 3, pp. 41–90, 1953.